Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 189 (2007) 1393–1401
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Edge-pancyclicity and Hamiltonian laceability of the balanced hypercubes ☆

Min Xu [a,*], Xiao-Dong Hu [a], Jun-Ming Xu [b]

[a] *Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100080, China*
[b] *Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China*

Abstract

The balanced hypercube BH_n is a variant of the hypercube Q_n. Huang and Wu proved that BH_n has better properties than Q_n with the same number of links and processors. In particularly, they showed that there exists a cycle of length 2^l in BH_n for all l, $2 \leqslant l \leqslant 2n$. In this paper, we improve this result by showing that BH_n is edge-pancyclic, which means that for arbitrary edge e, there exists a cycle of even length from 4 to 2^{2n} containing e in BH_n. We also show that the balanced hypercubes are Hamiltonian laceable.

Keywords: Cycles; Balanced hypercubes; Hypercubes; Edge-pancyclicity; Hamiltonian laceability

1. Introduction

The hypercube network has proved to be one of the most popular interconnection networks. The balanced hypercube, proposed by Huang and Wu [9], forms a hypercube variant. Like hypercubes, balanced hypercubes are bipartite graph and vertex-transitive [9,14]. The balanced hypercubes are superior to the hypercube in having smaller diameter of the hypercube, supporting an efficient reconfiguration without changing the adjacent relationship among tasks [14]. The variously desired properties of balanced hypercubes have been extensively investigated in the literature [9,14].

In interconnection networks, the problem of simulating one network by another is modelled as a graph embedding problem. There are several reasons why such an embedding is important [15]. For example, there are a number of efficient algorithms for solving some applications problems and best communication patterns for their executions. For these algorithms, the existence of certain topological structures guarantee the desired performance. Thus, for such applications, it is desired to provide logically a specific topological structure

☆ The work was supported in part by NNSF of China No. 10626053, No. 70221001, No. 10531070, No. 10671191 and No. 10301031. The work was supported in part by China Postdoctoral Science Foundation.
* Corresponding author.
E-mail address: xum@amss.ac.cn (M. Xu).

0096-3003/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2006.12.036

throughout the execution of the algorithm in the network design. On the other hand, some algorithms may be developed for another architecture for which it fits perfectly. In such a scenario one might wish to implement the same algorithm with little additional programming effort. For this purpose, the original architecture should be embedded into the new network.

In all the embedding problems, the cycle embedding problem is one of the most popular problems. It is to find a cycle of given length in graph. A graph G is called *pancyclic* [2] if there exists a cycle of every length from 3 to $|V(G)|$. A graph is *bipartite graph* if its vertex-set can be partitioned into two disjoint subsets such that each edge is incident to two vertices from different subsets. A bipartite graph G is called *bipancyclic* if there exists a cycle of every length from 4 to $|V(G)|$. The pancyclicity is an important metric in embedding cycles of any length into the topology of network. Large amount of related work appeared in the literature [4–7,10,12]. The concept of pancyclicity was extended to vertex-pancyclicity by Hobbs [8] and edge-pancyclicity by Alspach and Hare [1]. A graph G is called *vertex-pancyclic* if for any vertex u, there exists a cycle of every length from 3 to $|V(G)|$ containing u; and *edge-pancyclic* if for any edge e, there exists a cycle of every length from 3 to $|V(G)|$ containing e. Obviously, every edge-pancyclic graph is vertex-pancyclic. A bipartite graph G is *vertex-bipancyclic* if for any vertex u, there exists a cycle of every even length from 4 to $|V(G)|$ containing u. Similarly, a bipartite graph G is called *edge-bipancyclic* if for any edge e, there exists a cycle of every even length from 4 to $|V(G)|$ containing e. Wu and Huang [14] proved that the n-dimensional balanced hypercube BH_n is bipartite graph and there exists a cycle of length 2^l in BH_n for all l, $2 \leqslant l \leqslant 2n$. In this paper, we improve this result by showing that BH_n is edge-bipancyclic.

We also study the Hamiltonian laceability of balanced hypercubes. A path is called a *Hamilton path* if it contains all vertex of G. A graph G is said to be *Hamiltonian connected* if there exists a Hamiltonian path between any two vertices of G. It is easy to see that any bipartite graph with at least three vertices is not Hamiltonian connected. For this reason, Simmons [13] introduced the concept of Hamiltonian laceable for Hamiltonian bipartite graphs. A Hamiltonian bipartite graph is *Hamiltonian laceable* if there is Hamiltonian path between any two vertices in different bipartite sets. Obviously, the Hamilton cycle can be embedded in the Hamiltonian connected graphs. Then the Hamiltonian connectivity is also important metric in embedding Hamitonian cycle into the topology of network. There are many desirable results about the Hamiltonian connectivity of some interconnection networks [5,11]. In this paper, we prove that the balanced hypercubes are Hamiltonian laceable.

The rest of this paper is organized as follows. In Section 2, we give the definition and basic properties of the n-dimensional balanced hypercube BH_n. In Sections 3 and 4, we discuss the edge-bipancyclicity and Hamiltonian laceability of BH_n, respectively. In Section 5, we conclude the paper.

2. Balanced hypercubes

The architecture of an interconnection network is usually represented by a connected simple graph $G = (V, E)$, where the vertex-set V is the set of processors and the edge-set E is the set of communication links in the network. The edge joining two vertices x and y is denoted by (x, y). We will follow graph-theoretical terminologies and notations used in [3].

An n-dimensional *balanced hypercube* [14], denoted by BH_n, has 2^{2n} vertices, each of them has a unique n-component vector on $\{0, 1, 2, 3\}$ for an address, which is also called an n-bit string. A vertex $(a_0, a_1, \ldots, a_{n-1})$ is adjacent to the following $2n$ vertices, where $1 \leqslant i \leqslant n-1$,

$$((a_0+1) \bmod 4, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \tag{1}$$
$$((a_0-1) \bmod 4, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), \text{ and}$$
$$((a_0+1) \bmod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) \bmod 4, a_{i+1}, \ldots, a_{n-1}), \tag{2}$$
$$((a_0-1) \bmod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) \bmod 4, a_{i+1}, \ldots, a_{n-1}).$$

Fig. 1 demonstrates two balanced hypercubes of dimension one and two. Clearly, the balanced hypercube BH_n can be constructed from four copies of BH_{n-1} by adding a new dimension as the nth index of every vertex in BH_n.

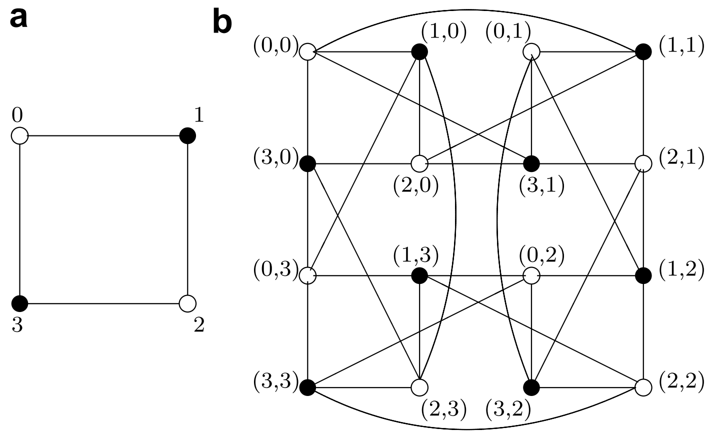

Fig. 1. Two balanced hypercubes: (a) BH_1, and (b) BH_2.

We use BH^i_{n-1} to denote the subgraph of BH_n which induced by the vertex-set $\{(a_0, a_1, \ldots, a_{n-2}, i) | a_j \in \{0, 1, 2, 3\}, 0 \leqslant j \leqslant n-2\}$ for $0 \leqslant i \leqslant 3$. Obviously, $BH^i_{n-1} \cong BH_{n-1}$ for $0 \leqslant i \leqslant 3$.

Wu and Huang [14] gave an equivalent definition of BH_n which can be recursively constructed as follows:

(1) BH_1 is a cycle with vertex-set $\{0, 1, 2, 3\}$.
(2) BH_{k+1} is constructed from four copies of BH_k: BH^0_k, BH^1_k, BH^2_k and BH^3_k. Every vertex $(a_0, a_1, \ldots, a_{k-1}, i)$ in BH^i_k $(0 \leqslant i \leqslant 3)$ has two extra adjacent vertices:
  (2.1) BH^{i+1}_k: $(a_0 + 1, a_1, \ldots, a_{k-1}, i+1)$ and $(a_0 - 1, a_1, \ldots, a_{k-1}, i+1)$ if a_0 is even.
  (2.2) BH^{i-1}_k: $(a_0 + 1, a_1, \ldots, a_{k-1}, i-1)$ and $(a_0 - 1, a_1, \ldots, a_{k-1}, i-1)$ if a_0 is odd.

Obviously, BH_n has 2^{2n} vertices, each of which has $2n$ adjacent vertices. Since BH_n is a bipartite graph, the vertex-set $V_1 = \{a = (a_0, a_1, \ldots, a_{n-1}) | a \in V(BH_n) \text{ and } a_0 \text{ is odd}\}$ and $V_2 = \{a = (a_0, a_1, \ldots, a_{n-1}) | a \in V(BH_n) \text{ and } a_0 \text{ is even}\}$ make the desired partition. In Fig. 1 (and other figures too), we use black nodes to denote the vertices in V_1 and white nodes to denote the vertices in V_2.

3. Edge-bipancyclicity of balanced hypercubes

In this section, we consider the edge-bipancyclicity of balanced hypercubes. We will prove a stronger result than that obtained by Wu and Huang [14]. For this purpose, we need the following lemma.

Lemma 1. *Let (u, v) be an edge of BH_n. Then (u, v) is contained in a cycle C of length 8 in BH_n such that $|E(C) \cap E(BH^i_{n-1})| = 1$ where $i = 0, 1, 2, 3$.*

Proof. Since BH_n is transitive, without loss of generality, we assume $u = (0, 0, \ldots, 0)$. We consider three different cases with respect to edge (u, v) in the following.

Case 1. $v = (v_0, 0, \ldots, 0)$ where $v_0 = \pm 1$. In this case it can be verified that

$$\begin{aligned}(v,u)\quad &+((0,0,\ldots,0,0),(v_0,0,\ldots,0,1))+((v_0,0,\ldots,0,1),(0,0,\ldots,0,1))\\ &+((0,0,\ldots,0,1),(v_0,0,\ldots,0,2))+((v_0,0,\ldots,0,2),(0,0,\ldots,0,2))\\ &+((0,0,\ldots,0,2),(v_0,0,\ldots,0,3))+((v_0,0,\ldots,0,3),(0,0,\ldots,0,3))\\ &+((0,0,\ldots,0,3),(v_0,0,\ldots,0,0)(=v))\end{aligned}$$

makes the desired cycle.

Case 2. $v = \left(v_0, \overbrace{0, \ldots, 0}^{i-1}, 1, \overbrace{0, \ldots, 0}^{n-i-1}\right)$ where $v_0 = \pm 1$ and $1 \leqslant i \leqslant n-2$. In this case it can be verified that

$$
\begin{aligned}
&(v,u)\\
&+((0,0,\ldots,0,0),(v_0,0,\ldots,0,1))\\
&+\left((v_0,0,\ldots,0,1),\left(0,\overbrace{0,\ldots,0}^{i-1},3,\overbrace{0,\ldots,0}^{n-i-2},1\right)\right)\\
&+((0,0,\ldots,0,3,0,\ldots,0,1),(v_0,0,\ldots,0,3,0,\ldots,0,2))\\
&+((v_0,0,\ldots,0,3,0,\ldots,0,2),(0,0,\ldots,0,2,0,\ldots,0,2))\\
&+((0,0,\ldots,0,2,0,\ldots,0,2),(v_0,0,\ldots,0,2,0,\ldots,0,3))\\
&+((v_0,0,\ldots,0,2,0,\ldots,0,3),(0,0,\ldots,0,1,0,\ldots,0,3))\\
&+((0,0,\ldots,0,1,0,\ldots,0,3),(v_0,0,\ldots,0,1,0,\ldots,0,0)(=v))
\end{aligned}
$$

makes the desired cycle.

Case 3. $v = \left(v_0,\overbrace{0,\ldots,0}^{n-2},1\right)$ where $v_0 = \pm 1$. In this case, it can be verified that

$$
\begin{aligned}
(u,v)\quad &+((v_0,0,\ldots,0,1),(0,0,\ldots,0,1)) + ((0,0,\ldots,0,1),(v_0,0,\ldots,0,2))\\
&+((v_0,0,\ldots,0,2),(0,0,\ldots,0,2)) + ((0,0,\ldots,0,2),(v_0,0,\ldots,0,3))\\
&+((v_0,0,\ldots,0,3),(0,0,\ldots,0,3)) + ((0,0,\ldots,0,3),(v_0,0,\ldots,0,0))\\
&+((v_0,0,\ldots,0,0),(0,0,\ldots,0,0)(=u))
\end{aligned}
$$

makes the desired cycle. The Lemma follows.

For simplicity of the presentation, we use $\langle v^0,u^0,v^1,u^1,v^2,u^2,v^3,u^3,v^0\rangle$ to denote the cycles constructed in Lemma 1, where $(u^0,v^0)\in E(BH^0_{n-1})$, $(u^1,v^1)\in E(BH^1_{n-1})$, $(u^2,v^2)\in E(BH^2_{n-1})$ and $(u^3,v^3)\in E(BH^3_{n-1})$. See Fig. 2. We are now ready to present one of the main results of this paper. $\square$

Theorem 1. *The balanced hypercube BH_n is edge-bipancyclic.*

Proof. We prove the theorem by mathematical induction on $n \geqslant 1$. The theorem is true for $n = 1$. For $n = 2$, since BH_2 is vertex-transitive, we only need to prove that each edge incident to $(0,0)$ is on a cycle of length $2i$ for $i = 2,3,\ldots,8$.

It can be verified that the union of the following two cycles of length $2i$ covers all edges adjacent to $(0,0)$ for $i = 2,3,\ldots,8$.

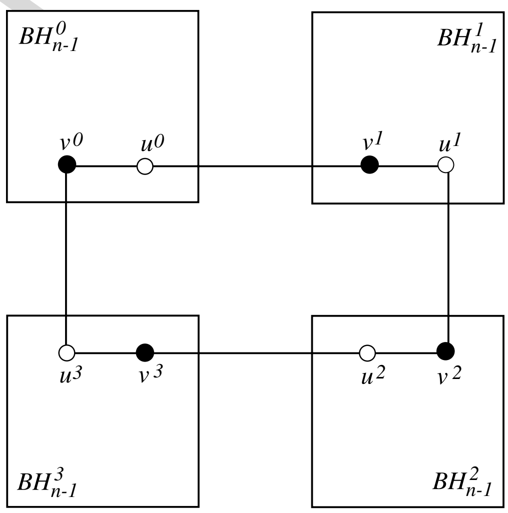

Fig. 2. The cycle $\langle v^0,u^0,v^1,u^1,v^2,u^2,v^3,u^3,v^0\rangle$.

$$\text{two cycles of length 4}\begin{cases}\langle(0,0),(1,0),(2,0),(3,0),(0,0)\rangle,\\ \langle(0,0),(3,1),(2,1),(1,1),(0,0)\rangle;\end{cases}$$

$$\text{two cycles of length 6}\begin{cases}\langle(0,0),(1,0),(2,0),(3,1),(0,1),(1,1),(0,0)\rangle,\\ \langle(0,0),(3,0),(2,0),(1,1),(2,1),(3,1),(0,0)\rangle;\end{cases}$$

$$\text{two cycles of length 8}\begin{cases}\langle(0,0),(1,0),(2,0),(3,1),(0,1),(3,2),(2,1),(1,1),(0,0)\rangle,\\ \langle(0,0),(3,0),(2,0),(1,1),(0,1),(1,2),(2,1),(3,1),(0,0)\rangle;\end{cases}$$

$$\text{two cycles of length 10}\begin{cases}\langle(0,0),(1,0),(2,0),(3,0),(0,3),(1,3),(0,2),(1,2),(2,1),(3,1),(0,0)\rangle,\\ \langle(0,0),(3,0),(0,3),(3,3),(2,2),(3,2),(0,2),(1,2),(2,1),(1,1),(0,0)\rangle;\end{cases}$$

$$\text{two cycles of length 12}\begin{cases}\langle(0,0),(1,0),(2,0),(3,0),(0,3),(1,3),(2,3),\\ (3,3),(2,2),(3,2),(0,1),(1,1),(0,0)\rangle,\\ \langle(0,0),(3,0),(2,0),(1,0),(2,3),(1,3),(0,3),\\ (3,3),(2,2),(3,2),(0,1),(3,1),(0,0)\rangle;\end{cases}$$

$$\text{two cycles of length 14}\begin{cases}\langle(0,0),(1,0),(2,0),(3,0),(0,3),(1,3),(2,3),(3,3),\\ (2,2),(3,2),(0,1),(3,1),(2,1),(1,1),(0,0)\rangle,\\ \langle(0,0),(3,0),(2,0),(1,0),(2,3),(1,3),(0,3),(3,3),\\ (2,2),(3,2),(0,1),(1,1),(2,1),(3,1),(0,0)\rangle;\end{cases}$$

$$\text{two cycles of length 16}\begin{cases}\langle(0,0),(1,0),(2,0),(3,0),(0,3),(1,3),(2,3),(3,3),(2,2),\\ (1,2),(0,2),(3,2),(0,1),(3,1),(2,1),(1,1),(0,0)\rangle,\\ \langle(0,0),(3,0),(2,0),(1,0),(2,3),(1,3),(0,3),(3,3),(2,2),\\ (1,2),(0,2),(3,2),(0,1),(1,1),(2,1),(3,1),(0,0)\rangle;\end{cases}$$

Thus the theorem is true for $n = 2$.

Assume now that the theorem is true for all $2 \leqslant k < n$. Let $e = ((0,0,\ldots,0),v)$ be any edge of BH_n incident to $(0,0,\ldots,0)$ and let ℓ be any even integer with $4 \leqslant \ell \leqslant 2^{2n}$, where $n \geqslant 3$. Since BH_n is vertex-transitive, to complete the proof of theorem, it suffices to show that there exists a cycle of length ℓ in BH_n containing e. We consider two cases separately in the following.

Case 1. The vertex v is in BH^0_{n-1}. If $4 \leqslant \ell \leqslant 2^{2(n-1)}$, by the induction hypothesis, there exists a cycle of length ℓ in $BH^0_{n-1} \subseteq BH_n$ that contains e. We consider two subcases separately in the following.

Case 1.1. $2^{2(n-1)} + 2 \leqslant \ell \leqslant 2^{2(n-1)} + 6$. For $n \geqslant 3$, we have $\ell - 6 \geqslant 4$. By the induction hypothesis, there exists a cycle C^0 of length $\ell - 6$ in BH^0_{n-1} containing e. Thus, we can choose an edge (u^0, v^0) in C^0 different from e where the first index of u is even. By Lemma 1, there exists a cycle $C' = \langle v^0, u^0, v^1, u^1, v^2, u^2, v^3, u^3, v^0\rangle$ of length 8 such that $(u^0, v^0) \in E(BH^0_{n-1}), (u^1, v^1) \in E(BH^1_{n-1})$, $(u^2, v^2) \in E(BH^2_{n-1})$ and $(u^3, v^3) \in E(BH^3_{n-1})$ (refer to Fig. 2). Let $P^0 = C^0 - (u^0, v^0)$ and $P' = C' - (u^0, v^0)$. Thus $P^0 + P'$ makes a cycle of length ℓ in BH_n containing e. Specially, let C to denote the cycle of length $2^{2(n-1)} + 6$ containing e.

Case 1.2. $2^{2(n-1)} + 8 \leqslant \ell \leqslant 2^{2n}$. Let $\ell' = \ell - 2^{2(n-1)}$. For $\ell' \geqslant 8$, we can write $\ell' = \ell_1 + \ell_2 + \ell_3$ where ℓ_1, ℓ_2, and ℓ_3 satisfy one of the following conditions:

$$\ell_1 = 2, \ell_2 = 2, \ell_3 \geqslant 4 \quad \text{or}$$
$$\ell_1 = 2, \ell_2 \geqslant 4, \ell_3 \geqslant 4 \quad \text{or}$$
$$\ell_1 \geqslant 4, \ell_2 \geqslant 4, \ell_3 \geqslant 4.$$

Consider the cycle C of length $2^{2(n-1)} + 6$ containing e. By the induction hypothesis, there exists a cycle C_1 of length ℓ_1 in BH^1_{n-1} containing (u^1, v^1) if $\ell_1 \geqslant 4$, there exists a cycle C_2 of length ℓ_2 in BH^2_{n-1} containing (u^2, v^2) if $\ell_2 \geqslant 4$, and there exists a cycle C_3 of length ℓ_3 in BH^3_{n-1} containing (u^3, v^3). Let

$$P_1 = \begin{cases} (u^1, v^1) & \text{if} \quad \ell_1 = 2, \\ C_1 - (u^1, v^1) & \text{if} \quad \ell_1 \geqslant 4; \end{cases}$$
$$P_2 = \begin{cases} (u^2, v^2) & \text{if} \quad \ell_2 = 2, \\ C_2 - (u^2, v^2) & \text{if} \quad \ell_2 \geqslant 4; \end{cases}$$
$$P_3 = C_3 - (u^3, v^3).$$

Then $P^0 + (u^0, v^1) + P_1 + (u^1, v^2) + P_2 + (u^2, v^3) + P_3 + (u^3, v^0)$ is a cycle of length ℓ in BH_n and contains edge e.

Case 2. The vertex v is in BH^1_{n-1}. In this case, note $v = (1, 0, \ldots, 0, 1)$ or $v = (3, 0, \ldots, 0, 1)$. We consider four cases separately in the following.

Case 2.1. $\ell = 4$. In this case, $\langle(0, 0, \ldots, 0, 0), (1, 0, \ldots, 0, 1), (2, 0, \ldots, 0, 1), (3, 0, \ldots, 0, 1), (0, 0, \ldots, 0, 0)\rangle$ is a cycle of length four in BH_n and contains the edges $((0, 0, \ldots, 0, 0),\ (1, 0, \ldots, 0, 1))$ and $((0, 0, \ldots, 0, 0),\ (3, 0, \ldots, 0, 1))$.

Case 2.2. $\ell = 6$. In this case, $\langle(0, 0, \ldots, 0, 0), (1, 0, \ldots, 0, 1), (0, 0, \ldots, 0, 1), (1, 0, \ldots, 0, 2), (2, 0, \ldots, 0, 1), (3, 0, \ldots, 0, 1), (0, 0, \ldots, 0, 0)\rangle$ is a cycle of length six in BH_n and contains the edges $((0, 0, \ldots, 0, 0),\ (1, 0, \ldots, 0, 1))$ and $((0, 0, \ldots, 0, 0),\ (3, 0, \ldots, 0, 1))$.

Case 2.3. $\ell = 8$. By Lemma 1, there exists desired cycle containing the edge $((0, 0, \ldots, 0, 0),\ (1, 0, \ldots, 0, 1))$ and $((0, 0, \ldots, 0, 0),\ (3, 0, \ldots, 0, 1))$, respectively. Specially, use $\langle v^0, u^0, v^1, u^1, v^2, u^2, v^3, u^3, v^0\rangle$ to denote the desired cycle containing (u^0, v^1) where $u^0 = u = (0, 0, \ldots, 0)$ and $v^1 = v = (1, 0, \ldots, 0, 1)$ or $v^1 = v = (3, 0, \ldots, 0, 1)$.

Case 2.4. $10 \leqslant \ell \leqslant 2^{2n}$. In this case, we can write $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3$ where ℓ_i satisfies one of the following conditions for $i = 0$, 1, 2, 3:

$$\ell_0 = 2, \ell_1 = 2, \ell_2 = 2, \ell_3 \geqslant 4 \quad \text{or}$$
$$\ell_0 = 2, \ell_1 = 2, \ell_2 \geqslant 4, \ell_3 \geqslant 4 \quad \text{or}$$
$$\ell_0 = 2, \ell_1 \geqslant 4, \ell_2 \geqslant 4, \ell_3 \geqslant 4 \quad \text{or}$$
$$\ell_0 \geqslant 4, \ell_1 \geqslant 4, \ell_2 \geqslant 4, \ell_3 \geqslant 4.$$

By the induction hypothesis, there exists a cycle C^0 of length ℓ_0 in BH^0_{n-1} that contains (v^0, u^0) if $\ell_0 \geqslant 4$, a cycle C_1 of length ℓ_1 in BH^1_{n-1} that contains (v^1, u^1) if $\ell_1 \geqslant 4$, a cycle C_2 of length ℓ_2 in BH^2_{n-1} that contains (v^2, u^2) if $\ell_2 \geqslant 4$, and a cycle C_3 of length ℓ_3 in BH^3_{n-1} that contains (v^3, u^3). Let

$$P_0 = \begin{cases} (v^0, u^0) & \text{if} \quad \ell_0 = 2, \\ C_0 - (v^0, u^0) & \text{if} \quad \ell_0 \geqslant 4; \end{cases}$$
$$P_1 = \begin{cases} (v^1, u^1) & \text{if} \quad \ell_1 = 2, \\ C_1 - (v^1, u^1) & \text{if} \quad \ell_1 \geqslant 4; \end{cases}$$
$$P_2 = \begin{cases} (v^2, u^2) & \text{if} \quad \ell_2 = 2, \\ C_2 - (v^2, u^2) & \text{if} \quad \ell_2 \geqslant 4; \end{cases}$$
$$P_3 = C_3 - (v^3, u^3).$$

Then $(u^0, v^1) + P_1 + (u^1, v^2) + P_2 + (u^2, v^3) + P_3 + (u^3, v^0) + P_0$ is a cycle of length ℓ in BH_n and contains e. Then Theorem 1 follows. □

From Theorem 1 we can deduce the following corollary due to Wu and Huang [14].

Corollary. *There exists a cycle of length* 2^l *in the balanced hypercube* BH_n *for all* l, $2 \leqslant l \leqslant 2n$.

4. Hamiltonian laceability of balanced hypercubes

In this section, we consider the Hamiltonian laceability of the balanced hypercubes and present another main result of this paper. For this purpose, we first prove the following lemma.

Lemma 2. *Let* $(a_0, a_1, \ldots, a_{n-1})$ *be any vertex of* BH_n, *then there exists a Hamiltonian cycle that contains the path* $\langle (a_0 - 1, a_1, \ldots, a_{n-2}, a_{n-1}), (a_0, a_1, \ldots, a_{n-2}, a_{n-1}), (a_0 + 1, a_1, \ldots, a_{n-2}, a_{n-1}), (a_0 + 2, a_1, \ldots, a_{n-2}, a_{n-1}) \rangle$.

Proof. We prove the lemma by mathematical induction on $n \geqslant 1$. The theorem is true for $n = 1$. Assume the theorem is true for all $1 \leqslant k < n$. Since $n \geqslant 2$, then the vertices $(a_0 - 1, a_1, \ldots, a_{n-2}, a_{n-1})$, $(a_0, a_1, \ldots, a_{n-2}, a_{n-1})$, $(a_1 + 1, a_1, \ldots, a_{n-2}, a_{n-1})$ and $(a_0 + 2, a_1, \ldots, a_{n-2}, a_{n-1})$ are in $BH_{n-1}^{a_{n-1}}$. Assume without loss of generality that $a_{n-1} = 0$. By induction hypothesis, there exists a cycle C_0 of length $2^{2(n-1)} - 1$ that contains the path $\langle (a_0 - 1, a_1, \ldots, a_{n-2}, 0), (a_0, a_1, \ldots, a_{n-2}, 0), (a_0 + 1, a_1, \ldots, a_{n-2}, 0), (a_0 + 2, a_1, \ldots, a_{n-2}, 0) \rangle$ in BH_{n-1}^0. Since $n \geqslant 2$, there exists an edge $(u^0, v^0) \in E(C_0)$ such that at most one of vertices u^0 and v^0 can be in $\{(a_0 - 1, a_1, \ldots, a_{n-2}, a_{n-1}), (a_0, a_1, \ldots, a_{n-2}, a_{n-1}), (a_0 + 1, a_1, \ldots, a_{n-2}, a_{n-1}), (a_0 + 1, a_1, \ldots, a_{n-2}, a_{n-1})\}$, where the first index of u^0 is even. By Lemma 1, there exists a cycle $C' = \langle v^0, u^0, v^1, u^1, v^2, u^2, v^3, u^3, v^0 \rangle$ of length 8 such that $(u^0, v^0) \in E(BH_{n-1}^0)$, $(u^1, v^1) \in E(BH_{n-1}^1)$, $(u^2, v^2) \in E(BH_{n-1}^2)$ and $(u^3, v^3) \in E(BH_{n-1}^3)$, (refer to Fig. 2). By Theorem 1, there exists a cycle $C_1$of length $2^{2(n-1)}$ in BH_{n-1}^1 containing (u^1, v^1), a cycle C_2 of length $2^{2(n-1)}$ in BH_{n-1}^2 containing (u^2, v^2) and a cycle C_3 of length $2^{2(n-1)}$ in BH_{n-1}^3 containing (u^3, v^3). Let $P_0 = C_0 - (u^0, v^0), P_1 = C_1 - (u^1, v^1), P_2 = C_2 - (u^2, v^2)$ and $P_3 = C_3 - (u^3, v^3)$. Then

$$P_0 + (u^0, v^1) + P_1 + (u^1, v^2) + P_2 + (u^2, v^3) + P_3 + (u^3, v^0),$$

is a Hamiltonian path of BH_n containing the path $\langle (a_0 - 1, a_1, \ldots, a_{n-2}, 0), (a_0, a_1, \ldots, a_{n-2}, 0), (a_0 + 1, a_1, \ldots, a_{n-2}, 0), (a_0 + 2, a_1, \ldots, a_{n-2}, 0) \rangle$. The lemma follows. □

Theorem 2. *The balanced hypercubes* BH_n *are Hamiltonian laceable for* $n \geqslant 1$.

Proof. We prove the theorem by induction on $n \geqslant 1$. The theorem is true for $n = 1$. Assume the theorem is true for all $1 \leqslant k <$ n. Suppose $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1})$ be any two vertices such that $x_0 + y_0$ is odd. Since BH_n is vertex-transitive, without loss of generality, we assume that $x_0 = 0$ and $x_{n-1} = 0$ which means that y_0 is odd and $y \in V_1$, $x \in V_2$. We will consider three cases.

Case 1. $y_{n-1} = 0$. In this case, x and y lie in BH_{n-1}^0. By induction hypothesis, there exists a path P_0 of length $2^{2(n-1)} - 1$ joining x and y in BH_{n-1}^0. Thus, we can choose an edge (u^0, v^0) in P_0 where the first index of u^0 is even. By Lemma 1 (and refer to Fig. 2), there exists a cycle $C' = \langle v^0, u^0, v^1, u^1, v^2, u^2, v^3, u^3, v^0 \rangle$ of length 8 such that $(u^0, v^0) \in E(BH_{n-1}^0), (u^1, v^1) \in E(BH_{n-1}^1), (u^2, v^2) \in E(BH_{n-1}^2)$ and $(u^3, v^3) \in E(BH_{n-1}^3)$. By Theorem 1, there exists a cycle C_1 of length $2^{2(n-1)}$ in BH_{n-1}^1 that contains (u^1, v^1), a cycle C_2 of length $2^{2(n-1)}$ in BH_{n-1}^2 that contains (u^2, v^2) and a cycle C_3 of length $2^{2(n-1)}$ in BH_{n-1}^3 that contains (u^3, v^3). Let $P'_0 = P_0 - (u^0, v^0), P_1 = C_1 - (u^1, v^1), P_2 = C_2 - (u^2, v^2)$ and $P_3 = C_3 - (u^3, v^3)$. Then $P'_0 + (u^0, v^1) + P_1 + (u^1, v^2) + P_2 + (u^2, v^3) + P_3 + (u^3, v^0)$ is a path of length $2^{2n} - 1$ in BH_n that joins x and y.

Case 2. $y_{n-1} = 1$ or $y_{n-1} = 3$. In this case, without loss of generality, we assume $y_{n-1} = 1$. Then x lies in BH_{n-1}^0 and y lies in BH_{n-1}^1. Let v^0 be an arbitrary vertex in BH_{n-1}^0 with the odd first index. By induction hypothesis, there exists a path P_0 of length $2^{2(n-1)} - 1$ that joins x and v^0 in BH_{n-1}^0. Let u^3 be a vertex in BH_{n-1}^3 such that $(v^0, u^3) \in E(BH_n)$. Then the first index of u^3 is even. Choose an arbitrary vertex v^3 with the odd first index in BH_{n-1}^3. By induction hypothesis, there exists a path P_3 of length $2^{2(n-1)} - 1$ that joins u^3 and v^3. Let u^2 be a vertex in BH_{n-1}^2 such that $(v^3, u^2) \in E(BH_n)$. Then the first index of u^2 is even. Choose an arbitrary vertex v^2 with the odd first index in BH_{n-1}^2. By induction hypothesis, there exists a path P_2 of length $2^{2(n-1)} - 1$ that joins u^2 and v^2. Let u^1 be a vertex in BH_{n-1}^1 such that $(v^2, u^1) \in E(BH_n)$. Then the first index of u^1 is even. By induction hypothesis, there exists a path P_1 of length $2^{2(n-1)} - 1$ that joins u^1 and y. Then $P_0 + (v^0, u^3) + P_3 + (v^3, u^2) + P_2 + (v^2, u^1) + P_1$ is a path of length $2^{2n} - 1$ in BH_n that joins x and y (see Fig. 3).

Case 3. $y_{n-1} = 2$. In this case, x lies in BH_{n-1}^0 and y lies in BH_{n-1}^2. Let $y^0 = (y_0, y_1, \ldots, y_{n-2}, 0)$ be a vertex in BH_{n-1}^0. By induction hypothesis, there exists a path P_0 of length $2^{2(n-1)} - 1$ that joins x and y^0 in BH_{n-1}^0. Let x^0 be the second-last vertex of P_0 and choose y^1 be a vertex in BH_{n-1}^1 such that $(y^1, x^0) \in E(BH_n)$. Then the first index of y^1 is odd. Let $x^1 = (y_0 + 1, y_1, \ldots, y_{n-2}, 1)$, then $(x^1, y) \in E(BH_n)$ and $(x^1, (y_0 + 2, y_1, \ldots, y_{n-2}, 2)) \in E(BH_n)$. By induction hypothesis, there exists a

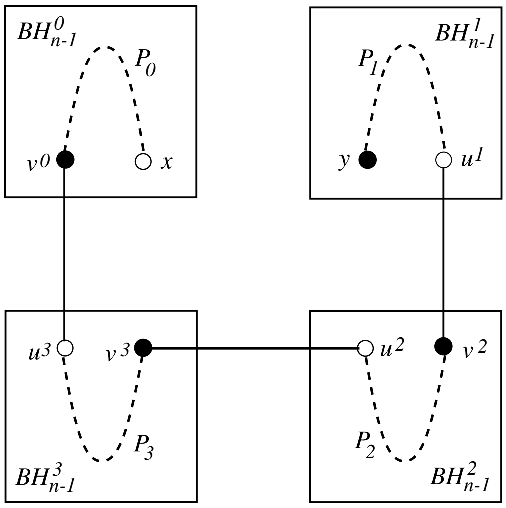

Fig. 3. The path of length $2^{2n}-1$ in BH_n that joins x and y.

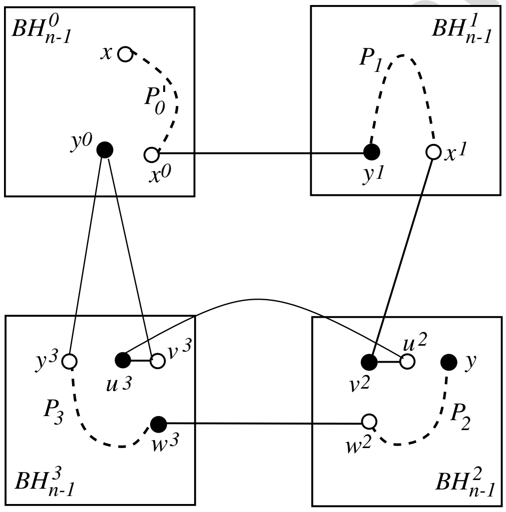

Fig. 4. The path of length $2^{2n}-1$ in BH_n joining x and y.

path P_1 of length $2^{2(n-1)}-1$ that joins y^1 and x^1. For convenience, let $u^2=(y_0+1,y_1,\ldots,y_{n-2},2)$, $v^2=(y_0+2,y_1,\ldots,y_{n-2},2)$, $w^2=(y_0+3,y_1,\ldots,y_{n-2},2)$, $y^3=(y_0-1,y_1,\ldots,y_{n-2},3)$, $u^3=(y_0,y_1,\ldots,y_{n-2},3)$, $v^3=(y_0+1,y_1,\ldots,y_{n-2},3)$ and $w^3=(y_0+2,y_1,\ldots,y_{n-2},3)$. By Lemma 2, there exists a cycle C_2 of length $2^{2(n-1)}$ in BH^2_{n-1} that contains the path $\langle y,u^2,v^2,w^2\rangle$ and exists a cycle C_3 of length $2^{2(n-1)}$ in BH^3_{n-1} that contains the path $\langle y^3,u^3,v^2,w^2\rangle$. Let $P'_0=P_0-(y^0,x^0), P_2=C_2-\langle y,u^2,v^2,w^2\rangle$ and $P_3=C_3-\langle y^3,u^3,v^2,w^2\rangle$. Then $P'_0+(x^0,y^1)+P_1+(x^1,v^2)+(v^2,u^2)+(u^2,u^3)+(u^3,v^3)+(v^3,y^0)+(y^0,y^3)+P_3+(w^3,w^2)+P_2$ is a path of length $2^{2n}-1$ in BH_n that joins x and y (see Fig. 4). The proof of Theorem 2 is complete. □

5. Conclusions

The balanced hypercube, proposed by Huang and Wu [9], forms a hypercube variant that give better performance with the same number of edges and vertices. It has been shown that there exists a cycle of length 2^l in BH_n for all l, $2\leqslant l\leqslant 2n$. In this paper, we improve this result by showing that BH_n is edge-pancyclic which means that for arbitrary edge e, there exists a cycle of even length from 4 to 2^{2n} each containing e in BH_n. We also show that the balanced hypercubes are Hamiltonian laceable.

Since the vertex-connectivity and the edge-connectivity of BH_n is $2n$, the balanced hypercube is still connected when some edges and vertices are broken. The cycle embedding problem of BH_n has been discussed regarding faulty vertices [14]. But regarding faulty edges, it is not known if there exists desired cycles in BH_n, this needs further study.

References

[1] B. Alspach, D. Hare, Edge-pancyclic block-intersection graphs, Discrete Math. 97 (1–3) (1991) 17–24.
[2] J.A. Bondy, Pancyclic graphs, J. Combin. Theory 11 (1971) 80–84.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan Press LTD, London and Basingstoke, 1976.
[4] K. Efe, A variation on the hypercube with lower diameter, IEEE Trans. Comput. 40 (11) (1991) 1312–1316.
[5] J. Fan, Hamilton-connectivity and cycle-embedding of the Möbius cubes, Inform. Process. Lett. 82 (2002) 113–117.
[6] J. Fan, X. Lin, X. Jia, Node-pancyclicity and edge-pancyclicity of crossed cubes, Inform. Process. Lett. 93 (2005) 133–138.
[7] A. Germa, M.C. Heydemann, D. Sotteau, Cycles in the cube-connected cycles graphs, Discrete Appl. Math. 83 (1998) 135–155.
[8] A. Hobbs, The square of a block is vertex pancyclic, J. Combin. Theory, B 20 (1) (1976) 1–4.
[9] K. Huang, J. Wu, Area efficient layout of balanced hypercubes, Int. J. High Speed Electron. Syst. 6 (4) (1995) 631–646.
[10] W.-T. Huang, W.-K. Chen, C.-H. Chen, Pancyclicity of Möbius cubes, in: Proc. 9th Int. Conf. Parallel and Distributed Systems (ICPADS'02), 17–20 December 2002, pp. 591–596.
[11] T.K. Li, C.H. Tsai, J.J.M. Tan, L.H. Hsu, Bipannevtivity and edge-falt-tolerant bipancyclicity of hypercubes, Inform. Process. Lett. 87 (2003) 107–110.
[12] Y. Saad, M.H. Schultz, Topological properties of hypercubes, IEEE Trans. Comput. 37 (7) (1988) 867–872.
[13] G. Simmons, Almost all n-dimensional rectangular lattices are Hamilton laceable, Congr. Numer. 21 (1978) 103–108.
[14] J. Wu, K. Huang, The balanced hypercubes: A cube-based system for fault-tolerant applications, IEEE Trans. Comput. 46 (4) (1997) 484–490.
[15] J.-M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.