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The bondage numbers of graphs with small crossing numbers☆

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Abstract

The bondage number $b(G)$ of a nonempty graph G is the cardinality of a smallest edge set whose removal from G results in a graph with domination number greater than the domination number $\gamma(G)$ of G. Kang and Yuan proved $b(G) \leqslant 8$ for every connected planar graph G. Fischermann, Rautenbach and Volkmann obtained some further results for connected planar graphs. In this paper, we generalize their results to connected graphs with small crossing numbers.

MSC: 05C69; 05C12

Keywords: Bondage number; Domination number; Crossing number; Planar graph

1. Introduction

For terminology and notation on graph theory not given here, the reader is referred to [7]. Let $G = (V, E)$ be a finite, undirected and simple graph. For each vertex $u \in V(G)$, let $N_G(u)$ be the neighborhood of u and $N_G(X) = \cup_{x \in X} N_G(x)$. We denote the degree of u by $d_G(u) = |N_G(u)|$, the maximum and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively, and the distance between the vertices x and y by $d_G(x, y)$. Let $n_i = n_i(G)$ be the number of vertices of degree i for $i = 1, 2, \ldots, \Delta(G)$. The girth of G, $g(G)$, is the length of the shortest cycle in G. If G has no cycles we define $g(G) = \infty$. For a subset $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X. The *crossing number* of G, $\mathrm{cr}(G)$, is the smallest number of pairwise intersections of its edges when G is drawn in the plane. If $\mathrm{cr}(G) = 0$, then G is a planar graph.

A subset D of $V(G)$ is called a *dominating set*, if $D \cup N(D) = V(G)$. The minimum cardinality of all dominating sets in G is called the *domination number*, and denoted by $\gamma(G)$. The *bondage number* of a nonempty graph G, $b(G)$, is the cardinality of a minimum set of edges whose removal from G results in a graph with domination number larger than $\gamma(G)$.

The first result on bondage numbers was obtained by Bauer et al. [1]. Dunbar et al. [2] conjectured that $b(G) \leqslant \Delta(G) + 1$ for any nontrivial planar graph G. Kang and Yuan [5] confirmed this conjecture for $\Delta(G) \geqslant 7$ by proving that $b(G) \leqslant \min\{8, \Delta(G) + 2\}$, and proved that $b(G) \leqslant 7$ for any connected planar graph without vertices of degree five. Fischermann et al. [3] generalized the latter result, and showed that the conjecture is valid for all connected planar

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graphs with $g(G) \geqslant 4$ and $\Delta(G) \geqslant 5$ as well as all planar graphs with $g(G) \geqslant 5$ unless they are 3-regular. We generalize these results to connected graphs with small crossing numbers.

The rest of the paper is organized as follows. In the next section, we recall some results to be used in our discussions. Our main results are given in Sections 3 and 4. In Section 3, we discuss the upper bound of $b(G)$ for a connected graph G with $g(G) \geqslant 4$. In Section 4, we discuss the upper bound of $b(G)$ for connected graph G with some degree constraints.

2. Some lemmas

In this section, we recall some useful known results on the bondage number.

Lemma 2.1 (*Bauer et al. [1], Teschner [6]*). *If G is a nontrivial graph, then $b(G) \leqslant d_G(u) + d_G(v) - 1$ for any two distinct vertices u and v with $d_G(u, v) \leqslant 2$ in G.*

Lemma 2.2 (*Bauer et al. [1]*). *Let G be a graph with $\delta(G) \geqslant 1$. Then $b(G) \leqslant 2$ if G is a tree, and $b(G) \leqslant \Delta(G) + \delta(G) - 1$ otherwise.*

Lemma 2.3 (*Hartnell and Rall [4], Teschner [6]*). *If G has edge-connectivity $\lambda(G) \geqslant 1$, then $b(G) \leqslant \Delta(G) + \lambda(G) - 1$.*

Lemma 2.4 (*Hartnell and Rall [4]*). *If G is a nontrivial graph, then $b(G) \leqslant d_G(u) + d_G(v) - 1 - |N_G(u) \cap N_G(v)|$ for any adjacent vertices u and v in G.*

The following two results about planar graphs are well-known (cf. [7]).

Lemma 2.5 (*Euler's Formula*). *If G is a planar graph with $n(G)$ vertices, $m(G)$ edges, $\omega(G)$ components and $\phi(G)$ regions, then $\phi(G) = m(G) - n(G) + \omega(G) + 1$.*

Lemma 2.6. *For a planar graph G, $m(G) \leqslant 3n(G) - 6$ if $n(G) \geqslant 3$ and $m(G) \leqslant 2n(G) - 4$ if G is bipartite and $n(G) \geqslant 3$.*

Lemma 2.7 (*Fischermann et al. [3]*). *If G is a planar graph with $3 \leqslant g(G) < \infty$ and the number $c(G)$ of cut-edges, then*

$$m(G) \leqslant \frac{g(G)(n(G) - 2) - c(G)}{g(G) - 2}.$$

Let F_1 be the graph with the vertex-set $\{u, u_1, u_2, u_3\}$ and the edge-set $\{uu_i |\ i = 1, 2, 3\} \cup \{u_1u_2\}$ and F_2 be the graph with the vertex-set $\{v, v_1, v_2, v_3, v_4\}$ and the edge-set $\{vv_i |\ i = 1, 2, 3, 4\} \cup \{v_1v_2, v_3v_4\}$. Furthermore, for every positive integer t, let $H_{2,t}$ be the graph obtained from the complete bipartite graph $K_{2,t}$ with the partite sets $\{x, y\}$ and $\{w_1, w_2, \ldots, w_t\}$ by adding an edge xy. Now we define $\mathscr{G} = \{C_4, C_5, F_1, F_2\} \cup \{H_{2,t} |\ t \geqslant 1\}$, where C_4 and C_5 are cycles of length 4 and 5, respectively.

Lemma 2.8 (*Fischermann et al. [3]*). *Let G be a connected planar graph with $3 \leqslant g(G) < \infty$. Then $G \notin \mathscr{G}$ if and only if*

$$3\phi(G) \leqslant 2m(G) - n_2(G) - n_1(G).$$

Lemma 2.9 (*Kang and Yuan [5]*). *G is a planar graph and $v \in V(G)$ with $d_G(v) \geqslant 2$. Let $E_v = \{xy |\ x, y \in N_G(v)$ and $xy \notin E(G)\}$. Then there is a subset $F \subseteq E_v$ such that $H = G + F$ is still a planar graph and $H[N_G(v)]$ is 2-connected when $d_G(v) \geqslant 3$, or connected when $d_G(v) = 2$.*

A spanning subgraph H of G is called a *maximum planar subgraph* of G if H is planar and contains as many edges as possible.

Lemma 2.10. *Let G be a graph with* $\mathrm{cr}(G) > 0$ *and H a maximum planar subgraph of G. Then*

(1) $0 < |E(G)| - |E(H)| \leqslant \mathrm{cr}(G)$;
(2) *H contains a cycle*;
(3) $\omega(H) = \omega(G)$;
(4) $H \notin \mathscr{G}$.

Proof. Let $E' \subset E(G)$ such that $H = G - E'$ is a planar graph and $|E'|$ is as small as possible. It is easy to verify that H is maximum and has required properties. □

3. Bounds with girth at least four

Fischermann et al. [3] showed the following results for a connected planar graph G:

$$b(G) \leqslant \begin{cases} 6 & \text{if } g(G) \geqslant 4, \\ 5 & \text{if } g(G) \geqslant 5, \\ 4 & \text{if } g(G) \geqslant 6, \\ 3 & \text{if } g(G) \geqslant 8. \end{cases}$$

In this section, we generalize this result to connected graphs with small crossing numbers. In our discussions, we will use the following notations.

Let $\Delta = \Delta(G)$, $m = m(G)$, $n = n(G)$, $n_i = n_i(G)$ and $\tau_i = n_i + n_{i+1} + \cdots + n_\Delta$ for $i = 1, 2, \ldots, \Delta$. Then

$$n = n_1 + n_2 + \cdots + n_\Delta \quad \text{and}$$
$$2m = n_1 + 2n_2 + 3n_3 + \cdots + \Delta n_\Delta. \tag{1}$$

Theorem 3.1. *Let G be a connected graph. Then*

$$b(G) \leqslant \begin{cases} 6 & \text{if } g(G) \geqslant 4 \text{ and } 2\mathrm{cr}(G) < n_1 + 2n_2 + 2n_3 + \sum_{i=8}^{\Delta} (i-7)n_i + 8, \\ 5 & \text{if } g(G) \geqslant 5 \text{ and } 6\mathrm{cr}(G) < 3n_1 + 6n_2 + 5n_3 + \sum_{i=7}^{\Delta} (3i-18)n_i + 20, \\ 4 & \text{if } g(G) \geqslant 6 \text{ and } 4\mathrm{cr}(G) < n_1 + 2n_2 + \sum_{i=6}^{\Delta} (2i-10)n_i + 12, \\ 3 & \text{if } g(G) \geqslant 8 \text{ and } 6\mathrm{cr}(G) < \sum_{i=5}^{\Delta} (3i-12)n_i + 16. \end{cases} \tag{2}$$

Proof. If G contains no cycles, then $b(G) \leqslant 2$ by Lemma 2.2, and so the theorem holds. Suppose that G contains cycles below, which implies $g(G) < \infty$. Let H be a maximum planar subgraph of G. By Lemma 2.10, $m(H) \geqslant m - \mathrm{cr}(G)$ and $4 \leqslant g(G) \leqslant g(H) < \infty$. Note that $c(H) \geqslant n_1(H) \geqslant n_1$ since H is still connected. Then it follows from Lemma 2.7 that

$$m - \mathrm{cr}(G) \leqslant m(H) \leqslant \frac{g(H)(n(H)-2) - n_1}{g(H)-2}.$$

Since the function $f(g) = (g(n-2) - n_1)/(g-2)$ is descending on the interval $[4, +\infty)$,

$$m - \mathrm{cr}(G) \leqslant \frac{g(n-2) - n_1}{g-2}, \tag{3}$$

where $g = g(G) \geqslant 4$. Substituting (1) into (3) yields

$$gn_1 + 4n_2 + (6-g)n_3 \geqslant \sum_{i=4}^{\Delta} (g(i-2) - 2i)n_i + 4g - 2(g-2)\mathrm{cr}(G) \tag{4}$$

To complete the proof of the theorem, by Lemma 2.1, it is sufficient to show that there are two vertices u and v with $d_G(u,v) \leqslant 2$ in G such that $d_G(u) + d_G(v) \leqslant 7, 6, 5$ or 4. To the end, we consider four cases depending on $g(G) \geqslant 4,\ 5,\ 6$ or 8.

Case 1: Suppose to the contrary that $d_G(u) + d_G(v) \geqslant 8$ for any two vertices u and v with $d_G(u,v) \leqslant 2$ in G when $g(G) \geqslant 4$. Then if $d_G(u) = 1$ then $d_G(v) \geqslant 7$; if $d_G(u) = 2$ then $d_G(v) \geqslant 6$; if $d_G(u) = 3$ then $d_G(v) \geqslant 5$. Thus,

$$\tau_5 \geqslant n_1 + 2n_2 + 3n_3, \quad \tau_6 \geqslant n_1 + 2n_2, \quad \tau_7 \geqslant n_1. \tag{5}$$

Substituting $g = 4$ and (5) into (4) yields

$$\begin{aligned} 2n_1 + 2n_2 + n_3 &\geqslant \tau_5 + \tau_6 + \tau_7 + \sum_{i=8}^{\Delta} (i-7)n_i + 8 - 2\mathrm{cr}(G) \\ &\geqslant 3n_1 + 4n_2 + 3n_3 + \sum_{i=8}^{\Delta} (i-7)n_i + 8 - 2\mathrm{cr}(G). \end{aligned}$$

That is,

$$2\mathrm{cr}(G) \geqslant n_1 + 2n_2 + 2n_3 + \sum_{i=8}^{\Delta} (i-7)n_i + 8,$$

which contradicts the condition given in (2).

Case 2: Suppose to the contrary that for any two vertices u and v with $d_G(u,v) \leqslant 2$ in G, $d_G(u) + d_G(v) \geqslant 7$. Then if $d_G(u) = 1$ then $d_G(v) \geqslant 6$; if $d_G(u) = 2$ then $d_G(v) \geqslant 5$; if $d_G(u) = 3$ then $d_G(v) \geqslant 4$. Thus

$$\tau_4 \geqslant n_1 + 2n_2 + 3n_3, \quad \tau_5 \geqslant n_1 + 2n_2, \quad \tau_6 \geqslant n_1. \tag{6}$$

Substituting $g = 5$ and (6) into (4) yields

$$\begin{aligned} 5n_1 + 4n_2 + n_3 &\geqslant 2\tau_4 + 3\tau_5 + 3\tau_6 + \sum_{i=7}^{\Delta} (3i-18)n_i + 20 - 6\mathrm{cr}(G) \\ &\geqslant 8n_1 + 10n_2 + 6n_3 + \sum_{i=7}^{\Delta} (3i-18)n_i + 20 - 6\mathrm{cr}(G). \end{aligned}$$

That is,

$$6\mathrm{cr}(G) \geqslant 3n_1 + 6n_2 + 5n_3 + \sum_{i=7}^{\Delta} (3i-18)n_i + 20,$$

which contradicts the condition given in (2).

Case 3: Suppose to the contrary that for any two vertices u and v with $d_G(u,v) \leqslant 2$ in G, $d_G(u) + d_G(v) \geqslant 6$. Then $d_G(v) \geqslant 5$ when $d_G(u) = 1$ and $d_G(v) \geqslant 4$ when $d_G(u) = 2$. Thus,

$$\tau_4 \geqslant n_1 + 2n_2, \quad \tau_5 \geqslant n_1. \tag{7}$$

Substituting $g=6$ and (7) into (4) yields

$$\begin{aligned}3n_1+2n_2 &\geqslant 2\tau_4+2\tau_5+\sum_{i=6}^{\Delta}(2i-10)n_i+12-4\mathrm{cr}(G)\\ &\geqslant 4n_1+4n_2+\sum_{i=6}^{\Delta}(2i-10)n_i+12-4\mathrm{cr}(G).\end{aligned}$$

That is,

$$4\mathrm{cr}(G)\geqslant n_1+2n_2+\sum_{i=6}^{\Delta}(2i-10)n_i+12,$$

which contradicts the condition given in (2).

Case 4: Suppose to the contrary that for any two vertices u and v with $d_G(u,v)\leqslant 2$ in G, $d_G(u)+d_G(v)\geqslant 5$. Then $d_G(v)\geqslant 4$ when $d_G(u)=1$ and $d_G(v)\geqslant 3$ when $d_G(u)=2$. Thus,

$$\tau_3\geqslant n_1+2n_2,\quad \tau_4\geqslant n_1. \tag{8}$$

Substituting $g=8$ and (8) into (4) yields

$$\begin{aligned}4n_1+2n_2 &\geqslant \tau_3+3\tau_4+\sum_{i=5}^{\Delta}(3i-12)n_i+16-6\mathrm{cr}(G)\\ &\geqslant 4n_1+2n_2+\sum_{i=5}^{\Delta}(3i-12)n_i+16-6\mathrm{cr}(G).\end{aligned}$$

That is,

$$6\mathrm{cr}(G)\geqslant \sum_{i=5}^{\Delta}(3i-12)n_i+16,$$

which contradicts the condition given in (2). The proof of the theorem is complete. □

The following corollary contains Fischermann et al.'s result for a planar graph mentioned in the beginning of this section.

Corollary 3.2. *For a connected graph G*,

$$b(G)\leqslant\begin{cases}6 & \text{if } g(G)\geqslant 4 \text{ and } \mathrm{cr}(G)\leqslant 3,\\ 5 & \text{if } g(G)\geqslant 5 \text{ and } \mathrm{cr}(G)\leqslant 4,\\ 4 & \text{if } g(G)\geqslant 6 \text{ and } \mathrm{cr}(G)\leqslant 2,\\ 3 & \text{if } g(G)\geqslant 8 \text{ and } \mathrm{cr}(G)\leqslant 2.\end{cases}$$

Proof. When $\mathrm{cr}(G)\leqslant 3$, it is clear that $2\mathrm{cr}(G)\leqslant 6<8\leqslant n_1+2n_2+2n_3+\sum_{i=8}^{\Delta}(i-7)n_i+8$. By Theorem 3.1, if $g(G)\geqslant 4$ and $\mathrm{cr}(G)\leqslant 3$, then $b(G)\leqslant 6$.

Assume $g(G)\geqslant 5$ and $\mathrm{cr}(G)\leqslant 4$. In order to show $b(G)\leqslant 5$, by Theorem 3.1, we need only to show that $3n_1+6n_2+5n_3+\sum_{i=7}^{\Delta}(3i-18)n_i>4$. Suppose to the contrary that $3n_1+6n_2+5n_3+\sum_{i=7}^{\Delta}(3i-18)n_i\leqslant 4$. Then $n_1+n_7\leqslant 1$, $n_2=n_3=0$ and $\Delta\leqslant 7$. If $n_1=1$ then $n_7=0$ and from (4), we should have $2n_4+5n_5+8n_6\leqslant 9$. Hence $n_5=1$ since $n_1=1$ and the number of odd vertices is even. Then $n_4\leqslant 2$ and $n_6=0$. However, such a graph does not exist. Therefore, $n_1=0$ and $n_7\leqslant 1$. From (4), we should have $2n_4+5n_5+8n_6+11n_7\leqslant 4$, a contradiction.

In the cases of $g(G)\geqslant 6$ or $g(G)\geqslant 8$, $\mathrm{cr}(G)\leqslant 2$ implies the conditions in (2) naturally. Thus, the conclusions follow from Theorem 3.1. □

Corollary 3.3. *Let G be a connected graph. Then*

(a) $b(G) \leqslant 6$ *if G is not 4-regular*, $\mathrm{cr}(G) = 4$ *and* $g(G) \geqslant 4$;
(b) $b(G) \leqslant \Delta(G) + 1$ *if G is not* 3*-regular*, $\mathrm{cr}(G) \leqslant 4$ *and* $g(G) \geqslant 5$;
(c) $b(G) \leqslant 4$ *if G is not* 3*-regular*, $\mathrm{cr}(G) = 3$ *and* $g(G) \geqslant 6$;
(d) $b(G) \leqslant 3$ *if* $\mathrm{cr}(G) = 3$, $g(G) \geqslant 8$ *and* $\Delta(G) \geqslant 5$.

Proof. (a) Assume $\mathrm{cr}(G) = 4$ and $g(G) \geqslant 4$. If $n_1 = n_2 = n_3 = 0$ then, from (4), G is 4-regular, which contradicts the hypothesis, which implies $n_1 + 2n_2 + 2n_3 \geqslant 1$. Thus, $2\mathrm{cr}(G) = 8 < n_1 + 2n_2 + 2n_3 + \sum_{i=8}^{\Delta}(i-7)n_i + 8$, and so $b(G) \leqslant 6$ by Theorem 3.1.

(b) If $\Delta(G) \geqslant 4$, then by Corollary 3.2, $b(G) \leqslant 5 \leqslant \Delta(G) + 1$. In the remaining case, $\delta(G) \leqslant 2$ since G is not 3-regular. Then by Lemma 2.2, $b(G) \leqslant \Delta(G) + \delta(G) - 1 \leqslant \Delta(G) + 1$.

(c) Assume $\mathrm{cr}(G) = 3$ and $g(G) \geqslant 6$. If $n_1 = n_2 = 0$ then, from (4), $\Delta = 3$, and so G is 3-regular, which contradicts the hypothesis. Therefore, $n_1 + 2n_2 \geqslant 1$. Thus, $4\mathrm{cr}(G) = 12 < n_1 + 2n_2 + \sum_{i=6}^{\Delta}(2i-10)n_i + 12$, and so $b(G) \leqslant 4$ by Theorem 3.1.

(d) The hypothesis that $\mathrm{cr}(G) = 3$ and $\Delta(G) \geqslant 5$ implies that the last condition in (2) holds clearly. Thus, when $g(G) \geqslant 8$, $b(G) \leqslant 3$ by Theorem 3.1. □

Remark 3.4. It is immediately obtained from Corollary 3.2 that, if G is a connected 3-regular graph with $g(G) \geqslant 6$ and $\mathrm{cr}(G) \leqslant 2$, then $b(G) \leqslant 4 = \Delta(G) + 1$.

Remark 3.5. From the proof of Theorem 3.1 and Corollaries 3.2, 3.3, it is easy to see that the results is still valid when each hypothesis on $g(G)$ is replaced by the same hypothesis on $g(H)$.

4. Bounds with degree constraints

In this section, we will generalize the results of Kang and Yuan [5] and Fischermann et al. [3] to graphs with small crossing numbers.

We need the following notations. For a connected graph G, let G_0 be a subgraph of G without isolated vertices, H_0 be a maximum planar subgraph of G_0, $E' = E(G_0)\backslash E(H_0)$ and $V_i = \{x \in V(G) |\ d_G(x) = i\}$. Let $E'_i = \{e \in E' |\ e$ is incident with some vertex in $V_i\}$ for $i = 1, 2, 3, 4$ and $E'_5 = \{e \in E' |\ e$ is incident with some vertex in $I\}$ for some subset $I \subseteq V_5$. Denote $|E'_i|$ by m_i for $i = 1, 2, 3, 4, 5$.

Suppose that $X = \{x_1, x_2, \ldots, x_k\}$ is a given independent set of G_0 with $d_{G_0}(x_i) \geqslant 2$ for each $1 \leqslant i \leqslant k$. By Lemma 2.9, there exists $F_i \subseteq E_{x_i} = \{xy |\ x, y \in N_{H_0}(x_i), x \neq y, xy \notin E(H_{i-1})\}$ such that $H_i = H_{i-1} + F_i$ is planar and $H_i[N_{H_0}(x_i)]$ is 2-connected (connected when $d_{H_0}(x_i) = 2$) for $i = 1, 2, \ldots, k$. Let $G_k = H_k + E'\backslash E(H_k) = H_k + E'\backslash \cup_{i=1}^{k} F_i$.

For any $x_i \in X$ with $d_G(x_i) = d \geqslant 2$ and $N_G(x_i) = \{v_1, v_2, \ldots, v_d\}$, it is clear that if x_i and v_j $(j = 1, 2, \ldots, d)$ are not incident with any edge in $E(G)\backslash E(G_0)$ for $i = 1, 2, \ldots, k$, then $d_{G_0}(x_i) = d$ and $d_{G_0}(v_j) = d_G(v_j)$. Suppose $d_{H_0}(x_i) = d - h$, then $0 \leqslant h \leqslant d - 1$ since neither H_0 nor G_0 contains isolated vertices. Suppose, without loss of generality, that

$$x_i v_j \begin{cases} \in E' & j = 1, 2, \ldots, h, \\ \notin E' & j = h+1, \ldots, d. \end{cases}$$

Lemma 4.1. *If* $d \geqslant 3$ *then*

$$d_{G_k}(v_j) \geqslant \begin{cases} b(G) + 1 - d & \text{if } 1 \leqslant j \leqslant h, \\ b(G) + 3 - d & \text{if } h+1 \leqslant j \leqslant d \text{ and } h = 0, 1, \ldots, d-3, \\ b(G) + 2 - d & \text{if } h+1 \leqslant j \leqslant d \text{ and } h = d-2, \\ b(G) + 1 - d & \text{if } h+1 \leqslant j \leqslant d \text{ and } h = d-1. \end{cases}$$

If $d=2$ *then*

$$d_{G_k}(v_j)\geqslant\begin{cases} b(G) & \text{if } j=1,2 \text{ and } h=0,\\ b(G)-1 & \text{if } j=1,2 \text{ and } h=1.\end{cases}$$

Proof. For any $v\in\{v_1,v_2,\ldots,v_d\}$, let $E'_v=\{e\in E'|e \text{ is incident with } v\}$, $l_v=|E'_v|$, $|N_G(v)\cap N_G(x_i)|=a$ and $|N_{H_0}(v)\cap N_{H_0}(x_i)|=b$. By Lemma 2.4,

$$d_G(v)\geqslant b(G)+1+a-d_G(x_i). \tag{9}$$

From the constructions above, it is clear that $b\leqslant a$. If $v\in N_{H_0}(x_i)$, then $|E'_v\cap F_i|\leqslant a-b$ and so $|E'_v\backslash F_i|\geqslant l_v-a+b$; otherwise, $|E'_v\cap F_i|=\emptyset$ and so $|E'_v\backslash F_i|=l_v$. Noting that $d_{G_k}(v)\geqslant d_{H_k}(v)+|E'_v\backslash F_i|$, we have

$$d_{G_k}(v)\geqslant\begin{cases} d_{H_k}(v)+l_v-(a-b) & \text{if } v\in N_{H_0}(x_i),\\ d_{H_k}(v)+l_v & \text{if } v\notin N_{H_0}(x_i).\end{cases} \tag{10}$$

Use δ_i to denote the minimum degree of $H_k[N_{H_0}(x_i)]$. It follows from the definition of H_k that

$$d_{H_k}(v)\geqslant\begin{cases} d_{H_0}(v)+\delta_i-b & \text{if } v\in N_{H_0}(x_i),\\ d_{H_0}(v) & \text{if } v\notin N_{H_0}(x_i).\end{cases} \tag{11}$$

Combining (9), (10), (11) with $d_{H_0}(v)=d_{G_0}(v)-l_v=d_G(v)-l_v$ we obtain

$$d_{G_k}(v)\geqslant\begin{cases} b(G)+1+\delta_i-d_G(x_i) & \text{if } v\in N_{H_0}(x_i),\\ b(G)+1+a-d_G(x_i) & \text{if } v\notin N_{H_0}(x_i).\end{cases}$$

We first consider $d\geqslant 3$.

If $1\leqslant j\leqslant h$ then $v_j\notin N_{H_0}(x_i)$, and so $d_{G_k}(v)\geqslant b(G)+1+a-d\geqslant b(G)+1-d$.

If $h\leqslant d-3$ and $h+1\leqslant j\leqslant d$, then $v_j\in N_{H_0}(x_i)$ and $\delta_i\geqslant 2$ by Lemma 2.9. Thus, $d_{G_k}(v)\geqslant b(G)+1+\delta_i-d\geqslant b(G)+3-d$.

If $h=d-2$ and $h+1\leqslant j\leqslant d$, then $v_j\in N_{H_0}(x_i)$ and $\delta_i\geqslant 1$ by Lemma 2.9. Thus, $d_{G_k}(v)\geqslant b(G)+1+\delta_i-d\geqslant b(G)+2-d$.

If $h=d-1$ and $j=d$, then $v_j\in N_{H_0}(x_i)$ and $d_{G_k}(v)\geqslant b(G)+1+\delta_i-d\geqslant b(G)+1-d$.

We now consider $d=2$. If $h=0$ then $h+1\leqslant j\leqslant d$ and $\delta_i\geqslant 1$ by Lemma 2.9. Thus, $d_{G_k}(v)\geqslant b(G)+1+\delta_i-d\geqslant b(G)+2-d=b(G)$. In the remaining cases $d_{G_k}(v)\geqslant b(G)+1-d=b(G)-1$.

The proof of the lemma is complete. □

Lemma 4.2. *Let* $A\subseteq V(G_0)$ *and* $E'_A=\{e\in E'|e \text{ is incident with some vertex in } A\}$, *then*

$$|N_{G_k}(A)|\geqslant|N_{H_k}(A)|\geqslant\frac{1}{2}\sum_{v\in A}d_{G_k}(v)-|A|+2-\frac{1}{2}|E'_A|.$$

Proof. Let $B=N_{H_k}(A)$, $C=N_{G_k}(A)$ and

$$E_{H_k}(A,B)=\{xy\in E(H_k)|x\in A,\ y\in B\},$$

$$E_{G_k}(A,C)=\{xy\in E(G_k)|\ x\in A,\ y\in C\}.$$

Then the induced subgraphs $H_k[E_{H_k}(A,B)]$ and $G_k[E_{G_k}(A,C)]$ are both bipartite graphs, the former is planar and obtained from the latter by deleting some edges in E'_A and then deleting possible isolated vertices. By Lemma 2.6, we have

$$|E_{G_k}(A,C)|\leqslant|E_{H_k}(A,B)|+|E'_A|\leqslant 2|A|+2|B|-4+|E'_A|,$$

and so the lemma follows. □

Theorem 4.3. *If G is a connected graph with* $\mathrm{cr}(G)<n_3(G)+n_4(G)+\frac{7}{2}$, *then* $b(G)\leqslant 8$.

Proof. Suppose to the contrary that $b(G) \geqslant 9$. We will deduce

$$2\mathrm{cr}(G) \geqslant 2n_3(G) + 2n_4(G) + 7, \tag{12}$$

which contradicts our hypothesis.

Let I be a maximum independent set in $G[V_5]$. Then I is a dominating set in $G[V_5]$. Since $b(G) \geqslant 9$, $d_G(u)+d_G(v) \geqslant 10$ for any two distinct vertices u and v with $d_G(u, v) \leqslant 2$ by Lemma 2.1. Thus, for any $x \in \cup_{i=1}^4 V_i$ and $y \in \cup_{i=1}^4 V_i \cup I$, since $d_G(x) + d_G(y) \leqslant 4 + 5 = 9$, we have $d_G(x, y) \geqslant 3$, which implies that $N_G(x) \cap N_G(y) = \emptyset$ and $xy \notin E(G)$. It follows that $(\cup_{i=1}^4 V_i) \cup I$ is an independent set of G and $|N_G(V_i)| = i|V_i| = in_i$ for $i = 1, 2, 3, 4$.

To obtain H_k and G_k mentioned in the beginning of this section, let $G_0 = G - V_1 - V_2$ and $X = V_3 \cup V_4 \cup I$. It is easy to observe that $N_{G_k}(x) = N_G(x)$ for every $x \in X$, and $m_3 + m_4 + m_5 \leqslant \mathrm{cr}(G)$ since X is independent. Let $V' = V(G_0) \backslash (X \cup N_{G_k}(X))$. Then $d_{G_k}(v) \geqslant 6$ for every $v \in V'$ since I is a dominating set of $G[V_5]$. If we can prove that

$$\sum_{v \in N_{G_k}(V_3)} d_{G_k}(v) \geqslant 25n_3 - 2m_3 \quad \text{if } V_3 \neq \emptyset, \tag{13}$$

$$\sum_{v \in N_{G_k}(V_4)} d_{G_k}(v) \geqslant 30n_4 - 2m_4 \quad \text{if } V_4 \neq \emptyset, \tag{14}$$

$$\sum_{v \in I \cup N_{G_k}(I)} d_{G_k}(v) \geqslant 6|I \cup N_{G_k}(I)| + 2 - 2m_5 \quad \text{if } I \neq \emptyset, \tag{15}$$

then, from (13)–(15), we have

$$\begin{aligned}
m(H_k) &\geqslant m(G_k) - \mathrm{cr}(G) = \frac{1}{2} \sum_{v \in V(G_k)} d_{G_k}(v) - \mathrm{cr}(G) \\
&= \frac{1}{2}\left[\sum_{v \in V_3 \cup N_{G_k}(V_3)} d_{G_k}(v) + \sum_{v \in V_4 \cup N_{G_k}(V_4)} d_{G_k}(v) \right. \\
&\quad \left. + \sum_{v \in I \cup N_{G_k}(I)} d_{G_k}(v) + \sum_{v \in V'} d_{G_k}(v) \right] - \mathrm{cr}(G) \\
&\geqslant \frac{1}{2}\{(3n_3 + 25n_3 - 2m_3) + (4n_4 + 30n_4 - 2m_4) \\
&\quad + (2 - 2m_5 + 6|I \cup N_{G_k}(I)|) \\
&\quad + 6(n(G_k) - 4n_3 - 5n_4 - |I \cup N_{G_k}(I)|)\} - \mathrm{cr}(G) \\
&= 3n(G_k) + 2n_3 + 2n_4 - m_3 - m_4 + 1 - m_5 - \mathrm{cr}(G) \\
&\geqslant 3n(G_k) + 2n_3 + 2n_4 + 1 - 2\mathrm{cr}(G),
\end{aligned}$$

that is,

$$2\mathrm{cr}(G) \geqslant 3n(G_k) + 2n_3 + 2n_4 + 1 - m(H_k). \tag{16}$$

Since H_k is planar, $m(H_k) \leqslant 3n(H_k) - 6 = 3n(G_k) - 6$ by Lemma 2.6. Substituting this inequality into (16) yields (12).

We now give the proofs of (13), (14) and (15).

We first prove (13). Assume $V_3 \neq \emptyset$ and let $V_3^{(h)} = \{x_i \in V_3 | d_{H_0}(x_i) = 3 - h\}$ and $n_3^{(h)} = |V_3^{(h)}|$ for $h = 0, 1, 2$. It is clear that x_i is incident with exact h edges in E'_3 for any $x_i \in V_3^{(h)}$, $0 \leqslant h \leqslant 2$. Thus $n_3 = n_3^{(0)} + n_3^{(1)} + n_3^{(2)}$ and

$m_3 = n_3^{(1)} + 2n_3^{(2)}$. Let $N_G(x_i) = \{v_1, v_2, v_3\}$ and $x_i v_j \in E'$ if $j \leqslant h$, $xv_j \notin E'$ if $j \geqslant h+1$. By Lemma 4.1 we have

$$d_{G_k}(v_j) \geqslant \begin{cases} 7 & \text{if } 1 \leqslant j \leqslant h, \\ 9 & \text{if } h+1 \leqslant j \leqslant d \text{ and } h=0, \\ 8 & \text{if } h+1 \leqslant j \leqslant d \text{ and } h=1, \\ 7 & \text{if } h+1 \leqslant j \leqslant d \text{ and } h=2. \end{cases}$$

Thus

$$\sum_{v \in N_{G_k}(V_3^{(h)})} d_{G_k}(v) \geqslant \begin{cases} 27n_3^{(0)} & \text{if } h=0, \\ 23n_3^{(1)} & \text{if } h=1, \\ 21n_3^{(2)} & \text{if } h=2. \end{cases}$$

It follows that

$$\begin{aligned} \sum_{v \in N_{G_k}(V_3)} d_{G_k}(v) &= \sum_{v \in N_{G_k}(V_3^{(0)})} d_{G_k}(v) + \sum_{v \in N_{G_k}(V_3^{(2)})} d_{G_k}(v) + \sum_{v \in N_{G_k}(V_3^{(3)})} d_{G_k}(v) \\ &\geqslant 27n_3^{(0)} + 23n_3^{(1)} + 21n_3^{(2)} \\ &\geqslant 25n_3 - 2n_3^{(1)} - 4n_3^{(2)} \\ &= 25n_3 - 2m_3, \end{aligned}$$

as required in (13).

Similarly, we can prove (14). Assume $V_4 \neq \emptyset$ and let $V_4^{(h)} = \{x_i \in V_4 | \ d_{H_0}(x_i) = 4-h\}$ and $n_4^{(h)} = |V_4^{(h)}|$ for $h = 0, 1, 2, 3$. It is clear that x_i is incident with exact h edges in E_4' for any $x_i \in V_4^{(h)}$, $0 \leqslant h \leqslant 3$. Thus $n_4 = n_4^{(0)} + n_4^{(1)} + n_4^{(2)} + n_4^{(3)}$ and $m_4 = n_4^{(1)} + 2n_4^{(2)} + 3n_4^{(3)}$. Let $N_G(x_i) = \{v_1, v_2, v_3, v_4\}$ and $x_i v_j \in E'$ for $j = 1 \leqslant h$, $xv_j \notin E'$ for $j \geqslant h+1$. By Lemma 4.1 we have

$$d_{G_k}(v_j) \geqslant \begin{cases} 6 & \text{if } 1 \leqslant j \leqslant h, \\ 8 & \text{if } h+1 \leqslant j \leqslant d \text{ and } h=0,1, \\ 7 & \text{if } h+1 \leqslant j \leqslant d \text{ and } h=2, \\ 6 & \text{if } h+1 \leqslant j \leqslant d \text{ and } h=3. \end{cases}$$

Thus

$$\sum_{v \in N_{G_k}(V_4^{(h)})} d_{G_k}(v) \geqslant \begin{cases} 32n_4^{(0)} & \text{if } h=0, \\ 30n_4^{(1)} & \text{if } h=1, \\ 26n_4^{(2)} & \text{if } h=2, \\ 24n_4^{(3)} & \text{if } h=3. \end{cases}$$

It follows that

$$\begin{aligned} \sum_{v \in N_{G_k}(V_4)} d_{G_k}(v) &\geqslant 32n_4^{(0)} + 30n_4^{(1)} + 26n_4^{(2)} + 24n_4^{(3)} \\ &\geqslant 30n_4 - 4n_4^{(2)} - 6n_4^{(3)} \\ &\geqslant 30n_4 - 2m_4, \end{aligned}$$

as required in (14).

We now prove (15). Assume $I \neq \emptyset$. By Lemma 4.2, $|N_{H_k}(I)| \geqslant \frac{3}{2}|I| + 2 - \frac{1}{2} m_5$. Note that there might be some vertices in $N_{G_k}(I)$ but not in $N_{H_k}(I)$. For every $v \in N_{G_k}(I)$ we define

$$t(v) = \begin{cases} 0 & \text{if } v \in N_{H_k}(I), \\ 1 & \text{if } v \notin N_{H_k}(I) \end{cases} \quad \text{and} \quad t = \sum_{v \in N_{G_k}(I)} t(v).$$

Then

$$|N_{G_k}(I)| = |N_{H_k}(I)| + t \geqslant \tfrac{3}{2}|I| + 2 - \tfrac{1}{2} m_5 + t. \tag{17}$$

Let $I_h = \{x \in I \mid d_{H_0}(x) = 5 - h\}$ for $h = 0, 1, 2, 3, 4$. It follows that $|I| = \sum_{h=0}^{4} |I_h|$, $m_5 = \sum_{h=1}^{4} h|I_h|$. For $x \in I_h$ and $N_G(x) = \{v_1, v_2, \ldots, v_5\}$, let $xv_j \in E'_5$ for $j = 1, \ldots, h$. Then $v_j \notin N_{H_0}(x)$ for $j = 1, \ldots, h$ and $v_j \in N_{H_0}(x)$ for $j = h + 1, \ldots, 5$. By Lemma 4.1 we obtain

$$d_{G_k}(v_j) \geqslant \begin{cases} 5 & \text{if } 1 \leqslant j \leqslant h, \\ 7 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 0, 1, 2, \\ 6 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 3, \\ 5 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 4. \end{cases} \tag{18}$$

Define

$$s(v) = 7 - d_{G_k}(v) \quad \text{and} \quad s = \sum_{v \in N_{G_k}(I)} s(v).$$

Then

$$\sum_{v \in N_{G_k}(I)} d_{G_k}(v) = \sum_{v \in N_{G_k}(I)} (7 - s(v)) = 7|N_{G_k}(I)| - s. \tag{19}$$

If we can prove

$$s - t \leqslant |I_1| + 2|I_2| + 5|I_3| + 6|I_4|, \tag{20}$$

then, from (17), (19) and (20), we have that

$$\begin{aligned}
&\sum_{v \in I \cup N_{G_k}(I)} d_{G_k}(v) - 6|I \cup N_{G_k}(I)| \\
&\quad \geqslant 5|I| + 7|N_{G_k}(I)| - s - 6(|I| + |N_{G_k}(I)|) \\
&\quad \geqslant |N_{G_k}(I)| - |I| - s \\
&\quad \geqslant \tfrac{3}{2}|I| + 2 - \tfrac{1}{2}m_5 - |I| - (s - t) \\
&\quad \geqslant 2 - \tfrac{1}{2}|I_1| - \tfrac{3}{2}|I_2| - \tfrac{9}{2}|I_3| - \tfrac{11}{2}|I_4| - \tfrac{1}{2}m_5 \\
&\quad \geqslant 2 - \tfrac{3}{2}(|I_1| + 2|I_2| + 3|I_3| + 4|I_4|) - \tfrac{1}{2}m_5 \\
&\quad = 2 - 2m_5
\end{aligned}$$

as required in (15).

We now establish (20). From (18) we have

$$s(v_j) - t(v_j) \leqslant \begin{cases} 1 & \text{if } 1 \leqslant j \leqslant h \text{ and } v_j \notin N_{H_k}(I), \\ 2 & \text{if } 1 \leqslant j \leqslant h \text{ and } v_j \in N_{H_k}(I), \\ 0 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 0, 1, 2, \\ 1 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 3, \\ 2 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 4, \end{cases} \tag{21}$$

For any $x \in I_h$ and $v_j \in N_{G_k}(x)$, define

$$r_x(v_j) = \begin{cases} 1 & \text{if } 1 \leqslant j \leqslant h, \\ 0 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 0, 1, 2, \\ 1 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 3, \\ 2 & \text{if } h+1 \leqslant j \leqslant 5 \text{ and } h = 4. \end{cases}$$

It is clear that

$$\sum_{j=1}^{5} r_x(v_j) = \begin{cases} 0 & \text{if } h = 0, \\ 1 & \text{if } h = 1, \\ 2 & \text{if } h = 2, \\ 5 & \text{if } h = 3, \\ 6 & \text{if } h = 4. \end{cases}$$

Thus, (20) follows from the following inequality:

$$\sum_{v \in N_{G_k}(I)} (s(v) - t(v)) \leqslant \sum_{x \in I} \sum_{v \in N_{G_k}(x)} r_x(v). \tag{22}$$

We now prove (22). Note that $r_x(v) \geqslant 0$ for every $x \in I$ and $v \in N_{G_k}(x)$. Then we need only to show that for every $v \in N_{G_k}(I)$, there exists $x \in I$ such that $v \in N_{G_k}(x)$ and $s(v) - t(v) \leqslant r_x(v)$. To this purpose, from (21) and the definition of $r_x(v_j)$, we need to consider only such a vertex $x \in I_h$ and its neighbor $v_j \in N_{G_k}(x)$ that $1 \leqslant j \leqslant h$ and $v_j \in N_{H_k}(I)$, for which $s(v_j) - t(v_j) \leqslant 2$ and $r_x(v_j) = 1$. In this case, however, $v_j \notin N_{H_0}(x) = N_{H_k}(x)$ and $v_j \in N_{H_k}(I)$. Thus, there exists another vertex $x' \in I$ such that $v \in N_{H_k}(x') = N_{H_0}(x')$, and by (21) $s(v_j) - t(v_j) \leqslant r_{x'}(v_j)$.

The proof of the theorem is complete. □

The result of Kang and Yuan [5, Theorem 3.1] is a special case of the following corollary for $\mathrm{cr}(G) = 0$.

Corollary 4.4. *If G is a connected graph with* $\mathrm{cr}(G) \leqslant 3$, *then* $b(G) \leqslant 8$.

Theorem 4.5. *Let G be a connected graph and* $I = \{v \in V(G) \mid d_G(v) = 5,\ d_G(u, v) \geqslant 3 \text{ if } d_G(u) \leqslant 3, \text{ and } d_G(u) \neq 4$ *for every* $u \in N_G(v)\}$. *Then* $b(G) \leqslant 7$ *if I is independent, has no vertex adjacent to vertices of degree* 6 *and*

$$\mathrm{cr}(G) < \max\left\{ \frac{5n_3(G) + |I| - 2n_4(G) + 28}{11}, \frac{7n_3(G) + 40}{16} \right\}.$$

Proof. Suppose to the contrary that $b(G) \geqslant 8$, then by Lemma 2.1, $d_G(u) + d_G(v) \geqslant 9$ for any two vertices u and v with $d_G(u, v) \leqslant 2$, which implies that $I = V_5 \backslash N_G(V_4)$. We first deduce

$$\mathrm{cr}(G) \geqslant \frac{5n_3 + |I| - 2n_4 + 28}{11}. \tag{23}$$

For any $x, y \in \cup_{i=1}^{4} V_i$, since $d_G(x) + d_G(y) \leqslant 4 + 4 = 8$, we have $d_G(x, y) \geqslant 3$, i.e. $N_G(x) \cap N_G(y) = \emptyset$ and $xy \notin E(G)$. Then $(\cup_{i=1}^{4} V_i) \cup I$ is a independent set of G by the hypothesis on I, and $|N_G(V_i)| = i|V_i| = in_i$ for $i = 1, 2, 3, 4$.

To obtain H_k and G_k, let $G_0 = G - V_1 - V_2$ and $X = V_3 \cup V_4 = \{x_1, x_2, \ldots, x_k\}$. It is easy to observe that $N_{G_k}(x) = N_G(x)$ for every $x \in X$, and $m_3 + m_4 + m_5 \leqslant \mathrm{cr}(G)$ since $V_3 \cup V_4 \cup I$ is independent. Let $Y = V_4 \cup I$ and $V' = V(G_0) \backslash (V_3 \cup N_{G_k}(V_3) \cup Y \cup N_{G_k}(Y))$. Then $d_{G_k}(v) \geqslant 6$ for every $v \in V'$ since $V_5 \subseteq N_{G_k}(V_4) \cup I$. If we can prove that

$$\sum_{v \in N_{G_k}(V_3)} d_{G_k}(v) \geqslant \frac{47}{2} n_3 - \frac{7}{2} m_3 \quad \text{if } V_3 \neq \emptyset, \tag{24}$$

and

$$\sum_{v \in Y \cup N_{G_k}(Y)} d_{G_k}(v) \geqslant 6|Y \cup N_{G_k}(Y)| + \frac{1}{2}|I| - n_4 + 2 - \frac{7}{2} m_4 - \frac{1}{2} m_5 \quad \text{if } Y \neq \emptyset, \tag{25}$$

then, from (24) and (25), we have

$$\begin{aligned} m(H_k) &\geqslant m(G_k) - \mathrm{cr}(G) = \frac{1}{2} \sum_{v \in V(G_k)} d_{G_k}(v) - \mathrm{cr}(G) \\ &= \frac{1}{2} \left[\sum_{v \in V_3 \cup N_{G_k}(V_3)} d_{G_k}(v) + \sum_{v \in Y \cup N_{G_k}(Y)} d_{G_k}(v) + \sum_{v \in V'} d_{G_k}(v) \right] - \mathrm{cr}(G) \\ &\geqslant \frac{1}{2} \{ (3n_3 + 23.5 n_3 - 3.5 m_3) \\ &\quad + \left(6|Y \cup N_{G_k}(Y)| + \frac{1}{2}|I| - n_4 + 2 - \frac{7}{2} m_4 - \frac{1}{2} m_5 \right) \\ &\quad + 6(n(G_k) - 4n_3 - |Y \cup N_{G_k}(Y)|) \} - \mathrm{cr}(G) \\ &= 3n(G_k) + \frac{5}{4} n_3 - \frac{1}{2} n_4 + \frac{1}{4}|I| + 1 - \frac{7}{4} m_3 - \frac{7}{4} m_4 - \frac{1}{4} m_5 - \mathrm{cr}(G) \\ &\geqslant 3n(G_k) + \frac{5}{4} n_3 - \frac{1}{2} n_4 + \frac{1}{4}|I| + 1 - \frac{11}{4} \mathrm{cr}(G). \end{aligned}$$

that is,

$$\tfrac{11}{4} \mathrm{cr}(G) \geqslant 3n(G_k) + \tfrac{5}{4} n_3 - \tfrac{1}{2} n_4 + \tfrac{1}{4}|I| + 1 - m(H_k). \tag{26}$$

Since H_k is planar, $m(H_k) \leqslant 3n(H_k) - 6 = 3n(G_k) - 6$ by Lemma 2.6. Substituting this inequality into (26) yields (23).

We now prove (24) and (25).

We first prove (24). Let $V_3^{(h)} = \{x \in V_3 | d_{H_0}(x) = 3 - h\}$ for $h = 0, 1, 2$ and $n_3^{(h)} = |V_3^{(h)}|$. It is clear that x is incident with exact h edge in E_3' for any $x_i \in V_3^{(h)}$. Thus $n_3 = n_3^{(0)} + n_3^{(1)} + n_3^{(2)}$ and $m_3 = n_3^{(1)} + 2n_3^{(2)}$. For $x \in V_3^{(h)}$ and $N_G(x) = \{v_1, v_2, v_3\}$, let $xv_j \in E'$ for $j = 1, \ldots, h$ and $xv_j \notin E'$ for $j = h + 1, \ldots, 3$. By Lemma 4.1 we obtain

$$d_{G_k}(v_j) \geqslant \begin{cases} 6 & \text{if } 1 \leqslant j \leqslant h, \\ 8 & \text{if } h + 1 \leqslant j \leqslant 3 \text{ and } h = 0, \\ 7 & \text{if } h + 1 \leqslant j \leqslant 3 \text{ and } h = 1, \\ 6 & \text{if } h + 1 \leqslant j \leqslant 3 \text{ and } h = 2. \end{cases} \tag{27}$$

Thus

$$\sum_{v\in N_{G_k}(V_3^{(h)})} d_{G_k}(v) \geqslant \begin{cases} 24n_3^{(0)} & \text{if } h=0,\\ 20n_3^{(1)} & \text{if } h=1,\\ 18n_3^{(2)} & \text{if } h=2. \end{cases}$$

It follows that

$$\begin{aligned}\sum_{v\in N_{G_k}(V_3)} d_{G_k}(v) &= \sum_{v\in N_{G_k}(V_3^{(0)})} d_{G_k}(v) + \sum_{v\in N_{G_k}(V_3^{(1)})} d_{G_k}(v) + \sum_{v\in N_{G_k}(V_3^{(2)})} d_{G_k}(v)\\ &\geqslant 24n_3^{(0)} + 20n_3^{(1)} + 18n_3^{(2)}\\ &\geqslant 23.5n_3 - 3.5n_3^{(1)} - 5.5n_3^{(2)}\\ &\geqslant 23.5n_3 - 3.5m_3\end{aligned}$$

as required in (24).

We now prove (25). Replacing A and E'_A in Lemma 4.2 by Y and $E'_Y = E'_4 \cup E'_5$, respectively, yields

$$|N_{G_k}(Y)| \geqslant |N_{H_k}(Y)| \geqslant n_4 + \tfrac{3}{2}|I| + 2 - \tfrac{1}{2}m_4 - \tfrac{1}{2}m_5. \tag{28}$$

If $x \in I$ and $v \in N_{G_k}(x)$, then $d_{G_k}(v) \geqslant 7$ by the hypothesis on I. Let $V_4^{(h)} = \{x \in V_4 | d_{H_0}(x) = 4 - h\}$ and $n_4^{(h)} = |V_4^{(h)}|$ for $h = 0, 1, 2, 3$. It follows that $n_4 = n_4^{(0)} + n_4^{(1)} + n_4^{(2)} + n_4^{(3)}$ and $m_4 = n_4^{(1)} + 2n_4^{(2)} + 3n_4^{(3)}$. For $x \in V_4^{(h)}$ and $N_G(x) = \{v_1, v_2, v_3, v_4\}$, let $xv_j \in E'$ for $j = 1, \ldots, h$ and $xv_j \notin E'$ for $j = h+1, \ldots, 5$. By Lemma 4.1 we obtain

$$d_{G_k}(v_j) \geqslant \begin{cases} 5 & \text{if } 1 \leqslant j \leqslant h,\\ 7 & \text{if } h+1 \leqslant j \leqslant 4 \text{ and } h = 0, 1,\\ 6 & \text{if } h+1 \leqslant j \leqslant 4 \text{ and } h = 2,\\ 5 & \text{if } h+1 \leqslant j \leqslant 4 \text{ and } h = 3. \end{cases}$$

Thus

$$\sum_{v\in N_{G_k}(V_4^{(h)})} d_{G_k}(v) \geqslant \begin{cases} 28n_4^{(0)} & \text{if } h=0,\\ 26n_4^{(1)} & \text{if } h=1,\\ 22n_4^{(2)} & \text{if } h=2,\\ 20n_4^{(3)} & \text{if } h=3. \end{cases} \tag{29}$$

Combining (29) with (28), we have

$$\begin{aligned}&\sum_{v\in Y\cup N_{G_k}(Y)} d_{G_k}(v) - 6|Y \cup N_{G_k}(Y)|\\ &\geqslant 4n_4 + 5|I| + 28n_4^{(0)} + 26n_4^{(1)} + 22n_4^{(2)} + 20n_4^{(3)} + 7(|N_{G_k}(Y)| - 4n_4)\\ &\quad - 6(n_4 + |I| + |N_{G_k}(Y)|)\\ &\geqslant |N_{G_k}(Y)| - |I| - 2n_4 - (2n_4^{(1)} + 6n_4^{(2)} + 8n_4^{(3)})\\ &\geqslant \tfrac{1}{2}|I| - n_4 + 2 - \tfrac{1}{2}m_4 - \tfrac{1}{2}m_5 - 3m_4,\end{aligned}$$

as required in (25).

Now (23) is valid. If we can show that $\mathrm{cr}(G) \geqslant \frac{1}{16}(7n_3 + 40)$, then

$$\mathrm{cr}(G) \geqslant \max\left\{\frac{5n_3 + |I| - 2n_4 + 28}{11}, \frac{7n_3 + 40}{16}\right\},$$

a contradiction to the hypothesis.

We now show $\mathrm{cr}(G) \geqslant \frac{1}{16}(7n_3 + 40)$. Lemma 2.6 implies for H_k that

$$\begin{aligned}
&3n_3(G_k) + 4n_4(G_k) + 5n_5(G_k) + \cdots + \Delta(G_k)n_{\Delta(G_k)}(G_k) - 2\mathrm{cr}(G)\\
&\quad = 2m(G_k) - 2\mathrm{cr}(G) \leqslant 2m(H_k) \leqslant 6n(H_k) - 12 = 6n(G_k) - 12\\
&\quad = 6n_3(G_k) + 6n_4(G_k) + 6n_5(G_k) + \cdots + 6n_{\Delta(G_k)}(G_k) - 12.
\end{aligned}$$

That is

$$3n_3(G_k) + 2n_4(G_k) + n_5(G_k) \geqslant \tau_7(G_k) + \tau_8(G_k) + 12 - 2\mathrm{cr}(G). \tag{30}$$

It is easy to observe that $n_3(G_k) = n_3$, $n_4(G_k) = n_4$, and vertices in I is still of degree five in G_k. Note that $d_{G_k}(v) \geqslant 6$ if $v \in V'$ and $d_{G_k}(v) \geqslant 7$ if $v \in N_{G_k}(I)$. Then it follows from (27) and (29) that

$$\begin{aligned}
&n_5(G_k) \leqslant |I| + n_4^{(1)} + 2n_4^{(2)} + 4n_4^{(3)},\\
&\tau_7(G_k) \geqslant 3n_3^{(0)} + 2n_3^{(1)} + 4n_4^{(0)} + 3n_4^{(1)},\\
&\tau_8(G_k) \geqslant 3n_3^{(0)}.
\end{aligned}$$

Substituting these into (30) yields

$$\begin{aligned}
3n_3 + |I| &\geqslant 6n_3^{(0)} + 2n_3^{(1)} + 4n_4^{(0)} + 2n_4^{(1)} - 2n_4^{(2)} - 4n_4^{(3)} - 2n_4 + 12 - 2\mathrm{cr}(G)\\
&\geqslant 5n_3 - 3n_3^{(1)} - 5n_3^{(2)} + 4n_4 - 2n_4^{(1)} - 6n_4^{(2)} - 8n_4^{(3)} - 2n_4 + 12 - 2\mathrm{cr}(G)\\
&\geqslant 5n_3 - 3m_3 + 2n_4 - 3m_4 + 12 - 2\mathrm{cr}(G)\\
&\geqslant 5n_3 + 2n_4 + 12 - 5\mathrm{cr}(G),
\end{aligned}$$

that is,

$$5\mathrm{cr}(G) \geqslant 2n_3 + 2n_4 + 12 - |I|. \tag{31}$$

Combining (31) with (23) yields $\mathrm{cr}(G) \geqslant \frac{1}{16}(7n_3 + 40)$.

The proof of the theorem is completed. □

The result of Fischermann et al. [3, Theorem 4.3] is a special case of the following corollary for $\mathrm{cr}(G) = 0$.

Corollary 4.6. *Let G be a connected graph with $\mathrm{cr}(G) \leqslant 2$. Then $b(G) \leqslant 7$ if $I = \{v \in V(G) | d_G(v) = 5, d_G(u, v) \geqslant 3$ if $d_G(u) \leqslant 3$ and $d_G(u) \neq 4$ for every $u \in N_G(v)\}$ is independent, and has no vertices adjacent to vertices of degree 6.*

Theorem 4.7. *Let G be a connected graph. Then $b(G) \leqslant 7$ if G satisfies*

(1) $5\mathrm{cr}(G) + n_5 < 2n_2 + 3n_3 + 2n_4 + 12$; *or*
(2) $7\mathrm{cr}(G) + 2n_5 < 3n_2 + 4n_4 + 24$.

Proof. Suppose to the contrary that $b(G) \geqslant 8$, then $d_G(u) + d_G(v) \geqslant 9$ for every pair u and v with $d_G(u, v) \leqslant 2$. Thus, for any $x, y \in \cup_{i=1}^4 V_i$, since $d_G(x) + d_G(y) \leqslant 4 + 4 = 8$, we have $d_G(x, y) \geqslant 3$, which implies that $N_G(x) \cap N_G(y) = \emptyset$ and $xy \notin E(G)$. It follows that $\cup_{i=1}^4 V_i$ is a independent set of G and $|N_G(V_i)| = i|V_i| = in_i$ for $i = 1, 2, 3, 4$.

To get H_k and G_k, let $G_0 = G - V_1$ and $X = V_2 \cup V_3 \cup V_4$. It is clear that $N_{G_k}(x) = N_G(x)$ for every $x \in X$, and $m_2 + m_3 + m_4 \leqslant \mathrm{cr}(G)$ since X is independent. For $i = 1, 2, \ldots, \Delta$ and $j = 2, 3, 4$ let $n_i = n_i(G)$ and

$$\tau_i^{(j)}(G_k) = |\{v \in V(G_k) | \, d_{G_k}(v) \geqslant i \quad \text{and} \quad v \in N_{G_k}(V_j)\}|.$$

Partition V_i into $V_i^{(h)} = \{x \in V_i | d_{H_0}(x) = i - h\}$ and denote $|V_i^{(h)}|$ by $n_i^{(h)}$ for $i = 2, 3, 4$, $h = 0, 1, \ldots, i - 1$. Then $n_i = \sum_{h=0}^{i-1} n_i^{(h)}$ and $m_i = \sum_{h=0}^{i-1} h n_i^{(h)}$.

For $x \in V_2^{(h)}$, $h = 0$ or 1, suppose $N_G(x) = \{v_1, v_2\}$. By Lemma 4.1 we obtain

$$d_{G_k}(v_j) \geqslant \begin{cases} 8 & \text{if } j = 1, 2 \text{ and } h = 0, \\ 7 & \text{if } j = 1, 2 \text{ and } h = 1, \end{cases}$$

which implies

$$\tau_7^{(2)}(G_k) \geqslant 2n_2, \quad \tau_8^{(2)}(G_k) \geqslant 2n_2^{(0)} = 2n_2 - 2n_2^{(1)}. \tag{32}$$

For $x \in V_3^{(h)}$, $0 \leqslant h \leqslant 2$, suppose $N_G(x) = \{v_1, v_2, v_3\}$. Let $xv_j \in E_3'$ for $j = 1, \ldots, h$. By Lemma 4.1 we have

$$d_{G_k}(v_j) \geqslant \begin{cases} 6 & \text{if } 1 \leqslant j \leqslant h, \\ 8 & \text{if } h + 1 \leqslant j \leqslant 3 \text{ and } h = 0, \\ 7 & \text{if } h + 1 \leqslant j \leqslant 3 \text{ and } h = 1, \\ 6 & \text{if } h + 1 \leqslant j \leqslant 3 \text{ and } h = 2, \end{cases}$$

which implies

$$\begin{aligned} &\tau_7^{(3)}(G_k) \geqslant 3n_3^{(0)} + 2n_3^{(1)} = 3n_3 - n_3^{(1)} - 3n_3^{(2)}, \\ &\tau_8^{(3)}(G_k) \geqslant 3n_3^{(0)} = 3n_3 - 3n_3^{(1)} - 3n_3^{(2)}. \end{aligned} \tag{33}$$

For $x \in V_4^{(h)}$, $0 \leqslant h \leqslant 3$, suppose $N_G(x) = \{v_1, v_2, v_3, v_4\}$. Let $xv_j \in E_4'$ for $j = 1, \ldots, h$. By Lemma 4.1 we have

$$d_{G_k}(v_j) \geqslant \begin{cases} 5 & \text{if } 1 \leqslant j \leqslant h, \\ 7 & \text{if } h + 1 \leqslant j \leqslant 4 \text{ and } h = 0, 1, \\ 6 & \text{if } h + 1 \leqslant j \leqslant 4 \text{ and } h = 2, \\ 5 & \text{if } h + 1 \leqslant j \leqslant 4 \text{ and } h = 3, \end{cases}$$

which implies

$$\tau_7^{(4)}(G_k) \geqslant 4n_4^{(0)} + 3n_4^{(1)} = 4n_4 - n_4^{(1)} - 4n_4^{(2)} - 4n_4^{(3)}. \tag{34}$$

It is easy to observe that $n_i(G_k) = n_i$ for $i = 2, 3, 4$, $n_5(G_k) \leqslant n_5$ and $\delta(G_k) \geqslant 2$. Lemma 2.10 guarantees that H_k satisfies the condition of Lemma 2.8. Then in view of Euler's Formula we obtain for H_k

$$6m(H_k) - 6n(H_k) + 12 = 6\phi(H_k) \leqslant 4m(H_k) - 2n_2(H_k) - 2n_1(H_k)$$

and it follows from $n(H_k) = n(G_k)$ that

$$2m(G_k) - 2\mathrm{cr}(G) + 2n_2(H_k) + 2n_1(H_k) + 12 \leqslant 6n(G_k).$$

That is,

$$\begin{aligned} &4n_2(G_k) + 3n_3(G_k) + 2n_4(G_k) + n_5(G_k) \\ &\quad \geqslant \tau_7(G_k) + \tau_8(G_k) + 2n_2(H_k) + 2n_1(H_k) + 12 - 2\mathrm{cr}(G). \end{aligned} \tag{35}$$

Note that $n_2(H_k) \geqslant n_2 - n_2^{(1)} + n_3^{(1)} + n_4^{(2)}$ and $n_1(H_k) = n_2^{(1)}$. Then substituting (32)–(34) into (35) yield

$$
\begin{aligned}
&4n_2 + 3n_3 + 2n_4 + n_5 \\
&\quad \geqslant 4n_2(G_k) + 3n_3(G_k) + 2n_4(G_k) + n_5(G_k) \\
&\quad \geqslant \tau_7^{(2)}(G_k) + \tau_7^{(3)}(G_k) + \tau_7^{(4)}(G_k) + \tau_8^{(2)}(G_k) + \tau_8^{(3)}(G_k) \\
&\qquad + 2n_2(H_k) + 2n_1(H_k) + 12 - 2\mathrm{cr}(G) \\
&\quad \geqslant 6n_2 + 6n_3 + 4n_4 - 2n_2^{(1)} - 2n_3^{(1)} - 6n_3^{(2)} \\
&\qquad - n_4^{(1)} - 2n_4^{(2)} - 4n_4^{(3)} + 12 - 2\mathrm{cr}(G) \\
&\quad \geqslant 6n_2 + 6n_3 + 4n_4 - 2m_2 - 3m_3 - \tfrac{4}{3}m_4 + 12 - 2\mathrm{cr}(G) \\
&\quad \geqslant 6n_2 + 6n_3 + 4n_4 - 5\mathrm{cr}(G) + 12, \qquad (36)
\end{aligned}
$$

that is,

$$5\mathrm{cr}(G) + n_5 \geqslant 2n_2 + 3n_3 + 2n_4 + 12,$$

which contradicts to condition (1). Using $n_i \geqslant \sum_{j=1}^{i-1} n_i^{(j)}$ for $i = 2, 3$ and (36), we obtain

$$
\begin{aligned}
&4n_2 + 3n_3 + 2n_4 + n_5 \\
&\quad \geqslant 6n_2 + 6n_3 + 4n_4 - \tfrac{3}{2}m_2 - \tfrac{1}{2}n_2 - \tfrac{3}{2}m_3 - 3n_3 - \tfrac{3}{2}m_4 + 12 - 2\mathrm{cr}(G) \\
&\quad \geqslant \tfrac{11}{2}n_2 + 3n_3 + 4n_4 + 12 - \tfrac{7}{2}\mathrm{cr}(G),
\end{aligned}
$$

that is, $7\mathrm{cr}(G) + 2n_5 \geqslant 3n_2 + 4n_4 + 24$, which contradicts to condition (2).

The proof of the theorem is complete. □

Remark 4.8. When $\mathrm{cr}(G) = 0$ condition (1) in Theorem 4.7 is just one in Theorem 4.4 in [3]. Using $n_i \geqslant \sum_{j=1}^{i} n_i^{(j)}$ we can obtain other similar conditions from (36). By considering the coefficient of $\mathrm{cr}(G)$, we believe that condition (2) in Theorem 4.7 is best possible.

Analogously, but much simpler, we can also prove the following proposition, which generalizes Proposition 4.5 in [3], omitted here for details.

Proposition 4.9. *Let G be a connected graph with no vertices of degree four and five. If* $\mathrm{cr}(G) \leqslant 2$, *then* $b(G) \leqslant 6$.

Theorem 4.10. *If G is a connected graph with* $\mathrm{cr}(G) \leqslant 4$ *and not* 4*-regular when* $\mathrm{cr}(G) = 4$, *then* $b(G) \leqslant \Delta(G) + 2$.

Proof. The result is valid in the following cases:

(1) $\Delta(G) \geqslant 6$ by Theorem 4.3;
(2) $\delta(G) \leqslant 3$ by Lemma 2.2;
(3) $\Delta = \delta = 4$ by Lemma 2.4 when $g(G) = 3$, or by Corollaries 3.2 and 3.3 when $g(G) \geqslant 4$.

In the remaining cases, suppose $\Delta(G) = 5$ and $\delta(G) = 4$ and so $V(G) = V_4 \cup V_5$. By Lemma 2.10, let H be the maximum planar subgraph of G. If $g(H) \geqslant 4$, then $b(G) \leqslant 6 < \Delta(G) + 2$ by Corollaries 3.2, 3.3 and Remark 3.5. Thus we assume $g(H) = 3$. If there are two distinct triangles having common edges in H, then there exist two vertices u and v such that $|N_H(u) \cap N_H(v)| \geqslant 2$. Note that $|N_G(u) \cap N_G(v)| \geqslant |N_H(u) \cap N_H(v)|$. Then $b(G) \leqslant 5 + 5 - 1 - 2 = 7 = \Delta(G) + 2$ by Lemma 2.4. Therefore, we assume that any two distinct triangles in H have no common edges. Let $n = n(G)$, $n_i = V_i(G)$ for $i = 4, 5$, $r = r(H)$ and r_3 be the number of triangles in H. Counting the number of edges of H, we have

$$2m(H) \geqslant 2m(G) - 2\mathrm{cr}(G) = 4n_4 + 5n_5 - 2\mathrm{cr}(G) = 5n - n_4 - 2\mathrm{cr}(G)$$

and

$$2m(H) \geqslant 3r_3 + 4(r - r_3) = 4r - r_3.$$

It follows from Euler's formula that

$$n_4 + r_3 \geqslant n + 8 - 2\mathrm{cr}(G). \quad (37)$$

Define T be the subgraph of H induced by all triangles in H. It is clear that $\Delta(T) \leqslant \Delta(H) \leqslant \Delta(G) = 5$. If $d_T(v) = 5$ for some $v \in V(T)$, then there are at least three triangles incident with v, two of which have a common edge. Thus $\Delta(T) \leqslant 4$. Then

$$3r_3 = m(T) \leqslant \tfrac{1}{2}\Delta(T)n(T) \leqslant 2n(T). \quad (38)$$

Combining (38) with (37), we have

$$n_4 + \tfrac{2}{3}n(T) \geqslant n + 8 - 2\mathrm{cr}(G).$$

Note that $n(T) \geqslant 1$ since $g(H) = 3$. Then by the hypothesis $\mathrm{cr}(G) \leqslant 4$ we have $n_4 + n(T) > n$, which implies that $V_4 \cap V(T) \neq \emptyset$. Let $v \in V_4 \cap V(T)$ and $u \in N_G(v) \cap V(T)$, then

$$\begin{aligned} b(G) &\leqslant d_G(v) + d_G(u) - 1 - |N_G(v) \cap N_G(u)| \\ &\leqslant 4 + 5 - 1 - 1 \leqslant \Delta(G) + 2. \end{aligned}$$

The proof of the theorem is complete. □

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