

# The total domination and total bondage numbers of extended de Bruijn and Kautz digraphs<sup>☆</sup>

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## Abstract

In this paper we consider the total domination number and the total bondage number for digraphs. The total bondage number, defined as the minimum number of edges whose removal enlarges the total domination number, measures to some extent the robustness of a network where a minimum total dominating set is required. We determine the total domination number and total bondage number of the extended de Bruijn digraph and the extended Kautz digraph, proposed by Shibata and Gonda in 1995, which generalize the classical de Bruijn digraph and the Kautz digraph.

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*Keywords:* Total domination; Total bondage number; Minimum total dominating set; Extended de Bruijn digraph; Extended Kautz digraph

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## 1. Introduction

It is well-known that an interconnection network can be modelled by a graph with vertices representing sites of the network and edges representing links between sites of the network. Therefore various problems in networks can be studied by graph theoretical methods. Now dominations have become one of the major areas in Graph Theory after more than 20 years' development. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real-world problems, such as facility location problems. Among the domination-type parameters that have been studied, two most fundamental ones are the domination number and the total domination number. The difference between them is whether each vertex in a dominating set can be viewed as dominated by itself. Hence these two parameters are useful in different problems.

The bondage number arises in the further consideration of link fault in a network. Such a fault possibly happens in the real world (hacking, experimental error, terrorism, etc.), and may break down the minimum dominating sets in the network. The minimum number of edges whose failure enlarges the domination number is defined as the bondage number. It measures to some extent the robustness of a network with respect to link failure. In view of the different uses of domination numbers and total domination numbers, in this paper we consider total bondage numbers and

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determine it for extended de Bruijn digraphs and extended Kautz digraphs, which are generalizations of classic de Bruijn digraphs and Kautz digraphs.

The reason of the special interest on de Bruijn networks and Kautz networks lies in their attractive features. The de Bruijn digraph and the Kautz digraph have been thought of as good candidates for the next generation of parallel system architectures after the hypercube networks [1]. Some computer systems based on the de Bruijn architecture have been built (see Pradhan [2]). Therefore de Bruijn digraphs and Kautz digraphs have attracted much attention, and many invariants and properties have been investigated. In [3] we determined their bondage number. In view of different applications of dominations and total dominations, we now consider their total bondage numbers and determine it in a wider sense, namely focus on a class of generalizations of them, called extended de Bruijn digraphs and extended Kautz digraphs, which have more flexible structure so that one can choose more suitable networks for prescribed requirements.

In this paper we mainly consider  $G = (V, E)$  as a digraph with the vertex-set  $V$  and the edge-set  $E$ . For a subset  $S \subset V$ , let  $E^+(S) = \{(u, v) \in E : u \in S, v \notin S\}$  and  $E^-(S) = \{(u, v) \in E : u \notin S, v \in S\}$ ; let  $N^+(S) = \{v \in V : \exists u \in S, (u, v) \in E^+(S)\}$  and  $N^-(S) = \{u \in V : \exists v \in S, (u, v) \in E^-(S)\}$ . If  $S = \{x\}$  we replace  $S$  by  $x$  for convenience. For  $v \in V$  and  $(u, v), (v, w) \in E$ ,  $u$  and  $w$  are called an *in-neighbor* and an *out-neighbor* of  $v$ , respectively. The *in-degree* and *out-degree* of  $v$  are the number of its in-neighbors and out-neighbors, and are denoted by  $d^-(v) = d_G^-(v)$  and  $d^+(v) = d_G^+(v)$ , respectively. The *degree* of  $v$  is  $d(v) = d_G(v) = d^+(v) + d^-(v)$ . Denote the maximum and the minimum degree of  $G$  by  $\Delta(G)$  and  $\delta(G)$ , the maximum and the minimum in-degree (resp. out-degree) of  $G$  by  $\Delta^-(G)$  and  $\delta^-(G)$  (resp.  $\Delta^+(G)$  and  $\delta^+(G)$ ). A digraph  $G$  is *d-regular* if  $\delta^-(G) = \delta^+(G) = \Delta^-(G) = \Delta^+(G) = d$ . We follow [4] for graph-theoretical terminology and notation not defined here.

In 1990, Fink et al. [5] introduced the bondage number of a nonempty undirected graph as the cardinality of a smallest set of edges whose removal results in a graph with domination number larger than that of  $G$ . Following this idea Kulli and Patwari [6] defined the total bondage number  $b_t(G)$  for a undirected graph  $G$ . A total dominating set for an undirected graph  $G$ , introduced by Cockayne et al. [7] in 1980, is a set  $T$  of vertices such that every vertex in  $G$  is adjacent to at least one vertex in  $T$ . The total domination number  $\gamma_t(G)$  is the minimum cardinality over all total dominating set, and the total bondage number  $b_t(G)$  is the cardinality of a minimum set of edges whose removal results in a graph with total domination number larger than  $\gamma_t(G)$ . The total domination number has received much attention (see [8], [9]). As far as we know, however, no research work on the total bondage number was reported in the literature except [6].

In this paper, we generalize the concept of total domination to digraphs. A total dominating set of a digraph  $G$  is a set  $T$  of vertices such that each vertex in  $G$  is an out-neighbor of some vertex (other than itself) in  $T$ . We say that a vertex *dominates* all its out-neighbors except itself. The total domination number  $\gamma_t(G)$  is the minimum cardinality over all total domination sets. It is easy to verify that  $\gamma_t(G)$  exists for a loopless digraph  $G$  if and only if  $\delta^-(G) \geq 1$ . The total bondage number  $b_t(G)$  of a digraph  $G$  is the cardinality of a minimum set  $E'$  of edges such that  $\gamma_t(G - E') > \gamma_t(G)$ . In this paper, we allow a digraph  $G$  to have loops, but all loops in  $G$  need not be considered, since loops have no effect on  $\gamma_t(G)$  and  $b_t(G)$ , according to their definitions.

The extended de Bruijn digraph  $EB(d, n; q_1, \dots, q_p)$  and the extended Kautz digraph  $EK(d, n; q_1, \dots, q_p)$  were introduced by Shibata and Gonda [10]. We shall determine their total domination numbers and total bondage numbers for general cases as follows:

$$\begin{aligned} \gamma_t(EB(d, n; q_1, \dots, q_p)) &= d^{n-p}, \\ b_t(EB(d, n; q_1, \dots, q_p)) &= d^p - 1; \end{aligned}$$

and

$$\begin{aligned} \gamma_t(EK(d, n; q_1, \dots, q_p)) &= d^{n-2p}(d + 1)^p, \\ b_t(EK(d, n; q_1, \dots, q_p)) &= d^p. \end{aligned}$$

As special cases, we obtain, for  $n \geq 2$ ,

$$\begin{aligned} \gamma_t(B(d, n)) &= d^{n-1}, & b_t(B(d, n)) &= d - 1 \quad \text{for } d \geq 2, \\ \gamma_t(K(d, n)) &= d^{n-1} + d^{n-2}, & b_t(K(d, n)) &= d \quad \text{for } d \geq 1. \end{aligned}$$

The rest of the paper is organized as follows. Section 2 presents some bounds of  $b_t(G)$  and some simple examples. Our main results for the extended de Bruijn digraph and the extended Kautz digraph are in Section 3 and Section 4, respectively.

## 2. Some bounds and examples

**Lemma 2.1.** *Let  $G$  be a loopless digraph with  $\delta^-(G) \geq 2$ . If there exist an edge  $(u, v)$  and a vertex  $w$  different from  $u$  and  $v$  in  $G$  such that  $X = (N^-(v) \cap N^-(w)) \setminus \{u\} \neq \emptyset$ , then*

- (a)  $b_t(G) \leq d^+(u) + d^-(v) + d^-(w) - |N^-(v) \cap N^-(w)| - 2$ ;
- (b)  $b_t(G) \leq d^+(u) + d^-(v) + d^-(w) - \min_{x \in X} \{|N^+(u) \cap N^+(x)|\} - 2$ .

**Proof.** Let  $Y = N^+(u) \cap N^+(x)$  for a given  $x \in X$  and  $Z = N^-(v) \cap N^-(w)$ , and let

$$E_1 = (E^+(u) \setminus \{(u, v)\}) \cup (E^-(v) \setminus \{(u, v)\}) \cup (E^-(w) \setminus \{(z, w) \in E : z \in Z\}),$$

$$E_2 = (E^+(u) \setminus \{(u, y) \in E : y \in Y\}) \cup (E^-(v) \setminus \{(u, v)\}) \cup (E^-(w) \setminus \{(x, w)\}).$$

We first prove the assertion (a). It is clear that a total dominating set of  $H = G - E_1$  exists since  $\delta^-(H) \geq 1$  by the hypothesis of  $\delta^-(G) \geq 2$ . Note that any minimum total dominating set  $T$  of  $H$  must contain  $u$  and some  $z \in Z$  in order to totally dominate  $v$  and  $w$ , respectively. If  $z = u$  then  $(u, w) \in E(H) \subseteq E(G)$ , which implies that  $(u, w) \notin E_1$ , a contradiction to the construction of  $E_1$ . Thus  $z \neq u$  and  $T' = T \setminus \{u\}$  remains a total dominating set of  $G$  since  $N_H^+(u) = \{v\} \subseteq N_H^+(z)$ . Thus

$$\gamma_t(H) = |T| = |T'| + 1 \geq \gamma_t(G) + 1,$$

which implies that

$$b(G) \leq |E_1| = d^+(u) + d^-(v) + d^-(w) - |Z| - 2.$$

Analogously, we can prove the assertion (b). It is clear that a total dominating set of  $H' = G - E_2$  exists since  $\delta^-(H') \geq 1$ . Let  $T$  be a minimum one. Then  $T$  must contain  $u$  and  $x$  in order to dominate  $v$  and  $w$ , respectively. But  $T' = T \setminus \{u\}$  is a total dominating set of  $G$  since  $N_{H'}^+(u) = Y \subseteq N_{H'}^+(x)$ . Thus,  $\gamma_t(H') = |T| \geq |T'| + 1 \geq \gamma_t(G) + 1$ , and so the result follows. ■

**Corollary 2.2.** *Let  $G$  be a loopless digraph. If  $\delta^+(G) \geq 2$  and  $\delta^-(G) \geq 2$ , then*

$$b_t(G) \leq \min\{\delta^+(G) + 2\Delta^-(G), \delta^-(G) + \Delta^+(G) + \Delta^-(G)\} - 3.$$

*In particular, if  $G$  is  $\delta$ -regular and  $\delta \geq 2$ , then  $b_t(G) \leq 3(\delta - 1)$ .*

**Proof.** Let  $u \in V(G)$  with  $d^+(u) = \delta^+(G)$  and  $v$  an out-neighbor of  $u$ . Since  $G$  is loopless,  $\delta^+(G) \geq 2$  and  $\delta^-(G) \geq 2$ , there is  $w \in V(G) \setminus \{u, v\}$  such that  $|N^-(v) \cap N^-(w)| \geq 1$ . Thus,  $b(G) \leq \delta^+(G) + 2\Delta^-(G) - 3$  by Lemma 2.1(a).

Let  $x \in V(G)$  with  $d^-(x) = \delta^-(G)$ . Similarly, by Lemma 2.1(b), we have  $b(G) \leq \delta^-(G) + \Delta^+(G) + \Delta^-(G) - 3$ . ■

Now we establish a lower bound of  $b_t(G)$ . Given an edge  $e$  and a total dominating set  $T$  of  $G$ , we say  $e$  supports  $T$  if  $e \in E_t^+(T) = \bigcup_{v \in T} E^+(v)$ . Denote by  $s(G)$  the minimum number of edges which support all minimum total dominating sets in  $G$ .

**Lemma 2.3.** *For a digraph  $G$ ,  $b_t(G) \geq s(G)$ .*

**Proof.** Let  $E' \subseteq E(G)$  with  $|E'| < s(G)$ . We show that  $\gamma_t(G - E') = \gamma_t(G)$ , which implies the result.

Since  $|E'| < s(G)$ , there exists a minimum total dominating set  $T$  not supported by any edge of  $E'$ . For any vertex  $v \in V(G)$ ,  $T$  contains a vertex  $u$  such that  $u \neq v$  and  $(u, v) \in E(G)$ . Then  $(u, v)$  supports  $T$  and so  $(u, v) \notin E'$ . Thus  $v$  is dominated by  $u$  in  $G - E'$ . Therefore  $T$  is still a total dominating set in  $G - E'$  and  $\gamma_t(G - E') = \gamma_t(G)$ . ■

We now give some examples.

**Example 2.4.** Let  $C_n$  and  $P_n$  be a directed cycle and a directed path with  $n$  vertices, respectively. Then  $\gamma_t(C_n) = n$ , and  $\gamma_t(P_n)$ ,  $b_t(P_n)$  and  $b_t(C_n)$  all do not exist.

**Proof.** It is clear that  $\gamma_t(P_n)$  and  $b_t(P_n)$  do not exist since  $P_n$  has a vertex of in-degree zero. Since  $C_n$  is 1-regular, any total dominating set must contain all vertices. Hence  $\gamma_t(C_n) = n$ . The removal of an edge from  $C_n$  results in a directed path  $P_n$ . It follows that  $b_t(C_n)$  does not exist. ■

**Example 2.5.** Let  $K_n$  be a complete digraph with vertex-set  $\{1, 2, \dots, n\}$  and edge-set  $\{(i, j) : 1 \leq i \neq j \leq n\}$ . Then

$$\begin{cases} \gamma_t(K_n) \text{ does not exist} & \text{if } n = 1; \\ \gamma_t(K_n) = 2 & \text{if } n \geq 2 \end{cases}$$

and

$$\begin{cases} b_t(K_n) \text{ does not exist} & \text{if } n = 1, 2; \\ b_t(K_n) = 3 & \text{if } n = 3; \\ n \leq b_t(K_n) \leq 2n - 3 & \text{if } n \geq 4. \end{cases}$$

**Proof.** It is a simple observation that the result is true for  $n = 1, 2$ .

Now assume  $n \geq 3$ . It is clear that  $\mathcal{T} = \{T_{ij} = \{i, j\} : i \neq j\}$  is a family of minimum total dominating sets. Each edge  $(x, y)$  in  $K_n$  supports  $n - 1$  sets in  $\mathcal{T}$ , i.e., all  $T_{ij}$  with one element equal to  $x$  (the other one is allowed to be  $y$ ). Thus, we need at least  $|\mathcal{T}|/(n - 1) = n$  edges to support  $\mathcal{T}$ . It follows from Lemma 2.3 that  $b_t(K_n) \geq n$ .

On the other hand, we can establish an upper bound for  $b_t(K_n)$ . If  $n = 3$  then  $E' = \{(1, 2), (2, 3), (3, 1)\}$  is a total bondage set since  $\gamma_t(K_n - E') = 3 > \gamma_t(K_n)$ . Thus  $b_t(K_3) \leq 3$ . If  $n \geq 4$ , it follows from Lemma 2.1 that  $b_t(K_n) \leq 3(n - 1) - 2 - (n - 2) = 2n - 3$ . The result follows. ■

### 3. Extended de Bruijn digraphs

The de Bruijn digraph  $B(d, n)$  has the vertex-set  $V = \{x_1 \dots x_n : 0 \leq x_i \leq d - 1\}$ ; there is a directed edge from  $x$  to  $y$  if  $x = x_1 x_2 \dots x_n$  and  $y = x_2 \dots x_n \alpha$ ,  $\alpha \in \{0, 1, \dots, d - 1\}$ . It is clear that  $B(d, n)$  has  $d^n$  vertices,  $d^{n+1}$  edges, and is  $d$ -regular.

The extended de Bruijn digraph  $EB(d, n; q_1, \dots, q_p)$  has the vertex-set  $V$  as a set of  $n$ -dimensional vectors on  $d$  elements divided into  $p$  blocks of sizes  $q_1, \dots, q_p$ , expressed as the following form

$$x = (x_{11}x_{12} \dots x_{1q_1})(x_{21}x_{22} \dots x_{2q_2}) \dots (x_{p1}x_{p2} \dots x_{pq_p}),$$

where  $0 \leq x_{ij} \leq d - 1$ , and  $q_1 + q_2 + \dots + q_p = n$ . The out-neighbors of  $x$  are those vertices having the form

$$(x_{12} \dots x_{1q_1} \alpha_1)(x_{22} \dots x_{2q_2} \alpha_2) \dots (x_{p2} \dots x_{pq_p} \alpha_p),$$

where  $0 \leq \alpha_i \leq d - 1$  for each  $i = 1, 2, \dots, p$ . Clearly,  $EB(d, n; n) = B(d, n)$  and  $EB(d, n; q_1, \dots, q_p)$  has  $d^n$  vertices,  $d^{n+p}$  edges and is  $d^p$ -regular.

In this section, we consider the total domination number and total bondage number of  $EB(d, n; q_1, \dots, q_p)$ . An easy lower bound  $\gamma_t(G) \geq \lceil \frac{|V(G)|}{\Delta^+(G)} \rceil$  helps us to determine  $\gamma_t(B(d, n))$ , and then we can generalize the result to  $EB(d, n; q_1, \dots, q_p)$ .

Suppose that  $(i_1 \dots i_p)$  and  $(j_1 \dots j_p)$  be two sequences on  $\{0, 1, \dots, d - 1\}$  with the property that there exists some  $k \in \{1, \dots, p\}$  such that  $i_k \neq j_k$  and  $q_k \geq 2$ . Let

$$\begin{aligned} T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)} &= \{(i_1 x_{12} \dots x_{1q_1}) \dots (i_p x_{p2} \dots x_{pq_p}) \in V\} \\ &\cup \{(j_1 i_1 \dots i_1) \dots (j_p i_p \dots i_p)\} \setminus \{(i_1 i_1 \dots i_1) \dots (i_p i_p \dots i_p)\}. \end{aligned} \tag{1}$$

It is clear that

$$\left| T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)} \right| = d^{n-p}.$$

**Theorem 3.1.** For any  $d \geq 2$  and  $q_k \geq 2$  for some  $k$  with  $1 \leq k \leq p$ ,

$$\gamma_t(EB(d, n; q_1, \dots, q_p)) = d^{n-p}$$

and  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$  defined in (1) is a minimum total dominating set in  $EB(d, n; q_1, \dots, q_p)$ .

**Proof.** Let  $G = EB(d, n; q_1, \dots, q_p)$ . Let

$$x = (x_{11}x_{12} \dots x_{1q_1}) \dots (x_{p1}x_{p2} \dots x_{pq_p})$$

be any vertex in  $G$  and

$$y = (i_1x_{11} \dots x_{1(q_1-1)}) \dots (i_px_{p1} \dots x_{p(q_p-1)}).$$

If  $y \neq (i_1 \dots i_1) \dots (i_p \dots i_p)$  then  $y \in T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$ ,  $x \neq y$  and  $x$  is dominated by  $y$ . If  $y = (i_1 \dots i_1) \dots (i_p \dots i_p)$  then

$$x = (i_1 \dots i_1x_{1q_1}) \dots (i_p \dots i_px_{pq_p}) \neq (j_1i_1 \dots i_1) \dots (j_pi_p \dots i_p)$$

since there exists some  $k \in \{1, \dots, p\}$  such that  $i_k \neq j_k$  and  $q_k \geq 2$ . Thus,  $x$  is dominated by  $(j_1i_1 \dots i_1) \dots (j_pi_p \dots i_p)$ , which is in  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$ . Therefore  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$  is a total dominating set for  $G$ . Since every vertex dominates at most  $d^p$  vertices in  $G$ , then

$$\gamma_t(G) \geq d^n/d^p = d^{n-p} = \left| T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)} \right|.$$

Thus,  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$  is minimum, and the theorem follows. ■

**Remark.** The condition in Theorem 3.1,  $q_k \geq 2$  for some  $k$ , is necessary. If  $q_k = 1$  for each  $k = 1, 2, \dots, p$ , then  $p = n$  and  $d^{n-p} = 1$ . But a single vertex dominates all vertices except itself. Hence  $\gamma_t(EB(d, n; 1, 1, \dots, 1)) = 2 \neq d^{n-p}$ .

Now we have a family of minimum total dominating sets of  $G = EB(d, n; q_1, \dots, q_p)$ . Then we can use Lemma 2.3 to determine  $b_t(G)$ .

**Theorem 3.2.** Let  $G = EB(d, n; q_1, \dots, q_p)$  with  $d \geq 2$ . If there are exactly  $0 \leq r \leq p - 1$  elements in  $\{q_1, \dots, q_p\}$  which are equal to 1, then

$$d^p - d^r \leq b_t(EB(d, n; q_1, \dots, q_p)) \leq d^p - 1.$$

**Proof.** Suppose that  $q_1 = \dots = q_r = 1$  and  $q_k \geq 2$  for  $k = r + 1, \dots, p$ , without loss of generality. Let  $\mathcal{P}$  be a set of sequences of length  $p$  on  $\{0, 1, \dots, d - 1\}$  and  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$  defined as (1). By Theorem 3.1,

$$\mathcal{T} = \left\{ T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)} : (i_1 \dots i_p), (j_1 \dots j_p) \in \mathcal{P}, i_k \neq j_k \text{ for some } k \in \{r + 1, \dots, p\} \right\}$$

is a family of minimum total dominating sets in  $G$  and  $|\mathcal{T}| = d^p d^r (d^{p-r} - 1)$ . Let  $E' \subseteq E(G)$  be a smallest set of edges which support  $\mathcal{T}$ . Note that an edge  $(x, y) \in E'$  with an end-vertex  $x = (x_{11}x_{12} \dots x_{1q_1}) \dots (x_{p1}x_{p2} \dots x_{pq_p})$  supports all  $T_{(x_{11} \dots x_{p1})}^{(j_1 \dots j_p)} \in \mathcal{T}$  and possibly some extra sets  $T_{(i_1 \dots i_r x_{(r+1)2} \dots x_{p2})}^{(j_1 \dots j_p)}$ 's, where  $j_1, \dots, j_p$  and  $i_1, \dots, i_r$  are taken from  $\{0, \dots, d - 1\}$  such that  $j_k \neq x_{k1}$  for some  $k \in \{r + 1, \dots, p\}$ . Thus  $(x, y)$  supports at most  $d^r (d^{p-r} - 1) + d^r = d^p$  sets in  $\mathcal{T}$ . It follows from Lemma 2.3 that

$$b(G) \geq s(G) \geq |E'| \geq \frac{d^{p+r}(d^{p-r} - 1)}{d^p} = d^p - d^r.$$

We now show that  $b_t(G) \leq d^p - 1$ . Let

$$x = (00 \dots 00) \dots (00 \dots 00),$$

$$y = (10 \dots 00) \dots (10 \dots 00),$$

$$z = (00 \dots 01) \dots (00 \dots 01)$$

be three vertices in  $G$  and

$$E' = (E^-(x) \setminus \{(y, x)\}) \cup \{(y, z)\}.$$

Then  $|E'| = d^p - 1$  since  $(x, x) \notin E'$  by the definition of  $E^-(x)$ . To dominate  $x$ , every minimum total dominating set  $T$  of  $H = G - E'$  must contain  $y$ . But  $y$  cannot dominate  $z$  in  $H$ . Since  $T$  is a minimum total dominating set of  $H$ , the number of out-neighbors of  $T$  is

$$|N^+(T)| = (|T| - 1)d^p + (d^p - 1) \geq |V(H)| = d^n,$$

which implies that  $|T| > d^{n-p}$ . It follows from Theorem 3.1 that

$$\gamma_t(H) = |T| > d^{n-p} = \gamma_t(G).$$

Therefore  $b_t(G) \leq |E'| = d^p - 1$ , and the theorem follows. ■

If  $r = 0$  then  $d^p - d^r = d^p - 1$ . And the extended de Bruijn digraph with  $p = 1$  is just the de Bruijn digraph  $B(d, n)$ . Then Theorems 3.1 and 3.2 yield the following corollaries immediately.

**Corollary 3.3.** *If  $d \geq 2$  and  $q_1, \dots, q_p \geq 2$ , then  $b_t(EB(d, n; q_1, \dots, q_p)) = d^p - 1$ .*

**Corollary 3.4.** *For any  $d \geq 2$  and  $n \geq 2$ ,*

$$\gamma_t(B(d, n)) = d^{n-1},$$

$$b_t(B(d, n)) = d - 1.$$

The condition in Corollary 3.4,  $d \geq 2$  and  $n \geq 2$ , is necessary. In fact,  $b_t(B(1, n))$  does not exist since  $B(1, n)$  has only one vertex, and  $b_t(B(d, 1)) = b_t(FK_d) = b_t(K_d) \neq d - 1$  by Example 2.5.

Note that the set defined in (1) is not a total dominating set of  $EB(d, n; q_1, \dots, q_p)$  if  $q_k = 1$  for each  $k = 1, 2, \dots, p$ . We propose the following problem.

**Problem 3.5.** Determine the total bondage number of  $EB(d, n; 1, \dots, 1)$ .

#### 4. Extended Kautz digraphs

We now consider another important class of networks, the Kautz digraph  $K(d, n)$ , with the vertex-set and edge-set defined as follows.

$$\begin{cases} V = \{x_1 \dots x_n : 0 \leq x_i \leq d, x_i \neq x_{i+1}, i = 1, \dots, n - 1\} & \text{and} \\ E = \{(x_1 x_2 \dots x_n, x_2 \dots x_n \alpha) : 0 \leq \alpha \leq d, \alpha \neq x_n\}. \end{cases}$$

$K(d, n)$  has  $d^{n-1}(d + 1)$  vertices,  $d^n(d + 1)$  edges, and is  $d$ -regular.

The vertex set of the extended Kautz digraph  $EK(d, n; q_1, \dots, q_p)$  is the set of  $n$ -dimensional vectors on  $d$  elements divided into  $p$  blocks of sizes  $q_1, \dots, q_p$ , expressed as the following form

$$x = (x_{11}x_{12} \dots x_{1q_1})(x_{21}x_{22} \dots x_{2q_2}) \dots (x_{p1}x_{p2} \dots x_{pq_p}),$$

where  $0 \leq x_{ij} \leq d$ ,  $x_{ij} \neq x_{i(j+1)}$ , and  $q_1 + q_2 + \dots + q_p = n$ . The out-neighbors of  $x$  are those vertices having the form

$$(x_{12} \dots x_{1q_1} \alpha_1)(x_{22} \dots x_{2q_2} \alpha_2) \dots (x_{p2} \dots x_{pq_p} \alpha_p),$$

where  $\alpha_i \in \{0, 1, \dots, d\}$  with  $\alpha_i \neq x_{iq_i}$  for each  $i = 1, 2, \dots, p$ . Clearly,  $EK(d, n; n) = K(d, n)$  and  $EK(d, n; q_1, \dots, q_p)$  has  $d^{n-p}(d + 1)^p$  vertices,  $d^n(d + 1)^p$  edges and is  $d^p$ -regular.

In this section we consider  $G = EK(d, n; q_1, \dots, q_p)$  with  $q_k \geq 2$  for each  $k = 1, 2, \dots, p$ . The technique is similar to Section 3. In order to determine  $b_t(G)$ , we first construct a family of minimum total dominating sets in  $G$ . For any given  $0 \leq i_1, \dots, i_p \leq d$ , choose  $j_1, \dots, j_p$  such that  $i_k \neq j_k$  for each  $k = 1, 2, \dots, p$ . For  $t = 0, 1, \dots, p$ , let  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}(t)$  be the set of all vertices  $(x_{11} \dots x_{1q_1}) \dots (x_{p1} \dots x_{pq_p})$  with the property that there

exists a subset  $I \subseteq \{1, 2, \dots, p\}$  with  $|I| = t$  such that  $x_{k1} = i_k$  if  $k \notin I$ , and  $x_{k1} = j_k \neq i_k = x_{k2}$  if  $k \in I$ . It is not difficult to observe that

$$\left| T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}(t) \right| = \binom{p}{t} d^{n-p-t} \quad \text{for each } t = 0, 1, \dots, p$$

since  $p + t$  coordinates are fixed in every vertex of  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}(t)$ . Let

$$T_{(i_1, \dots, i_p)}^{(j_1, \dots, j_k)} = \bigcup_{t=0}^p T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}(t). \tag{2}$$

**Theorem 4.1.** *If  $q_k \geq 2$  for each  $k = 1, 2, \dots, p$ , then*

$$\gamma_t(EK(d, n; q_1, \dots, q_p)) = d^{n-2p}(d + 1)^p,$$

and  $T_{(i_1, \dots, i_p)}^{(j_1, \dots, j_k)}$  defined in (2) is a minimum total dominating set.

**Proof.** Let  $v = (x_{11} \dots x_{1q_1}) \dots (x_{p1} \dots x_{pq_p})$  be any vertex in  $EK(d, n; q_1, \dots, q_p)$ . Assume  $x_{k1} = i_k$  if  $k \leq t$  and  $x_{k1} \neq i_k$  if  $k \geq t + 1$ ,  $0 \leq t \leq p$ , without loss of generality. Then  $v$  is dominated by

$$u = (j_1 i_1 x_{12} \dots x_{1(q_1-1)}) \dots (j_t i_t x_{t2} \dots x_{t(q_t-1)}) \\ (i_{t+1} x_{(t+1)1} \dots x_{(t+1)(q_{t+1}-1)}) \dots (i_p x_{p1} \dots x_{p(q_p-1)})$$

and  $u \in T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}(t) \subseteq T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$ . Therefore  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$  is a total dominating set.

On the other hand, every vertex in  $EK(d, n; q_1, \dots, q_p)$  dominates  $d^p$  vertices, which implies that

$$\begin{aligned} \gamma_t(EK(d, n; q_1, \dots, q_p)) &\geq d^{n-p}(d + 1)^p/d^p = d^{n-2p}(d + 1)^p \\ &= \sum_{t=0}^p \binom{p}{t} d^{n-p-t} = \sum_{t=0}^p \left| T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}(t) \right| \\ &= \left| T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)} \right|. \end{aligned}$$

Thus,  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$  is a minimum total dominating set, and so the theorem follows. ■

**Theorem 4.2.** *If  $d \geq 1$  and  $q_k \geq 2$  for each  $k = 1, 2, \dots, p$  then*

$$b_t(EK(d, n; q_1, \dots, q_p)) = d^p.$$

**Proof.** Let  $G = EK(d, n; q_1, \dots, q_p)$ . Let

$$\mathcal{T} = \left\{ T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)} : 0 \leq i_k \leq d - 1, j_k = d, k = 1, \dots, p \right\},$$

where  $T_{(i_1 \dots i_p)}^{(j_1 \dots j_p)}$  is defined as (2). Clearly,  $|\mathcal{T}| = d^p$  and is a family of minimum total dominating sets for  $G$  by Theorem 4.1. It is easy to verify that  $T \cap T' = \emptyset$  and, hence,  $E_t^+(T) \cap E_t^+(T') = \emptyset$ , if  $T, T' \in \mathcal{T}$  and  $T \neq T'$ . Hence every edge supports at most one set in  $\mathcal{T}$ . It follows from Lemma 2.3 that  $b(G) \geq s(G) \geq |\mathcal{T}| = d^p$ .

On the other hand, suppose that  $x \in V(G)$  and  $y, z \in N^+(x)$ . Let  $E' = (E^-(y) \setminus \{(x, y)\}) \cup \{(x, z)\}$ . Then  $|E'| = d^p$ . To dominate  $y$ , any minimum total dominating set  $T$  of  $H = G - E'$  must contain  $x$ . But  $x$  cannot dominate  $z$  in  $H$ . Since  $T$  is a total dominating set of  $H$ , the number of out-neighbors of  $T$

$$|N^+(T)| = (|T| - 1)d^p + (d^p - 1) \geq |V(H)| = d^{n-p}(d + 1)^p,$$

which implies that  $|T| > d^{n-2p}(d + 1)^p$ . It follows from Theorem 4.1 that

$$\gamma_t(H) = |T| > d^{n-2p}(d + 1)^p = \gamma_t(G),$$

and so  $b(G) \leq |E'| = d^p$ . The theorem follows. ■

Note that  $EK(d, n; n) = K(d, n)$ . Theorems 4.1 and 4.2 yield the results for  $K(d, n)$  immediately.

**Corollary 4.3.** For any  $d \geq 1$  and  $n \geq 2$ ,

$$\gamma(K(d, n)) = d^{n-1} + d^{n-2},$$

$$b_t(K(d, n)) = d.$$

The condition  $n \geq 2$  cannot be improved. In fact,  $K(d, 1)$  is isomorphic to  $K_{d+1}$  and  $b_t(K_{d+1}) \neq d$  by Example 2.5.

Like the extended de Bruijn digraph, the set defined in (2) is not a minimum total dominating set of  $EK(d, n; q_1, \dots, q_p)$  if there exists some  $k \in \{1, 2, \dots, p\}$  such that  $q_k = 1$ . The following problem is still open.

**Problem 4.4.** Determine the total domination number and the total bondage number of  $EK(d, n; q_1, \dots, q_p)$  if there exists at least one  $q_k = 1$  for  $k \in \{1, 2, \dots, p\}$ .

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### References

- [1] J.C. Bermond, C. Peyrat, De Bruijn and Kautz networks: A competitor for the hypercube? in: F. Andre, J.P. Verjus (Eds.), Hypercube and Distributed Computers, Elsevier Science, BV, Amsterdam, 1989, pp. 278–293.
- [2] D.K. Pradhan, Fault-tolerant VLSI architectures based on de Bruijn graphs (Galileo in the mid nineties), DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 5 (1991) 183–195.
- [3] J. Huang, J.-M. Xu, The bondage numbers of extended de Bruijn and Kautz digraphs, Comput. Math. Appl. 51 (2006) 1137–1147.
- [4] J.-M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer, Dordrecht, 2001.
- [5] J.F. Fink, M.S. Jacobson, L.F. Kinch, J. Roberts, The bondage number of a graph, Discrete Math. 86 (1990) 47–57.
- [6] V.R. Kulli, D.K. Patwari, The total bondage number of a graph, in: V.R. Kulli (Ed.), Advances in Graph Theory, Vishwa, Gulbarga, 1991, pp. 227–235.
- [7] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi, Total domination in graphs, Networks 10 (1980) 211–219.
- [8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
- [9] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.
- [10] Y. Shibata, Y. Gonda, Extension of de Bruijn graph and Kautz graph, Comput. Math. Appl. 30 (1995) 51–61.