Fault-tolerant analysis of a class of networks ☆

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Abstract

In this paper, we explore the 2-extraconnectivity of a special class of graphs $G(G_0, G_1; M)$ proposed by Chen et al. [Y.-C. Chen, J.J.M. Tan, L.-H. Hsu, S.-S. Kao, Super-connectivity and super edge-connectivity for some interconnection networks, Applied Mathematics and Computation 140 (2003) 245–254]. As applications of the results, we obtain that the 2-extraconnectivities of several well-known interconnection networks, such as hypercubes, twisted cubes, crossed cubes, Möbius cubes and locally twisted cubes, are all equal to $3n - 5$ when their dimension n is not less than 8. That is, when $n \geqslant 8$, at least $3n - 5$ vertices must be removed to disconnect any one of these n-dimensional networks provided that the removal of these vertices does not isolate a vertex or an edge.

1. Introduction

It is well known that when the underlying topology of an interconnection network is modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are the two important features determining reliability and fault tolerance of the network [18]. These two parameters, however, have a obvious deficiency, that is,

they tacitly assume that all vertices adjacent to or all edges incident with the same vertex of G can potentially fail at the same time, which happens almost impossible in practical applications of networks. In other words, in the definitions of κ and λ, absolutely no restrictions are imposed on the components of $G - S$. Consequently, these two measurements are inaccurate for large-scale processing systems in which all processors adjacent to or all links incident with the same processor cannot fail at the same time. To compensate for this shortcoming, it would seem natural to generalize the notion of the classical connectivity by imposing some conditions or restrictions on the components of $G - S$. Haray [10] first considered this problem by introducing the concept of the conditional connectivity.

Given a graph G and a graph-theoretical property $\mathscr{P}$, he defined the conditional connectivity $\kappa(G; \mathscr{P})$ (resp.,

edge-connectivity $\lambda(G; \mathscr{P})$) as the minimum cardinality of a set of vertices (resp., edges), if any, whose deletion disconnects G and every remaining component has property $\mathscr{P}$. Clearly, $\kappa(G)$ (resp., $\lambda(G)$) is a special case of $\kappa(G; \mathscr{P})$ (resp., $\lambda(G; \mathscr{P})$) when no condition is restricted to $\mathscr{P}$. The existence and value of $\kappa(G; \mathscr{P})$ ($\lambda(G; \mathscr{P})$) varies depending on the different choice of the property $\mathscr{P}$.

Fàbrega and Fiol [9] considered $\kappa(G; \mathscr{P}_h)$ (resp., $\lambda(G; \mathscr{P}_h)$) for a non-negative integer h and a graph G, where $\mathscr{P}_h$ is the property of having more than h vertices. They called this type of connectivity as the h-extraconnectivity (resp., h-edge-extraconnectivity) of G, denoted by $\kappa_h(G)$ (resp., $\lambda_h(G)$). In other words, $\kappa_h(G)$ (resp., $\lambda_h(G)$) is the minimum cardinality of a set of vertices (resp., edges) of G, if any, whose deletion disconnects G and every remaining component has more then h vertices.

Clearly, $\kappa_0(G) = \kappa(G)$ and $\lambda_0(G) = \lambda(G)$ for any graph G if G is not a complete graph. Thus, the h-extraconnectivity is a generalization of the classical connectivity and can provides more accurate measures for the reliability and the tolerance of a large-scale parallel processing system, and so has received much research attention (see, for example, [6–8,12–16,20–22,24,26–28]) for $h = 1$ in recent years. However, a few of results for $h \geqslant 2$ are known in the present literature, for example, [29].

Fàbrega et al. [9] studied the extraconnectivity of some class of graphs with large girth. But the girth of most of the well-known interconnection networks is small. Thus we are interested in the study of the extraconnectivity of the well-known interconnection networks with small girth. In this paper, we consider a class of graphs, denoted by $G(G_0, G_1; M)$ and proposed by Chen et al. [3]. In [23], Xu and Wang studied the 1-extraconnectivity and the 1-edge-extraconnectivity of $G(G_0, G_1; M)$. In [29], Zhu et al. studied the 2-edge-extraconnectivity of $G(G_0, G_1; M)$. In this paper, we study the 2-extraconnectivity of $G(G_0, G_1; M)$. As applications, we obtain the 2-extraconnectivity of the above-mentioned well-known networks is equal to $3n - 5$ if $n \geqslant 8$.

The remaining of this paper is organized as follows. In Section 2, we give some definitions and lemmas. In Section 3, we study that 2-extraconnectivity of $G(G_0, G_1; M)$. Then, in Section 4, as applications of these results, we determine 2-extraconnectivity of the hypercube Q_n, the twisted cube TQ_n, the Möbius cube MQ_n, the locally twisted cube LTQ_n and the crossed cube CQ_n. Conclusions are in Section 5.

2. Definitions and lemmas

For the terminology and notation not defined here, we follow [19]. For a graph $G = (V, E)$ and $S \subset V$ or $S \subset G$, we use $N_G(S)$ to denote the set of neighbors of S in $G - S$. For any vertex v, we use $d_G(x)$ to denote the degree of x. A graph G is k-regular if $d_G(x) = k$ for every $x \in V(G)$. A subset $S \subset V(G)$ is called a vertex-cut if $G - S$ is disconnected. The connectivity of G, denoted by $\kappa(G)$, is the minimum cardinality of all vertex-cuts in G.

Definition 2.1. Let G be a simple graph. A vertex-cut S of G is called a restricted h-vertex-cut if each component of $G - S$ has more than h vertices. The h-extraconnectivity of G, denoted by $\kappa_h(G)$, is defined as $\min\{|S|:\ S$ is a restricted h-vertex-cut of $G\}$ if G has restricted h-vertex cuts.

Clearly, when $h = 0$, a restricted 0-vertex-cut is just a vertex-cut. Thus, the concept of the h-extraconnectivity is a generalization of the concept of the traditional connectivity. In this paper we explore a special class of graphs $G(G_0, G_1; M)$ proposed by Chen et al. [3].

Definition 2.2. Let G_1 and G_2 be two graphs with the same order, and M be any arbitrary perfect matching between vertices of G_1 and G_2. The graph $G(G_1, G_2; M)$ is defined as a graph with the vertex-set $V = V(G_1) \cup V(G_2)$ and the edge-set $E = E(G_1) \cup E(G_2) \cup M$.

Lemma 2.3. *(See [3].) If both G_1 and G_2 are $(k-1)$-regular $(k-1)$-connected graphs with n vertices, then $G(G_1, G_2; M)$ is k-regular k-connected.*

Definition 2.4. For a graph G, the *cn*-number of G, $cn(G)$, is the smallest integer ℓ for which any two vertices in G share at most ℓ common neighboring vertices.

From the definition, we have the following result immediately.

Lemma 2.5. $cn(G(G_0, G_1; M)) = 2$ if $cn(G_0) = cn(G_1) = 2$.

3. Main result

In this section, we present our main result, which can be stated as follows.

Theorem 3.1. *Let $G = G(G_0, G_1; M)$. If both G_0 and G_1 are $(k-1)$-regular and triangle-free, $\kappa_1(G_0) =*

$\kappa_1(G_1) = 2k - 4$, and $cn(G_0) = cn(G_1) = 2$, then $\kappa_2(G) = 3k - 5$ for $k \geqslant 8$ and $\kappa_2(G) \geqslant 3k - 5$ for $6 \leqslant k \leqslant 7$.

Proof. By Lemma 2.5, $cn(G) = 2$ and so there exist two vertices x and y in G such that $|N_G(x) \cap N_G(y)| = 2$. Let $T = (x, z, y)$ be a path of length 2 between x and y. By Lemma 2.3, G is k-regular. Since both G_0 and G_1 contain no triangles, it is easy to check that G contains no triangles, and so $N_G(z) \cap (N_G(x) \cup N_G(y)) = \emptyset$. Thus

$$\big|N_G(T)\big| = (k-1) + (k-2) + (k-2) = 3k - 5.$$

Let $H = G - (T \cup N_G(T))$. Since $\kappa_1(G_0) = \kappa_1(G_1) = 2k - 4$, $|V(G)| = 2|V(G_1)| \geqslant 2(\kappa_1(G_0) + 4) = 4k$, and so $V(H) \neq \emptyset$. Since $cn(G) = 2$, $|N_G(u) \cap N_G(T)| \leqslant 6$ for any $u \in V(H)$ and $|N_G(e) \cap N_G(T)| \leqslant 12$ for any $e \in E(H)$. Since $6 < k = |N_G(u)|$ and $12 < 2k - 2 = |N_G(e)|$ for $k \geqslant 8$, there are neither isolated vertices nor isolated edges in $G - N_G(T)$. Thus, $N_G(T)$ is a restricted 2-vertex-cut of G, and so

$$\kappa_2(G) \leqslant \big|N_G(T)\big| = 3k - 5 \quad \text{for } k \geqslant 8. \qquad (1)$$

We now prove

$$\kappa_2(G) \geqslant 3k - 5 \quad \text{for } k \geqslant 5. \qquad (2)$$

Suppose that there is a subset $F \subset V(G)$ with $|F| \leqslant 3k - 6$ such that there are neither isolated vertices nor isolated edges in $G - F$. We want to prove that $G - F$ is connected. To the end, let $F_0 = F \cap V(G_0)$ and $F_1 = F \cap V(G_1)$. Clearly, $F_0 \cap F_1 = \phi$. Without loss of generality, we may suppose that $|F_0| \geqslant |F_1|$. Then

$$|F_1| \leqslant \frac{3k-6}{2} < \frac{4k-10}{2} = 2k - 5 \quad \text{for } k \geqslant 5.$$

We now prove that $G - F$ is connected by two steps. In Step 1, we prove that any vertex in $G_0 - F_0$ can be connected to a vertex in $G_1 - F_1$. In Step 2, we prove that $G_1 - F_1$ is connected in $G - F$.

Step 1. Any vertex in $G_0 - F_0$ can be connected to some vertex in $G_1 - F_1$.

Let u_0 be any vertex in $G_0 - F_0$ and $u_0 u_1 \in M$. If $u_1 \notin F_1$, then we are done. So suppose that $u_1 \in F_1$. Since u_0 is not an isolated vertex in $G - F$, there exist a vertex $v_0 \in N_{G_0}(u_0) - F_0$, suppose $v_0 v_1 \in M$, if $v_1 \notin F_1$, then we are done, so suppose that $v_1 \in F_1$. Since $u_0 v_0$ is not an isolated edge in $G - F$, there exist a vertex $w_0 \in N_{G_0}(u_0, v_0) - F_0$. Suppose that $w_0 w_1 \in M$, if $w_1 \notin F_1$, then we are done, so suppose $w_1 \in F_1$. The vertex w_0 may be adjacent to u_0 or v_0, without loss of generality, we suppose that w_0 is adjacent to v_0, then $N_{G_0}(w_0) \cap N_{G_0}(v_0) = \emptyset$ and $N_{G_0}(u_0) \cap N_{G_0}(v_0) = \emptyset$

since G_0 contains no triangles. And since $cn(G_0) = 2$, so $N_{G_0}(w_0) \cap N_{G_0}(u_0) \leqslant 2$, thus

$$\big|N_{G_0}(u_0, v_0, w_0)\big| \geqslant (k-2) + (k-3) + (k-3)$$
$$= 3k - 8.$$

Let $N = N_{G_0}(u_0, v_0, w_0) \cup \{u_0, v_0, w_0\}$. Then $|N| \geqslant 3k - 5$.

Let $M' = \{e \in M \mid e \text{ is incident with a vertex in } N\}$. Then $|M'| = |N| \geqslant 3k - 5$ since M is a perfect matching in G. Since $|F| \leqslant 3k - 6$, there exists at least one edge of M' whose two end-vertices both are not in F. Let $e \in M'$ be such an edge. Since u_0, v_0, w_0 all are not in F, u_0 can be connected to $G_1 - F_1$ via the edge e.

Step 2. $G_1 - F_1$ is connected in $G - F$.

If there are no isolated vertices in $G_1 - F_1$, then it is clear that $G_1 - F_1$ is connected since $|F_1| < 2k - 5 < 2k - 4 = \kappa_1(G_1)$ by our hypothesis.

Suppose now that there exists an isolated vertex, say u_1, in $G_1 - F_1$. Since any two vertices in G_1 can share at most two common neighbors by our hypothesis, at least $2k - 4$ vertices are to be removed to get two isolated vertices in G_1. Since $|F_1| < 2k - 5$, there is just one isolated vertex u_1 in $G_1 - F_1$. Let $F_1' = F_1 \cup \{u_1\}$. Then $G_1 - F_1'$ contains no isolated vertices, and so $G_1 - F_1'$ is connected since $|F_1'| < 2k - 5 + 1 = 2k - 4 = \kappa_1(G_1)$ by our hypothesis. We only need to show that u_1 can be connected to some vertex in $G_1 - F_1'$ via some vertices in $G_0 - F_0$.

Since there are no isolated edges in $G - F$, $N_{G_0}(u_0) - F_0 \neq \emptyset$. If there is some $v_0 \in N_{G_0}(u_0) - F_0$ with $v_0 v_1 \in M$ such that $v_1 \notin F_1$, then we are done. So assume $v_1 \in F_1$ for any $v_0 \in N_{G_0}(u_0) - F_0$ with $v_0 v_1 \in M$. Note $|N_{G_0}(u_0, v_0)| = 2(k-2) = 2k - 4$ since G_0 contains no triangles. Since $|F_1| \geqslant |N_{G_1}(u_1)| = k - 1$ and $|F_0| = |F| - |F_1| \leqslant 3k - 6 - (k-1) = 2k - 5$, we have $|N_{G_0}(u_0, v_0)| > |F_0|$, that is, $N_{G_0}(u_0, v_0) - F_0 \neq \emptyset$. If there is some $w_0 \in N_{G_0}(u_0, v_0) - F_0$ with $w_0 w_1 \in M$ such that $w_1 \notin F_1$, then we are done. So assume $w_1 \in F_1$ for any $w_0 \in N_{G_0}(u_0, v_0) - F_0$ with $w_0 w_1 \in M$. Since G_0 contains no triangles and $cn(G_0) = 2$, $|N_{G_0}(u_0, v_0, w_0)| \geqslant 3k - 8 > 2k - 5 \geqslant |F_0|$ for $k \geqslant 4$, that is, $N_{G_0}(u_0, v_0, w_0) - F_0 \neq \emptyset$. If there is some $x_0 \in N_{G_0}(u_0, v_0, w_0) - F_0$ with $x_0 x_1 \in M$ such that $x_1 \notin F_1$, then we are done. So assume $x_1 \in F_1$ for any $x_0 \in N_{G_0}(u_0, v_0, w_0) - F_0$ with $x_0 x_1 \in M$. Since G_0 contains no triangles and $cn(G_0) = 2$, it is easy to compute that $|N_{G_0}(u_0, v_0, w_0, x_0)| \geqslant 4(k-1) - 6 - 4 = 4k - 14$.

Let $N = N_{G_0}(u_0, v_0, w_0, x_0) \cup \{v_0, w_0, x_0\}$. Then $|N| \geqslant 4k - 11$. Let $M' = \{e \in M \mid e \text{ is incident with a vertex in } N\}$. Then $|M'| = |N|$ since M is a perfect

matching in G. Since $|M'| \geqslant 4k - 11 > 3k - 6 \geqslant |F|$ for $k \geqslant 6$, there exists at least one edge of M' whose two end-vertices both are not in F. Let $e \in M'$ be such an edge. Since u_0, v_0, w_0, x_0 all are not in F, u_0 can be connected to $G_1 - F_1$ via the edge e.

Combining Step 1 with Step 2 fields the inequality (2). It follows from (1) and (2) that $\kappa_2(G) = 3k - 5$ for $k \geqslant 8$ and $\kappa_2(G) \geqslant 3k - 5$ for $6 \leqslant k \leqslant 7$. This completes the proof of the theorem. $\square$

4. Applications

Topologies of many interconnection networks can be easily viewed as $G(G_0, G_1; M)$ for some k-regular graphs G_0 and G_1, see Chen et al. [3], such as the hypercube Q_n [11,17], the twisted cube TQ_n [1,2], the cross cube CQ_n [5], the Möbius cube MQ_n [4] and the locally twisted cube LTQ_n [25]. For these networks, Chen et al. [3] proved that they are super connected and super edge-connected for $n \geqslant 3$; Xu and Wang [23] determined their 1-extraconnectivities and the 1-edge-extraconnectivities both are $2n - 2$ for $n \geqslant 3$; Zhu et al. [29] determined their 2-edge-extraconnectivities all are $3n - 4$ for $n \geqslant 4$. Applying Theorem 3.1, we immediately obtain that their 2-extraconnectivities all are $3n - 5$ for $n \geqslant 8$. The proofs are omitted here for details.

5. Conclusions

In this paper, we concentrate on a stronger measure of reliability and fault-tolerance of a network called the 2-extraconnectivity, which not only compensate for shortcoming but also generalize the classical connectivity. For a special class of graphs $G(G_0, G_1; M)$, we explore its 2-extraconnectivity. As applications of our result, we immediately determine the 2-extraconnectivities of some well-known networks, such as the hypercube Q_n, the twisted cube TQ_n, the cross cube CQ_n, the Möbius cube MQ_n and the locally twisted cube LTQ_n, all are $3n - 5$ for $n \geqslant 8$. In other words, for $n \geqslant 8$, at least $3n - 5$ vertices must be removed to disconnect any one of these networks provided that the removal of these vertices does not isolate a vertex or an edge. When these networks are used to model the topological structure of a large-scale parallel processing system, our results provide more accurate measurements for reliability and fault tolerance of the system.

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