A New Upper Bound On Forwarding Index of Graphs*

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Abstract

To measure the efficiency of a routing in network, Chung *et al* [The forwarding index of communication networks. IEEE Trans. Inform. Theory, 33 (2) (1987), 224-232] proposed the notion of forwarding index and established an upper bound $(n-1)(n-2)$ of this parameter for a connected graph of order n. This note improves this bound as $(n-1)(n-2) - \left(2n-2-\Delta\left\lfloor 1+\frac{n-1}{\Delta}\right\rfloor\right)\left\lfloor\frac{n-1}{\Delta}\right\rfloor$, where Δ is the maximum degree of the graph G. This bound is best possible in the sense that there is a graph G attaining it.

Keywords Forwarding index, Routing, Path, Network

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1 Introduction

Throughout this paper, a graph $G=(V,E)$ always means a simple connected graph (without loops and multiple edges) of order n. The symbols $K_{1,\,n-1}$ denotes a star graph. A routing ρ in G is a set of $n(n-1)$ fixed paths for all ordered pairs (x,y) of vertices of G. The path $\rho(x,y)$ specified by ρ carries the data transmitted from the source x to the destination y. It is possible that the fixed paths specified by a given routing ρ passing through some vertex are too many, which means that the routing

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ρ loads the vertex too much. The load of any vertex is limited by the capacity of the vertex, for otherwise it would affect efficiency of transmission, even result in malfunction of the network. In order to measure the load of a vertex, Chung, Coffman, Reiman and Simon [1] proposed the notion of forwarding index.

Let G be a graph with a given routing ρ, x a vertex of G. The load of x with respect to ρ, denoted by $\xi(G,\rho,x)$, is defined as the number of paths specified by ρ admitting x as an inner vertex. The forwarding index of G with respect to ρ is defined as

$$\xi(G,\rho) = \max\{\xi(G,\rho,x) :\ x \in V(G)\}.$$

The forwarding index of G is defined as

$$\xi(G) = \min\{\xi(G,\rho) :\ \rho \text{ is a routing of } G\}.$$

It has been explained in [1] that maximizing network capacity reduces to minimizing forwarding index of the network. Thus, it is very desirable to determine the forwarding index for a given graph. Many authors showed interest in this subject, see, for example, [1, 2, 3, 4]. However, Saad [4] proved that for any graph determining the forwarding index problem is NP-complete. So it is of interest to determine the tight bound of forwarding index in terms of some basic graph parameters. Chung *et al* [1] established a tight upper bound, that is, $\xi(G) \leq (n-1)(n-2)$ for any connected graph G of order n.

In this note we give an improve bound by proving

$$\xi(G) \leq (n-1)(n-2) - \left(2n-2-\Delta\left\lfloor 1+\frac{n-1}{\Delta}\right\rfloor\right)\left\lfloor\frac{n-1}{\Delta}\right\rfloor$$

for any graph G of order n and maximum degree Δ. This bound is best possible in the sense that there is a graph G attaining the upper bound.

2 Main Results

Before giving the proof of the main result, we first prove several lemmas.

Lemma 1 If the sum of Δ positive integers is n, then the minimum sum of their squares is $\Delta q^2 + 2rq + r$, where $q = \lfloor \frac{n}{\Delta} \rfloor$ and $r = n - \Delta q$.

The proof is a routine exercise and omitted here.

Lemma 2 For any positive integers n and Δ with $n > \Delta$, let

$$B(n,\Delta) = (n-1)(n-2) - \left(2n-2-\Delta\left\lfloor 1+\frac{n-1}{\Delta}\right\rfloor\right)\left\lfloor\frac{n-1}{\Delta}\right\rfloor.$$

If $d \leq \Delta$, then $B(n,d) \leq B(n,\Delta)$.

Proof Let $q = \lfloor \frac{n-1}{\Delta} \rfloor$, $r = n-1-\Delta q$, and let $q' = \lfloor \frac{n-1}{d} \rfloor$, $r' = n-1-dq'$. Then

$$B(n,\Delta) = (n-1)^2 - \Delta q^2 - 2rq - r,$$
$$B(n,d) = (n-1)^2 - dq'^2 - 2r'q' - r'.$$

Thus, we have

$$\begin{aligned} B(n,\Delta) - B(n,d) = & \quad (dq'^2 - \Delta q^2) + 2r'q' + r' - 2rq - r \\ = & \quad (n-1)(q'-q) + r'q' + r' - rq - r. \end{aligned} \tag{1}$$

Since $d \leq \Delta$, we have $q \leq q'$. If $q' = q$, then $r \leq r'$, and from (1) we have

$$B(n,\Delta) - B(n,d) = (q+1)(r'-r) \geq 0.$$

If $q' > q$, then from (1), we have

$$\begin{aligned} B(n,\Delta) - B(n,d) = & \quad (n-1)(q'-q) + r'q' + r' - rq - r \\ \geq & \quad (n-1) - (rq + r) \\ > & \quad (n-1) - (\Delta q + r) = 0, \end{aligned}$$

the latter inequality holds since $\Delta > r \geq 0$. The lemma follows. ∎

Lemma 3 Let T be a connected spanning subgraph of a connected graph G. Then $\xi(G) \leq \xi(T)$.

Proof Let T be a connected spanning subgraph of G and ρ a routing in T such that $\xi(T) = \xi(T,\rho)$. Define a routing ρ' in G as follows: For any two distinct x and y of G, choose $\rho'(x,y) = xy$ if $xy \in E(G) \setminus E(T)$, and $\rho'(x,y) = \rho(x,y)$ otherwise. By definition, we have $\xi(G) \leq \xi(G,\rho') = \xi(T,\rho) = \xi(T)$. ∎

Theorem 1 $\xi(G) \leq B(n,\Delta)$ for any connected graph G of order n and maximum degree Δ. This bound is best possible in the sense that there is a graph G such that $\xi(G) = B(n,\Delta)$.

Proof Let G be a connected graph of order n and maximum degree Δ. Then $n > \Delta$. If $\Delta = 1$, then $G = K_2$ and $\xi(K_2) = 0 = B(2,1)$. Suppose $\Delta \geq 2$ below. We first show $\xi(G) \leq B(n,\Delta)$. Let T be a spanning tree of G. Then $\xi(G) \leq \xi(T)$ by Lemma 3, and there exists the unique routing ρ in T such that $\xi(T) = \xi(T,\rho)$. Let x be a vertex x in T such that $\xi(T) = \xi_x(T,\rho)$, and let the vertex-degree of x in T be d. Then $d \geq 2$ and $T - x$ contains exactly d components. Let T_i be a component of order n_i in $T - x$ for $i = 1, 2, \cdots, d$. Since any path connecting two vertices in different components must pass through x, by definition, we have

$$\xi_x(T,\rho) = \sum_{i \neq j} n_i n_j = \sum_{i=1}^{d} n_i(n-1-n_i) = (n-1)^2 - \sum_{i=1}^{d} n_i^2.$$

Since $n_1 + n_2 + \cdots + n_d = n - 1$, from Lemma 1, Lemma 2 and Lemma 3, we have

$$\xi(G) \leq \xi(T) = \xi_x(T, \rho) \leq B(n, d) \leq B(n, \Delta).$$

For given integers n and Δ with $n > \Delta$, let $q = \lfloor \frac{n}{\Delta} \rfloor$ and $r = n - \Delta q$. We now construct a connected graph G with order n and maximum degree Δ as follows: Let G_i be a connected graph with a vertex x_i of degree at most $\Delta - 1$ and others at most Δ, and G_i has order q for $i = 1, 2, \cdots, \Delta - r$, and order $q + 1$ for $i = \Delta - r + 1, \Delta - r + 2, \cdots, \Delta$. The graph G is constructed from $G_1 \cup G_2 \cup \cdots \cup G_\Delta$ by adding a new vertex x and Δ new edges connecting x and x_i for $i = 1, 2, \cdots, \Delta$. It is easy to see that for any routing ρ in G and for any two vertices u in G_i and v in G_j, the route $\rho(u, v)$ must pass through the vertex x if $i \neq j$, and do not pass through x if $i = j$. It follows that

$$\begin{aligned} \xi_x(G, \rho) = & \quad (n-1)(n-2) - (\Delta - r)q(q-1) - rq(q+1) \\ = & \quad (n-1)^2 - \Delta q^2 - 2rq - r = B(n, \Delta). \end{aligned}$$

Noting $\xi(G_i) \leq (q-1)(q-2)$ for $i = 1, 2, \cdots, \Delta - r$ and $\xi(G_i) \leq q(q-1)$ for $i = \Delta - r + 1, \Delta - r + 2, \cdots, \Delta$, all of which are less than q^2 $(\leq \xi_x(G, \rho))$ as $\Delta \geq 2$, we have $\xi(G) = \xi_x(G, \rho) = B(n, \Delta)$.

The theorem follows. ∎

Considering a special case of $\Delta = n - 1$ in Theorem 1, we obtain Chung *et al*'s bound [1] immediately.

Corollary 1 For any connected graph G of order n, $\xi(G) \leq (n-1)(n-2)$, and the star $K_{1,n-1}$ attains this bound. ∎

3 Other Results

Chung *et al* [1] proposed the following problem: Given Δ and n, determine $\xi_{\Delta,n}$, the minimum of $\xi(G)$ taken over all graphs of order n with maximum degree at most Δ. This problem was solved asymptotically in [1], determined $\xi_{\Delta,n}$ for $n \leq 15$ and $(n+4)/3 \leq \Delta \leq n-1$ in [2].

In this section, we consider the following problem: Given δ and n, and $\xi_{\delta,n}$ was determined, the minimum of $\xi(G)$ taken over all graphs of order n with minimum degree δ.

Theorem 2 $\xi_{\delta,n} = \left\lceil \frac{2(n-1-\delta)}{\delta} \right\rceil$ for any n and δ with $n > \delta \geq 1$.

Proof Assume that G is a connected graph with order n and minimum degree δ, and x is a vertex in G of degree δ. Let N be the set of neighbors of x in G and N^c be the set of vertices not adjacent to x in G. Then $|N| = \delta$ and $|N^c| = n - 1 - \delta$. For any routing ρ of G, any path between x

and any vertex in N^c specified by ρ must pass through a vertex in N. The total number of such paths is $2(n-1-\delta)$ and the maximum number of the passing through a vertex y in N can not be less than the average number. Thus, we have

$$\xi(G) \geq \xi_y(G,\rho) \geq \left\lceil \frac{2(n-1-\delta)}{\delta} \right\rceil.$$

Because of the arbitrary choice of G, we have $\xi_{\delta,n} \geq \left\lceil \frac{2(n-1-\delta)}{\delta} \right\rceil$.

For any integers n and δ with $n > \delta$, we construct a graph H obtained from a complete graph K_n by deleting $(n-1-\delta)$ edges at one vertex. It is easily to verify that the graph H satisfies $\xi(H) = \left\lceil \frac{2(n-1-\delta)}{\delta} \right\rceil$. ∎

For the edge-forwarding index, $\pi(G)$, of a connected graph G, proposed by Heydemann, Meyer, Sotteau [3], we can similarly define the parameter $\pi_{\delta,n}$ and prove $\pi_{\delta,n} = \left\lceil \frac{2(n-1)}{\delta} \right\rceil$, and the details are omitted here.

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