The Restricted Edge-Connectivity of de Bruijn Undirected Graphs*

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Abstract

A connected graph is said to be super edge-connected if every minimum edge-cut isolates a vertex. The restricted edge-connectivity λ' of a connected graph is the minimum number of edges whose deletion results in a disconnected graph such that each component has at least two vertices. It has been shown by A. H. Esfahanian and S. L. Hakimi [On computing a conditional edge-connectivity of a graph. Information Processing Letters, 27(1988), 195-199] that $\lambda'(G) \leq \xi(G)$ for any graph of order at least four that is not a star, where $\xi(G) = \min\{d_G(u) + d_G(v) - 2 : \ uv$ is an edge in $G\}$. A graph G is called λ'-optimal if $\lambda'(G) = \xi(G)$. This paper proves that the de Bruijn undirected graph $UB(d, n)$ is λ'-optimal except $UB(2, 1)$, $UB(3, 1)$ and $UB(2, 3)$ and, hence, is super edge-connected for $n \geq 1$ and $d \geq 2$.

Keywords: Edge-connectivity, Restricted edge-connectivity, Super edge-connected, de Bruijn graphs, λ'-optimal

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1 Introduction

It is well-known that when the underlying topology of an interconnection network is modelled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the connectivity of G is an important measurement for fault-tolerance of the network. In this paper, we consider the edge-connectivity $\lambda(G)$ as a measurement for fault-tolerance of G.

Suppose that all vertices are perfectly reliable and that all edges fail independently with the same probability p. The parameter

$$R(G,p) = 1 - \sum_{i=\lambda}^{\varepsilon} c_i p^i (1-p)^{\varepsilon - i} \tag{1}$$

is an important measurement of global reliability of G, where $\varepsilon = |E(G)|$, $\lambda = \lambda(G)$, and c_i is the number of edge-cuts of cardinality i in G. It has been proved by Ball [1] that the computation of $R(G,p)$ is NP-hard for a general graph G. To minimize c_λ in (1), Bauer *et al* [2] suggested to investigate super edge-connected graphs. A connected graph G is said to be *super edge-connected*, if every minimum edge-cut isolates a vertex of G. Many well-known graphs are shown to be super edge-connected. In particular, Lü and Zhang [10] proved that the de Bruijn undirected graph $UB(d,n)$ is super edge-connected.

A very natural question is how many edges must be removed to disconnect a graph such that every component of the resulting graph contains no isolated vertices. In 1988, Esfahanian and Hakimi [5] proposed the concept of the restricted edge-connectivity. The *restricted edge-connectivity* of G, denoted by $\lambda'(G)$, is defined as the minimum number of edges whose deletion results in a disconnected graph and contains no isolated vertices. In general, $\lambda'(G)$ does not always exist for a connected graph G. For example, $\lambda'(G)$ does not exist if G is a star $K_{1,n}$ or a complete graph K_3. We write $\lambda'(G) = \infty$ if $\lambda'(G)$ does not exist. In [5], Esfahanian and Hakimi showed that if G has at least four vertices then $\lambda'(G)$ does not exist if and only if G is a star and that if $\lambda'(G)$ exists then

$$\lambda'(G) \le \xi(G), \tag{2}$$

where $\xi(G)$ is the minimum edge-degree of G, i.e. $\xi(G) = \min\{d(u) + d(v) - 2 : uv \text{ is an edge in } G\}$, and the symbol $d(x)$ denotes the degree of the vertex x in G.

A graph G is called λ'-optimal if $\lambda'(G) = \xi(G)$. Several sufficient conditions for graphs to be λ'-optimal were given for example by Hellwig and Volkmann [8] for graphs with diameter 2, Ueffing and Volkmann [14] for the cartesian product of graphs, Xu and Xu [19] for transitive graphs. It is clear that G is super edge-connected if $\lambda'(G) > \lambda(G)$. Recently, Hellwig and Volkmann [9] have showed that a λ'-optimal graph G is super edge-connected if its minimum degree $\delta(G) \geq 3$.

The new parameter λ' in conjunction with λ can provide more accurate measures for fault tolerance of a large-scale parallel processing system and, thus, has received much attention of many researchers (see, for example, [4] $\sim$ [11], [13] $\sim$ [15], [17] $\sim$ [19]).

As competitors for the hypercube, the de Bruijn and the Kautz graphs have received much attention of many researchers [3]. Recently, Xu and Lü [18] have shown that the de Bruijn digraph $B(d,n)$ and the Kautz digraph $K(d,n)$ are λ'-optimal. Very recently, Fan *et al* [6] have shown that the Kautz undirected graph $UK(d,n)$ is λ'-optimal. In this paper, we consider the same problem for the de Bruijn undirected graph $UB(d,n)$. In [10], Lü and Zhang proved that $\lambda'(UB(2,3)) = 3$ and $\lambda'(UB(2,n)) = 4$ for $n \geq 4$. We determine the rest of $\lambda'(UB(d,n))$ and state these results as the following theorem.

Theorem For any de Bruijn undirected graph $UB(d,n)$ with $n \geq 1$ and $d \geq 2$,

$$\lambda'(UB(d,n)) = \begin{cases} \infty & \text{for } n = 1 \text{ and } 2 \leq d \leq 3; \\ 2d-4 & \text{for } n = 1 \text{ and } d \geq 4; \\ 4d-5 & \text{for } n = 2 \text{ and } d \geq 2, \text{or } n = 3 \text{ and } d = 2; \\ 4d-4 & \text{for } n \geq 3 \text{ and } d \geq 3, \text{or } n \geq 4 \text{ and } d = 2. \end{cases}$$

Corollary The de Bruijn undirected graph $UB(d,n)$ is λ'-optimal except $UB(2,1)$, $UB(3,1)$ and $UB(2,3)$ and, hence, is super edge-connected for $n \geq 1$ and $d \geq 2$.

Although the main result of this paper is in parallel with the result in [6], which asserts that the Kautz undirected graphs are λ'-optimal, the technique used in [6] can not be used to prove the result in this paper.

The proofs of the theorem and the corollary are in Section 3. Some definitions and lemmas used in the proofs of the theorem and the corollary are given in Section 2.

2 Definitions and Lemmas

We follow [16] for graph-theoretical terminology and notation not defined here.

Let $G = (V,E)$ be a digraph, and $\{X,Y\}$ a partition of V. We use $E(X,Y)$ to denote the set of directed edges in G from X to Y. A digraph G is said to be d-regular if the out-degree and the in-degree of every vertex in G are equal to d. The following property of a regular digraph is simple and useful, and the detail proof can be found in [16, Example 1.4.1].

Lemma 1 If G is a regular digraph, then $|E(X,Y)| = |E(Y,X)|$ for any partition $\{X,Y\}$ of $V(G)$.

The well-known de Bruijn digraph is a class of important digraphs, which has been widely used in the design and analysis of interconnection networks [3]. We recall the definition of the de Bruijn digraph $B(d,n)$ for given integers $n \geq 1$ and $d \geq 2$.

The *de Bruijn digraph* $B(d,n)$ has the vertex-set $V = \{x_1x_2\cdots x_n : x_i \in \{0,1,\ldots,\, d-1\},\ i = 1,2,\ldots,n\}$ and the directed edge-set E, where for $x,y \in V$, if $x = x_1x_2\cdots x_n$, then $(x,y) \in E \Leftrightarrow y = x_2x_3\cdots x_n\alpha$, where $\alpha \in \{0,1,\ldots,d-1\}$.

Clearly, $B(d,1)$ is a complete digraph of order d plus a self-loop at every vertex. It has been shown that $B(d,n)$ is d-regular and the connectivity $\kappa = d-1$. For more properties of de Bruijn digraphs, the reader is referred to Xu [15, Section 3.2].

A pair of directed edges is said to be *symmetric* if they have the same end-vertices but different orientations. The de Bruijn digraph $B(d,n)$ contains $\binom{d}{2}$ pairs of symmetric directed edges. It is clear that if there is a pair of symmetric directed edges between two vertices x and y in $B(d,n)$, then the coordinates of x (resp. y) are alternately in two different components a and b. It is also clear that the directed distance between two end-vertices of different pairs of symmetric directed edges in $B(d,n)$ is equal to either $n-1$ or n. Moreover, two end-vertices of different pairs of symmetric directed edges have no vertex in common if and only if $n \geq 2$. For a vertex u in $B(d,n)$, we use $O(u)$ and $I(u)$ to denote the sets of out-neighbors and in-neighbors of u in $B(d,n)$ apart from u, respectively, that is, $O(u) = \{v \in V(B(d,n)) :\ v \neq u \text{ and } (u,v) \in E(B(d,n))\}$ and $I(u) = \{w \in V(B(d,n)) :\ w \neq u \text{ and } (w,u) \in E(B(d,n))\}$. The following fact is a simple observation, and the proof is left to the reader.

Lemma 2 Let (x,y) be a directed edge in $B(d,n)$, $O = O(x) \setminus \{y\}$ and $I = I(y) \setminus \{x\}$. Then every vertex in O is an out-neighbor of all vertices in I and $E(I,O)$ contains at most a pair of symmetric directed edges in $B(d,n)$ if and only if $n \geq 3$.

The *de Bruijn undirected graph*, denoted by $UB(d,n)$, is obtained from $B(d,n)$ by deleting the orientation of all directed edges and omitting multiple edges and loops. An edge in $UB(d,n)$ is said to be *singular* if it corresponds to a pair of symmetric directed edges in $B(d,n)$, and a vertex is said to be *singular* if it has a self-loop in $B(d,n)$.

Clearly, $UB(d,1)$ is a complete graph of order d. The structural properties of the de Bruijn undirected graph are first studied by Pradhan and Reddy [12].

Lemma 3 For $d \geq 2$ and $n \geq 2$, the de Bruijn undirected graph $UB(d,n)$ has the following properties:

(a) $UB(d,n)$ has the minimum degree $2d-2$, the maximum degree $2d$ if $n \geq 3$ and $2d-1$ if $n=2$.

(b) $\xi(UB(d,n)) = \begin{cases} 4d-5 & \text{for } n=2; \\ 4d-4 & \text{for } n \geq 3. \end{cases}$

(c) $UB(d,n)$ has the diameter n.

(d) $UB(d,n)$ has the connectivity $\kappa = 2d-2$ and the edge-connectivity $\lambda = 2d-2$.

Use V_{2d-i} to denote the set of vertices in $UB(d,n)$ of degree $2d-i$ for $i=0,1,2$. Clearly, a vertex $u \in V_{2d-2}$ if and only if it is singular and $u \in V_{2d-1}$ if and only if it is an end-vertex of some singular edge. By definition, $|O(u)| = |I(u)| = d-1$ if and only if $u \in V_{2d-2}$. Moreover, $|O(u)| = |I(u)| = d$ and $|O(u) \cap I(u)| = 1$ if $u \in V_{2d-1}$. The following fact is a simple observation.

Lemma 4 In $UB(d,n)$, the distance between two vertices in V_{2d-2} is n; the distance between a vertex in V_{2d-2} and a vertex in V_{2d-1} is at least $n-1$. Moreover, a singular edge is incident with a singular vertex, or two singular edges are adjacent if and only if $n=1$.

A set of edges F in a connected graph G is called a *nontrivial edge-cut* if $G-F$ is disconnected and contains no isolated vertices. If such an edge-cut exists, then the *restricted edge-connectivity* $\lambda'(G)$ of G is the minimum number of edges over all nontrivial edge-cuts of G. A nontrivial edge-cut F is called a λ'-*cut* if $|F| = \lambda'(G)$.

By the result of Esfahanian and Hakimi [5] mentioned in Introduction, for any integers $d \geq 2$ and $n \geq 1$ the restricted edge-connectivity $\lambda'(UB(d,n))$ exists except $UB(2,1)$ and $UB(3,1)$. When $\lambda'(UB(d,n))$ exists, let F be a λ'-cut of $UB(d,n)$, and then there is a partition $\{X,Y\}$ of $V(UB(d,n))$ such that $F = E[X,Y]$ with $|X| \geq 2$ and $|Y| \geq 2$, where $E[X,Y]$ denotes the set of edges between X and Y in $UB(d,n)$.

The following two lemmas are key results to the proof of our main theorem stated in Introduction.

Lemma 5 If $|X| \leq 2d-2$ or $|Y| \leq 2d-2$, then

$$|F| \geq \begin{cases} 4d-5 & \text{for } n=2; \\ 4d-4 & \text{for } n \geq 3. \end{cases}$$

Proof Without loss of generality, we assume $|X| \leq 2d-2$. Let

$$t=|X|,\quad t_i=|V_{2d-i}\cap X|\quad \text{for } i=0,1,2.$$

Then $2\le t=t_0+t_1+t_2\le 2d-2$. Noting that the function $f(t)=-t^2+2dt$ is convex on the interval $[2,2d-2]$ and $f(t)\ge f(2)=f(2d-2)=4d-4$, in order to prove the lemma, we only need to show that

$$|F|\ge\begin{cases}-t^2+2dt-1 & \text{for } n=2;\\ -t^2+2dt & \text{for } n\ge 3.\end{cases}$$

In fact, since any two vertices in V_{2d-2} are not adjacent when $n\ge 2$ by Lemma 4, the subgraph $G[X]$ induced by X is not a complete graph and can be obtained by deleting at least $\frac{1}{2}\,t_2(t_2-1)$ edges from K_t, which implies that

$$|E(G[X])|\le\frac{1}{2}\,t(t-1)-\frac{1}{2}\,t_2(t_2-1).$$

Thus, we have

$$\begin{aligned}|F| &= \sum_{x\in X}d(x)-2|E(G[X])|\\ &\ge (2d-2)t_2+(2d-1)(t-t_2)-(t(t-1)-t_2(t_2-1))\\ &= -t^2+2dt+t_2(t_2-2)\\ &\ge \begin{cases}-t^2+2dt-1 & \text{for } t_2=1;\\ -t^2+2dt & \text{for } t_2\ne 1.\end{cases}\end{aligned}$$

Thus, we only need to consider the case of $t_2=1$ and $n\ge 3$. By Lemma 4, the distance between a vertex in V_{2d-2} and a vertex in V_{2d-1} is at least $n-1\ge 2$, which implies that

$$|E(G[X])|\le\frac{1}{2}\,t(t-1)-t_1.$$

Thus, we have

$$\begin{aligned}|F| &= \sum_{x\in X}d(x)-2|E(G[X])|\\ &\ge (2d-1)t_1+(2d-2)+2d(t-t_1-1)-(t^2-t-2t_1)\\ &= -t^2+2dt+t+t_1-2\\ &\ge -t^2+2dt\end{aligned}$$

as required, so the lemma follows. ∎

Lemma 6 Let X_F and Y_F be the sets of end-vertices of F in X and Y, respectively. If $|X_F|\ge 2d-2$ and $|Y_F|\ge 2d-2$ then, for $d\ge 3$,

$$|F|\ge\begin{cases}4d-5 & \text{for } n=2;\\ 4d-4 & \text{for } n\ge 3.\end{cases}$$

Proof Let s be the number of singular edges in F. Since $B(d,n)$ is a regular digraph, by Lemma 1, we have

$$|F| = |E[X,Y]| = \begin{aligned} &2|E(X,Y)| - s \\ =\ &2|E(Y,X)| - s. \end{aligned} \tag{3}$$

In addition, suppose $E' \subseteq E(X,Y)$(or $E(Y,X)$) and that s' is the number of singular edges in F induced by the edges in E'. Let $E'' = E(X,Y) \setminus E'$ (or $E'' = E(Y,X) \setminus E'$). Then, from (3), we have

$$|F| = |E[X,Y]| \geq 2|E'| - s' + |E''|. \tag{4}$$

We prove the lemma by contradiction, that is, assume

$$|F| < \begin{cases} 4d-5 & \text{for } n = 2; \\ 4d-4 & \text{for } n \geq 3, \end{cases} \tag{5}$$

and then our aim is to deduce a contradiction.

For convenience' sake, for any $u \in V(B(d,n))$ and $T \subset V(B(d,n))$, let $O_T(u) = O(u) \cap T$ and $I_T(u) = I(u) \cap T$. Let U_X be the set of vertices in X_F each of which has exactly one neighbor in Y_F and U_Y be the set of vertices in Y_F each of which has exactly one neighbor in X_F. From the assumption (5), $U_X \neq \emptyset$ and $U_Y \neq \emptyset$ since $|X_F| \geq 2d-2$ and $|Y_F| \geq 2d-2$. Without loss of generality, choose a vertex $u_x \in U_X$, and denote its unique neighbor in Y_F by u_y.

If $u_y \in V_{2d-2}$, then $u_x \notin V_{2d-2}$ since any two vertices in V_{2d-2} are not adjacent when $n \geq 2$ by Lemma 4. We first assume $u_y \in O(u_x)$. In this case, $|O(u_y)| = d-1$ and $O(u_y) = O_X(u_x)$, which implies that u_y contributes $(d-1)$ directed edges to $E(Y,X)$. Since u_y is not an isolated vertex of $UB(d,n) - F$, $I_Y(u_y) \neq \emptyset$. Moreover, for any $z_y \in I_Y(u_y)$, $z_y \notin V_{2d-2}$ and $O_X(z_y) = O_X(u_x)$. This fact implies that z_y also contributes $(d-1)$ directed edges to $E(Y,X)$. Denote $E' = (\{u_y, z_y\}, X)$. So, $E' \subseteq E(Y,X)$ and $|E'| = 2(d-1)$. We denote the number of singular edges in F incident with u_y or z_y by s'. Then $s' \leq 1$ and the equality holds only if $n = 2$ by Lemma 4. It follows from (4) that

$$|F| \geq 2|E'| - s' = \begin{cases} 4d-4 & \text{if } s' = 0; \\ 4d-5 & \text{if } s' = 1, \end{cases}$$

which contradicts (5).

Similarly, we can also obtain a contradiction if $u_y \in I(u_x)$. Thus, we state the above result as the following claim.

Claim 1 For any vertex in $U_X \cup U_Y$ its only neighbor in another part is not in V_{2d-2}.

We now assume $u_y \notin V_{2d-2}$ by Claim 1 and still assume $u_y \in O(u_x)$. Then $|I(u_y)| = d$. Let $k = |I_Y(u_y)|$ and s_u be the number of singular edges between $I_Y(u_y)$ and $O_X(u_x)$ in F. Clearly, $k \leq d-1$. It follows from Lemma 2 that

$$|E(I_Y(u_y), O_X(u_x))| = k|\,O_X(u_x)| \quad \text{and } s_u \leq \begin{cases} k & \text{for } n=2; \\ 1 & \text{for } n \geq 3. \end{cases} \tag{6}$$

Let $E' = E(Y,X) \setminus E(I_Y(u_y), O_X(u_x))$. It follows from (4) and (6) that

$$\begin{aligned} |F| &\geq 2|E(I_Y(u_y), O_X(u_x))| - s_u + |E'| \\ &\geq \begin{cases} 2k|\,O_X(u_x)| - k + |E'| & \text{for } n=2; \\ 2k|\,O_X(u_x)| - 1 + |E'| & \text{for } n \geq 3. \end{cases} \end{aligned} \tag{7}$$

When $k \geq 1$ we have $u_x \notin V_{2d-2}$, otherwise $I(u_y) = I(u_x) \subseteq X$, contradicting the assumption of $k\,(\geq 1)$. Thus, $|\,O(u_x)| = d$ and $|\,O_X(u_x)| = d-1$. It follows from (7) that

$$|F| \geq \begin{cases} k(2d-3) + |E'| & \text{for } n=2; \\ 2k(d-1) - 1 + |E'| & \text{for } n \geq 3. \end{cases} \tag{8}$$

If $|E'| \geq 1$ and $k \geq 2$, then from (8), we have

$$|F| \geq \begin{cases} k(2d-3) + 1 \geq 4d-5 & \text{for } n=2; \\ 2k(d-1) \geq 4d-4 & \text{for } n \geq 3, \end{cases}$$

which contradicts (5).

We now assume $|E'| = 0$. In this case, $I_Y(u_y)$ is a vertex-cut in $B(d,n)$, and thus we have

$$d-1 \geq k = |I_Y(u_y)| \geq \kappa(B(d,n)) = d-1,$$

which implies $k = d-1$. By (8), when $d \geq 4$ we have

$$|F| \geq \begin{cases} (d-1)(2d-3) > 4d-5 & \text{for } n=2; \\ 2(d-1)(d-1) - 1 > 4d-4 & \text{for } n \geq 3, \end{cases}$$

which contradicts (5).

When $d = 3$ we have $k = 2$. If $n = 2$, let $u_x = x_1x_2$ and $u_y = x_2x_3$, then $u_y \notin V_{2d-2}$ by Claim 1, which means $x_2 \neq x_3$. Moreover, $x_2 \neq x_1$ since $u_x \notin V_{2d-2}$. Thus, the vertex $u_z = x_2x_2 \in O(u_x) \cap I(u_y) \cap X$ and $u_z \neq u_x$. It follows that

$$2 = d-1 = k = |I_Y(u_y)| = d - |I_X(u_y)| = d-2 = 1,$$

a contradiction. We now assume $n \geq 3$. By (6), $I_Y(u_y)$ contributes four arcs to $E(Y,X)$, at most one of which is a singular edge. By the above discussion, u_y and $I_Y(u_y)$ contribute five edges to F, and denote the set of the five edges by E_1. To obtain a contradiction, we only need to prove $|F| \geq 8$, so we only need to show that there are at least three other edges between X and Y besides the edges in E_1. Since $|F_Y - \{u_y\} - I_Y(u_y)| = |F_Y| - 1 - k \geq 2d - 2 - 1 - k = 1$, there is a vertex different from u_y and $I_Y(u_y)$ in F_Y. Without loss of generality, we assume the vertex is v_y and that $v_x \in F_X$ is one of its in-neighbors since $|E'| = 0$. Clearly $v_x \neq u_x$. If $v_x \in V_4$, then $v_y \notin V_4$. And $v_x \notin O_X(u_x)$, otherwise $v_x \in I_X(u_y)$ contradicting $|I_Y(u_y)| = k = d - 1$. Noting that $|E'| = 0$, $I(v_x) \subseteq X$. And $I(v_y) = I_X(v_x) \cup \{v_x\} \subseteq X$. So v_y contributes $|I(v_y)| = d = 3$ other edges to $E[X, Y]$, different from the five edges in E_1. If $v_x \notin V_4$, $|O(v_x)| = d = 3$. Since u_x has exactly one neighbor in Y, $(O(v_x) \setminus \{v_y\}) \cap O(u_x) = \emptyset$. Similarly, we have $(I(v_y) \setminus \{v_x\}) \cap I(u_y) = \emptyset$. If $O(v_x) \subseteq F_Y$, v_x contributes $d = 3$ other edges to F. If $O_X(v_x) \neq \emptyset$, then $I_Y(v_y) = \emptyset$ since $|E'| = 0$. Next we consider two cases according as the vertex v_y is in V_4 or not. When $v_y \notin V_4$, $|I_X(v_y)| = |I(v_y)| = d$. So, we get $d = 3$ other edges between X and Y. When $v_y \in V_4$, $O_X(v_y) = O_X(v_x) \neq \emptyset$, contradicting the assumption of E'. We always obtain a contradiction when $k \geq 2$, and so $k \leq 1$.

Similarly, we can also show that $|O_Y(u_y)| \leq 1$ if $u_y \in I(u_x)$. Thus, we state the above result as the following claim.

Claim 2 For any vertex u in $U_X \cup U_Y$ its only neighbor v in another part has at most one in-neighbor if $v \in O(u)$ or at most one out-neighbor if $v \in I(u)$ in the part which contains the vertex v.

We still consider $u_y \in O(u_x)$. By Claim 2, $k = |I_Y(u_y)| \leq 1$. In this case, $|I_X(u_y)| \geq d - 1 \geq 2$, so $u_y \notin U_Y$. Since $U_Y \neq \emptyset$, choose a vertex $v_y \in U_Y$ and assume v_x is its unique neighbor in F_X. Clearly, $v_x \neq u_x$ since $u_x \in U_X$, and $v_x \notin V_{2d-2}$ by Claim 1.

Assume $k = 1$. We first assume $v_x \in I(v_y)$. Since $O(z) = O(u_x)$ for any vertex $z \in I(u_y)$, $v_x \notin I(u_y)$ and $O_X(v_x) \cap O_X(u_x) = \emptyset$. Let $k' = |O_X(v_x)|$. Then $k' \leq 1$ by Claim 2. If $k' = 1$, then $v_y \notin V_{2d-2}$. So, $|E(I_Y(v_y), O_X(v_x))| = k'|I_Y(v_y)| = d - 1$. Thus, we have

$$
\begin{aligned}
|F| &\geq |E(I_X(u_y), \{u_y\})| + |E(I_Y(u_y), O_X(u_x))| \\
&\quad + |E(\{v_x\}, O_Y(v_x))| + |E(I_Y(v_y), O_X(v_x))| \\
&= (d-1) + (d-1) + (d-1) + (d-1) \\
&= 4d - 4,
\end{aligned}
$$

which contradicts (5). If $k' = 0$, then v_x contributes d directed edges to $E(X, Y)$. Since $|I_X(u_y)| = d - 1$, u_y contributes $d - 1$ directed edges to $E(X, Y)$. Denote $E' = E(I_X(u_y) \cup \{v_x\}, Y)$ and suppose s' is the number of singular edges in F induced by the edges in E'. Then $|E'| = (d-1) + d =$

$2d-1$ and $s' \le 2$. Thus, it follows from (4) that

$$|F| \ge 2|E'| - s' \ge 2(2d-1) - 2 = 4d - 4,$$

which contradicts (5).

We now assume $v_x \in O(v_y)$. Similarly, $v_x \notin O_X(u_x)$ and $I_X(v_x) \cap I(u_y) = \emptyset$. We can derive a contradiction in the same way for $k' = |I_X(v_x)| \ge 1$. When $k' = 0$, then $I(v_x) \subseteq Y$ and v_x contributes d directed edges to $E(Y, X)$. By (6), the only vertex in $I_Y(u_y)$ contributes $d-1$ directed edges to $E(Y, X)$. Denote $E' = E(I_Y(u_y) \cup I(v_x), X)$ and suppose s' is the number of singular edges in F induced by the edges in E'. Then $|E'| = (d-1) + d = 2d - 1$ and $s' \le 2$. Thus, it follows from (4) that

$$|F| \ge 2|E'| - s' \ge 2(2d-1) - 2 = 4d - 4,$$

which contradicts (5). So, $|I_Y(u_y)| = 0$.

Similarly, we can also show that $|O_Y(u_y)| = 0$ if $u_y \in I(u_x)$. Thus, we obtain the following claim.

Claim 3 For any vertex u in $U_X \cup U_Y$ its only neighbor v in another part has no in-neighbors if $v \in O(u)$ or no out-neighbors if $v \in I(u)$ in the part which contains the vertex v.

We still first assume $u_y \in O(u_x)$. By Claim 1 and Claim 3, $u_y \notin V_{2d-2}$ and $k = |I_Y(u_y)| = 0$. In this case, u_y contributes d directed edges to $E(X, Y)$. Similarly, there is a vertex $v_y \in U_Y$ and assume v_x is its unique neighbor in F_X. Clearly, $v_x \ne u_x$ since $u_x \in U_X$, and $v_x \notin V_{2d-2}$ by Claim 1.

If $v_x \in I(v_y)$, then $v_x \notin I(u_y)$. By Claim 3, $O(v_x) \subseteq F_Y$, which means v_x contributes d directed edges to $E(X, Y)$. Denote $E' = E(I(u_y) \cup \{v_x\}, Y)$ and suppose s' is the number of singular edges in F induced by the edges in E'. So, $|E'| = 2d$ and $s' \le 2$. It follows from (4) that

$$|F| \ge 2|E'| - s' \ge 2(2d) - 2 = 4d - 2 > 4d - 4,$$

which contradicts (5).

If $v_x \in O(v_y)$, then $I_X(v_x) \cap I(u_y) = \emptyset$. By Claim 3, $I(v_x) \subseteq F_Y$. Let $T_1 = Y_F - \{u_y\} - I(v_x)$ and T_2 be the set of vertices in $I(v_x) \setminus \{v_y\}$ which are adjacent to or from some vertex in $X_F \setminus \{u_x, v_x\}$. Then $T = \{u_y, v_x\} \cup T_1 \cup T_2$ is a vertex-cut of $UB(d, n)$ since there is no path between u_x and v_y in $UB(d, n) - T$. By Lemma 3, $|T| = 2 + |T_1| + |T_2| \ge \kappa(UB(d, n)) = 2d - 2$, which implies that $|T_1| + |T_2| \ge 2d - 4$. Now, u_y and v_x contribute $2d$ edges to F, and every vertex in $T_1 \cup T_2$ contributes at least one other edge to F. Thus we have

$$|F| \ge 2d + |T_1| + |T_2| \ge 2d + (2d-4) = 4d - 4,$$

which contradicts (5).

Since all possible cases can deduce a contradiction to (5), the lemma follows. ∎

Lemma 7([10]) For de Bruijn undirected graphs $UB(d,n)$, $\lambda'(UB(2,n)) = 4$ for $n \geq 4$, $\lambda'(UB(2,3)) = 3$, and $2d-2 < \lambda'(UB(d,n)) \leq 4d-4$ for $d \geq 3$.

3 Proof of Theorem

In this section, we give proofs of the theorem and the corollary stated in Introduction.

Proof of Theorem Note that $UB(d,1)$ is a complete graph of order d. It is easy to see that $\lambda'(UB(d,1))$ does not exist for $d = 2,3$ and that $\lambda'(UB(d,1)) = 2d-4$ for $d \geq 4$.

Clearly, $\lambda'(UB(2,2)) = 3$. By Lemma 7, we only need to show that for $d \geq 3$ and $n \geq 2$,

$$\lambda'(UB(d,n)) = \begin{cases} 4d-5 & \text{for } n = 2; \\ 4d-4 & \text{for } n \geq 3. \end{cases}$$

By (2) and Lemma 3 (b), we have

$$\lambda'(UB(d,n)) \leq \xi(UB(d,n)) = \begin{cases} 4d-5 & \text{for } n = 2; \\ 4d-4 & \text{for } n \geq 3. \end{cases}$$

To complete the proof, we only need to prove

$$\lambda'(UB(d,n)) \geq \begin{cases} 4d-5 & \text{for } n = 2; \\ 4d-4 & \text{for } n \geq 3. \end{cases} \tag{9}$$

To the end, let F be a λ'-cut of $UB(d,n)$ and $\{X,Y\}$ a partition of $V(UB(d,n))$ such that $F = E[X,Y]$. Let X_F and Y_F be the sets of end-vertices of F in X and Y, respectively.

If $|X| \leq 2d-2$ or $|Y| \leq 2d-2$, then (9) follows by Lemma 5. Assume both $|X| \geq 2d-1$ and $|Y| \geq 2d-1$ below.

We claim both $|X_F| \geq 2d-2$ and $|Y_F| \geq 2d-2$. In fact, if $|X_F| < 2d-2$, then $X \setminus X_F \neq \emptyset$ and disconnects to Y in $G - X_F$, which implies $\kappa(G) < 2d-2$, contradicting Lemma 3 (d). Similarly, we have $|Y_F| \geq 2d-2$. It follows from Lemma 6 that (9) holds.

The proof of the theorem is complete. ∎

Proof of Corollary It is a simple observation from Theorem and Lemma 3 (b) that $\lambda'(UB(d,n)) = \xi(UB(d,n))$ except $UB(2,1)$, $UB(3,1)$ and $UB(2,3)$ and, hence, $UB(d,n)$ is λ'-optimal.

Since $UB(d,1)$ is a complete graph of order d, $\lambda'(UB(d,1)) = \infty > d-1 = \lambda(UB(d,1))$ for $2 \leq d \leq 3$ and $\lambda'(UB(d,1)) = 2d-4 > d-1 = \lambda(UB(d,1))$ for $d \geq 4$. Thus, $UB(d,1)$ is super edge-connected for $d \geq 2$.

For $n \geq 2$ and $d \geq 2$, by Lemma 3, $\lambda(UB(d,n)) = \delta(UB(d,n)) = 2d-2$ and, by Theorem, $\lambda'(UB(d,n)) > 2d-2 = \lambda(UB(d,n))$. Thus, $UB(d,n)$ is super edge-connected for $n \geq 2$ and $d \geq 2$. ∎

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