On Distance Connected Domination Numbers of Graphs *

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Abstract Let k be a positive integer and $G = (V, E)$ be a connected graph of order n. A set $D \subseteq V$ is called a k-dominating set of G if each $x \in V(G) - D$ is within distance k from some vertex of D. A connected k-dominating set is a k-dominating set that induces a connected subgraph of G. The connected k-domination number of G, denoted by $\gamma_k^c(G)$, is the minimum cardinality of a connected k-dominating set. Let δ and Δ denote the minimum and the maximum degree of G, respectively. This paper establishes that $\gamma_k^c(G) \leq \max\{1, n - 2k - \Delta + 2\}$, and $\gamma_k^c(G) \leq (1+o_\delta(1))n\frac{\ln[m(\delta+1)+2-t]}{m(\delta+1)+2-t}$, where $m = \lceil \frac{k}{3} \rceil$, $t = 3\lceil \frac{k}{3} \rceil - k$, and $o_\delta(1)$ denotes a function that tends to 0 as $\delta \to \infty$. The later generalizes the result of Caro *et al*'s in [Connected domination and spanning trees with many leaves. SIAM J. Discrete Math. 13 (2000), 202-211] for $k = 1$.

Keywords: domination, connected k-domination number, distance.

AMS Subject Classification: 05C69 05C12

1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to [2] or [13]. Let $G = (V, E)$ be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order, the maximum degree and the minimum degree of vertices of G are denoted by $n(G)$, $\Delta(G)$ and $\delta(G)$, respectively. The distance $d_G(x, y)$ between two vertices

* The work was supported by NNSF of China (No.10271114 and No.10301031).

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x and y is the length of a shortest xy-path in G. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and for $v \in V(G)$, $d_G(v, S) = \min_{u \in V(S)}\{d_G(v, u)\}$. The eccentricity $e_G(v)$ of v is $\max_{x \in V(G)}\{d_G(v, x)\}$. The radius $\mathrm{rad}(G)$ is the smallest eccentricity of a vertex in G. Let k be a positive integer. For every vertex $x \in V(G)$, the k-neighborhood $N_k(x)$ of x is defined by $N_k(x) = \{y \in V(G) : \ d_G(x, y) \leq k, x \neq y\}$, and $N_1(x)$ is usually called the neighborhood of x in G.

A set D of vertices in G is called a k-dominating set of G if every vertex of $V(G)-D$ is within distance k from some vertex of D. A k-dominating set D is called to be connected if $G[D]$ is connected. The minimum cardinality among all k-dominating sets (resp. connected k-dominating sets) of G is called the k-domination number (resp. connected k-domination number) of G and is denoted by $\gamma_k(G)$ (resp. $\gamma_k^c(G)$). The concept of the k-dominating set was first introduced by Chang and Nemhauser [4, 5].

Since the distance versions of domination have a strong background of applications, many efforts have been made by several authors to consider the distance parameters (see, for example, [4] $\sim$ [10], [12, 14]).

It is quite difficult to determine the value of $\gamma_k(G)$ or $\gamma_k^c(G)$ for any given graph G. In this paper, we prove that for any nontrivial connected graph G with order n, $\gamma_k^c(G) = \min\gamma_k^c(T)$, where the minimum is taken over all spanning trees T of G. We also get two upper bounds for $\gamma_k^c(G)$ in terms of the maximum degree $\Delta = \Delta(G)$, that is,

$$\gamma_k^c(G) \leq \max\{1, n - 2k - \Delta + 2\},$$

and the minimum degree $\delta = \delta(G)$, that is,

$$\gamma_k^c(G) \leq (1 + o_\delta(1))n\frac{\ln[m(\delta+1) + 2 - t]}{m(\delta+1) + 2 - t},$$

where $m = \lceil \frac{k}{3} \rceil$, $t = 3\lceil \frac{k}{3} \rceil - k$, and $o_\delta(1)$ denotes a function that tends to 0 as $\delta \to \infty$. The later generalizes the result of Caro *et al*'s [3] for $k = 1$, that is,

$$\gamma_1^c(G) \leq (1 + o_\delta(1))n\frac{\ln(\delta+1)}{\delta+1}.$$

The method used here is a generalization and refinement of theirs.

2 Elementary Results

Theorem 1 Let G be a nontrivial connected graph, and k be a positive integer. Then $\gamma_k^c(G) = \min\gamma_k^c(T)$, where the minimum is taken over all spanning trees T of G.

Proof Let G be a nontrivial connected graph and T be a spanning tree of G. Then any connected k-dominating set of T is also a connected k-dominating set of G. Therefore, $\gamma_k^c(G) \leq \gamma_k^c(T)$. Thus we have that $\gamma_k^c(G) \leq \min\gamma_k^c(T)$, where the minimum is taken over all spanning trees T of G.

Now we show the reverse inequality. If G is a tree, then the theorem holds trivially. So we may assume that G is a connected graph containing cycles. Let D be a minimum connected k-dominating set of G and C be a cycle in G. If we can prove that D is also a connected k-dominating set of $G - e$ for some cycle edge $e \in E(C)$, then $\gamma_k^c(G - e) \leq |D| = \gamma_k^c(G)$. By applying this process a finite number of times, we have $\gamma_k^c(T) \leq \gamma_k^c(G)$ for some spanning tree T of G. Thus, we have that $\min\gamma_k^c(T) \leq \gamma_k^c(G)$, where the minimum is taken over all spanning trees T of G.

If $V(C) \subseteq V(D)$, then obviously $G[D] - e$ for any $e \in E(C)$ is also connected and the vertices in $V(G) - D$ are also all within distance k to D.

If $V(C) \nsubseteq V(D)$, then we select an edge xy in C such that $d_G(x, D) + d_G(y, D) = \max\{d_G(u, D) + d_G(v, D) :\ uv \in E(C)\}$. Now we will show that D is a connected k-dominating set of $G - \{xy\}$.

First for any two adjacent vertices u and v in G, we have $|d_G(u, D) - d_G(v, D)| \leq 1$. Then if w is a vertex in $V(C)$ such that $d_G(w, D) = \max\{d_G(v, D) :\ v \in V(C)\}$, we have that $w = x$ or $w = y$. Without loss of generality, suppose that $d_G(x, D) = \max\{d_G(v, D) :\ v \in V(C)\}$.

Let z be another neighbor of x different from y in $V(C)$. So we immediately have that $d_G(z, D) \leq d_G(y, D)$. Thus, we get the distance between a vertex in $V(G) - D$ and D is not influenced by deleting the edge $\{xy\}$. That is to say, $d_{G-xy}(v, D) = d_G(v, D)$ for all vertices v in $V(G)$. Hence, D is also a connected k-dominating set of $G - e$ for some cycle edge e. ∎

Proposition 2 Let $G = (V, E)$ be a nontrivial connected graph, and k be a positive integer. If $\mathrm{rad}(G) \leq k$, then $\gamma_k^c(G) = 1$.

3 Main Results

Theorem 3 Let G be a connected graph of order $n \geq 2$ with maximum degree $\Delta = \Delta(G)$, and k be a positive integer, then

$$\gamma_k^c(G) \leq \max\{1, n - 2k - \Delta + 2\}.$$

Proof By Theorem 1, it is sufficient to show that $\gamma_k^c(T) \leq \max\{1, n - 2k - \Delta + 2\}$, for any spanning tree T with maximum degree $\Delta = \Delta(T)$.

If $\mathrm{rad}(T) \leq k$, then by Theorem 2, we get $\gamma_k^c(T) = 1$. So we may assume that $\mathrm{rad}(T) > k$. Let P be a longest path in T with end-vertices u and

v. Then there exists two vertices x and y of P such that $d_T(x,u) = k$ and $d_T(y,v) = k$. Let P_{xy} be the xy-subpath of P, and let $D' = V(P) - V(P_{xy})$. Let $D = V(T) - (D' \cup \mathscr{L}(T))$, where $\mathscr{L}(T)$ is the set of leaves of $V(T)$. Thus D must contain a connected k-dominating set of T. Since $u, v \in D' \cap \mathscr{L}(T)$, and $\mathscr{L}(T) \geq \Delta$, we have

$$\begin{aligned}\gamma_k^c(T) &\leq |V(T)| - |D' \cup \mathscr{L}(T)| \\ &\leq |V(T)| - |D'| - |\mathscr{L}(T)| + |D' \cap \mathscr{L}(T)| \\ &\leq n - 2k - \Delta + 2\end{aligned}$$

as required. ∎

We use probabilistic method to give an upper bound of $\gamma_k^c(G)$ in terms of the minimum degree $\delta = \delta(G)$ below. This bound improves the results of Caro *et al* [3] for $k = 1$ and the method is a generalization and refinement of theirs.

For an event A and for a random variable Z of an arbitrary probability space, $P[A]$ and $E[Z]$ denote the probability of A, the expectation of Z, respectively.

Lemma 4 (Xu, Tian and Huang [14]) Let S be a k-dominating set of a connected graph G. If $G[S]$ has h components, then

$$\gamma_k^c(G) \leq |S| + 2(h-1)k.$$

Theorem 5 Let G be a nontrivial connected graph of order n with minimum degree δ, then

$$\gamma_k^c(G) < n\frac{72k + 20km^2 + 17 + 0.5\sqrt{\ln q} + \ln q}{q}, \tag{1}$$

where $q = m(\delta+1) + 2 - t$, $m = \lceil \frac{k}{3} \rceil$ and $t = 3\lceil \frac{k}{3} \rceil - k$.

Proof Let $k = 3m - t$, where $m \geq 1$, $0 \leq t \leq 2$. For $\delta(G) < 72\lfloor \frac{k}{m} \rfloor + 20km$, we immediately have $\gamma_k^c(G) \leq n$, and the theorem holds. We assume that $\delta(G) \geq 72\lfloor \frac{k}{m} \rfloor + 20km \geq 92$ below. Let $p = \frac{\ln q}{q}$, where $q = m(\delta+1) + 2 - t$, and let us pick, randomly and independently, each vertex of V with probability p. Let X be the set of vertices picked. Let Y be the random set of all vertices that are not picked and have no k-neighbors in X. By the choice of Y, $X \cup Y$ is a k-dominating set of G.

Claim 1 $d_G(X,Y) = k+1$.

Proof of Claim 1. It is clear from the choice of Y that $d_G(X,Y) \geq k+1$. Now let $a \in X$, $b \in Y$ be two vertices whose distance in G is the smallest, that is, $d_G(a,b) = d_G(X,Y)$. Let P be any shortest path from a to b and let v be the second-last vertex on P. Then $v \notin Y$. If $d_G(a,b) \geq k+2$, then v has no k-neighbors in X. By definition of Y, we should get $v \in Y$, a contradiction. ∎

Let $\alpha = |X|$, $\beta = |Y|$ and P_{XY} denote one shortest path from X to Y. By Claim 1, we have $|V(P_{XY})| = k+2$. Let μ denote the number of components in $G[X]$. Then $X \cup Y \cup V(P_{XY})$ is a subgraph of G having at most $\mu + \beta - 1$ components. By Lemma 4, we have

$$\gamma_k^c(G) \leq \alpha + \beta + k + 2(\mu + \beta - 1 - 1)k = \alpha + (2k+1)\beta + 2k\mu - 3k.$$

In order to prove (1), it therefore suffices to show that with positive probability,

$$\alpha + (2k+1)\beta + 2k\mu - 3k < n\frac{72k + 20km^2 + 17 + 0.5\sqrt{\ln q} + \ln q}{q}. \quad (2)$$

Claim 2 $|N_k(v)| \geq m(\delta+1) + 1 - t$ for any $v \in V(G)$.

Proof of Claim 2 Let $X_i(v) = \{u \in V(G) : d_G(u,v) = i\}$.

If $v \in X \cup Y$, then by $d_G(X,Y) = k+1$ and G is connected, $X_i(v) \neq \emptyset$ for $i = 1, \cdots, k$. Clearly, $|X_1(v)| \geq \delta$. For $2 \leq i \leq k-2$, we have that $|X_i(v)|+|X_{i+1}(v)|+|X_{i+2}(v)| \geq \delta+1$. In fact, for any $u \in X_{i+1}(v)$, $N_1(u) \subseteq X_i(v) \cup X_{i+1}(v) \cup X_{i+2}(v)$, thus, $|X_i(v)| + |X_{i+1}(v)| - 1 + |X_{i+2}(v)| \geq \delta$. So, we have

$$\begin{aligned} |N_k(v)| &= |X_1(v)| + |X_2(v)| + \cdots + |X_k(v)| \\ &\geq \delta + \left\lfloor \frac{k-1}{3} \right\rfloor (\delta+1) + \left(k - 1 - 3\left\lfloor \frac{k-1}{3} \right\rfloor\right) \\ &= \delta + (m-1)(\delta+1) + (2-t) \\ &= m(\delta+1) + 1 - t. \end{aligned}$$

Let $v \in V(G) - (X \cup Y)$. If $d_G(v,Y) \geq k$ or $d_G(v,X) \geq k$, using the same discussion as above we get $|N_k(v)| \geq m(\delta+1)+1-t$. Now suppose that $d_G(v,Y) < k$ and $d_G(v,X) < k$. Since $d_G(X,Y) = k+1$, there must exist a shortest path between a vertex $a \in X$ and a vertex $b \in Y$ through v such that $d_G(a,b) \geq k+1$, $d_G(v,b) < k$ and $d_G(a,v) < k$. We only consider the worst case $d_G(a,b) = k+1$, and let P_{ab} denote the shortest path from a to b passing through v.

Let v_1 and v_2 be two neighbors of v on P_{ab} from b to v and from a to v, respectively. Let $d_G(b,v_1) = \ell_1$, $d_G(a,v_2) = \ell_2$. Thus, $\ell_1 + \ell_2 = k-1$. We only consider three cases. The other one are analogue or immediate by symmetry.

If $\ell_1 \equiv 1 (\text{mod } 3)$, $\ell_2 \equiv 1 (\text{mod } 3)$, then $k \equiv 0 (\text{mod } 3)$, that is, $k = 3m$, $t = 0$.

$$\begin{aligned} |N_k(v)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + 2 \\ &= \delta + \frac{\ell_1 + \ell_2 - 2}{3}(\delta+1) + 2 \end{aligned}$$

$$
\begin{aligned}
&= \delta + \frac{k-3}{3}(\delta+1) + 2 \\
&= \delta + (m-1)(\delta+1) + 2 \\
&= m(\delta+1) + 1
\end{aligned}
$$

If $\ell_1 \equiv 1(\mathrm{mod}\ 3)$, $\ell_2 \equiv 2(\mathrm{mod}\ 3)$, then $k \equiv 1(\mathrm{mod}\ 3)$, that is, $k = 3m-2$, $t = 2$. Notice $\ell_2 \equiv 2(\mathrm{mod}\ 3)$ and $d_G(v,a) < k$, then $N_1(a) \subseteq N_k(v)$, thus $|X_{\ell_2-1}(v_2)| + |X_{\ell_2}(v_2)| + |X_{\ell_2+1}(v_2)| \geq \delta + 1$. So we have

$$
\begin{aligned}
|N_k(v)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor \right)(\delta+1) + 1 + (\delta+1) \\
&= \delta + \frac{\ell_1 - 1 + \ell_2 - 2}{3}(\delta+1) + \delta + 2 \\
&= \delta + \frac{k-4}{3}(\delta+1) + \delta + 2 \\
&= m(\delta+1) \\
&> m(\delta+1) + 1 - t
\end{aligned}
$$

If $\ell_1 \equiv 2(\mathrm{mod}\ 3)$, $\ell_2 \equiv 2(\mathrm{mod}\ 3)$, then $k \equiv 2(\mathrm{mod}\ 3)$, that is, $k = 3m-1$, $t = 1$. By the discussion as above, we also get $|X_{\ell_1-1}(v_1)| + |X_{\ell_1}(v_1)| + |X_{\ell_1+1}(v_1)| \geq \delta + 1$. Thus, we have,

$$
\begin{aligned}
|N_k(v)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor \right)(\delta+1) + 2(\delta+1) \\
&= \delta + \frac{\ell_1 - 2 + \ell_2 - 2}{3}(\delta+1) + 2\delta + 2 \\
&= \delta + \frac{k-5}{3}(\delta+1) + 2\delta + 2 \\
&= m(\delta+1) + \delta \\
&> m(\delta+1)
\end{aligned}
$$

The Claim 2 follows. ∎

Claim 3 $P\left[\beta > 17\frac{n}{q}\right] < 0.059$.

Proof of Claim 3 For each vertex v, the probability that $v \in Y$ is that $P[v \in Y] = (1-p)^{|N_k(v)|+1}$. By Claim 2, we already have that $|N_k(v)| \geq m(\delta+1)+1-t$ for any $v \in V(G)$ and since β can be written as a sum of n indicator random variables χ_v, where $\chi_v = 1$ if $v \in Y$ and $\chi_v = 0$ otherwise, it follows that the expectation of β satisfies $E[\beta] \leq n(1-p)^q$. By using Taylor's formula,

$$
\left(1 - \frac{\ln q}{q}\right)^q < \left(e^{-\frac{\ln q}{q}}\right)^q = \frac{1}{q},
$$

we have $E[\beta] < \frac{n}{q}$. By Markov's inequality, for any $s > 0$, $P[\beta > s] < \frac{E[\beta]}{s}$, we have,

$$P\left[\beta > 17\frac{n}{q}\right] < \frac{1}{17} < 0.059$$

as required. ∎

Claim 4 $P\left[\alpha > n\frac{\ln q}{q} + n\frac{0.5\sqrt{\ln q}}{q}\right] < 0.892.$

Proof of Claim 4 Since α can also be written as a sum of n indicator random variables that each having probability p of success, we also have $E[\alpha] = np = n\frac{\ln q}{q}$. We use an inequality attributed to Chernoff in [1], that is, for any $s \geq 0$:

$$P[\alpha > E[\alpha] + s] \leq \exp\left\{\frac{-s^2}{2(E[\alpha] + \frac{s}{3})}\right\}.$$

Take $s = n\frac{0.5\sqrt{\ln q}}{q}$ to this inequality, we have

$$\begin{aligned} & P\left[\alpha > n\frac{\ln q}{q} + n\frac{0.5\sqrt{\ln q}}{q}\right] \\ \leq\ & \exp\left(-\frac{n}{8q + \frac{4}{3}q\frac{1}{\sqrt{\ln q}}}\right) \\ <\ & \exp\left(-\frac{1}{8 + 1.34\frac{1}{\sqrt{\ln[93m+2-t]}}}\right) \\ \leq\ & \exp\left(-\frac{1}{8 + 1.34\frac{1}{\sqrt{\ln 93}}}\right) < 0.892. \end{aligned}$$

Here $n \geq |N_k(v)| + 1 \geq q$. The Claim 4 follows. ∎

Like [3], we say that a vertex $v \in V(G)$ is weakly dominated if v has fewer than $\frac{1}{8m^2}\ln q$ neighbors in X. Let $N_1^X(v)$ denote the set of neighbors of v in X. Let $\mathscr{D}$ denote the set of weakly dominated vertices in X.

Claim 5 $P\left[|\mathscr{D}| > 19n\frac{\ln q}{q^{1.34}}\right] < 0.047.$

Proof of Claim 5 First we have, for any $v \in V(G)$,

$$\begin{aligned} E\left[|N_1^X(v)|\right] &= |N_1(v)|p \geq \delta p \\ &= \frac{\delta}{q}\ln q \\ &\geq \frac{92}{93m + 2 - t}\ln q \\ &\geq \frac{92}{93m+2}\ln q, \end{aligned}$$

where $\frac{\delta}{q}$ is an increasing function for δ. By using linearity of expectation and another inequality of Chernoff [1], that is, for any $s \geq 0$,

$$P\left[|N_1^X(v)| < E[|N_1^X(v)|] - s\right] \leq \exp\left(-\frac{s^2}{2E[|N_1^X(v)|]}\right),$$

we have,

$$\begin{aligned}
& P\left[|N_1^X(v)| < \tfrac{1}{8m^2}\ln q\right] = P\left[m|N_1^X(v)| < \tfrac{1}{8m}\ln q\right] \\
\leq\; & P\left[m|N_1^X(v)| < \tfrac{(93m+2)}{8m\times 92}E[|N_1^X(v)|]\right] \\
=\; & P\left[m|N_1^X(v)| - E[m|N_1^X(v)|] < -\left(m - \tfrac{(93m+2)}{8m\times 92}\right)E[|N_1^X(v)|]\right] \\
<\; & \exp\left(-\frac{\left(m-\frac{(93m+2)}{8m\times 92}\right)^2 E^2[|N_1^X(v)|]}{2mE[|N_1^X(v)|]}\right) \\
=\; & \exp\left(-\frac{\left(m-\frac{(93m+2)}{8m\times 92}\right)^2 E[|N_1^X(v)|]}{2m}\right) \\
\leq\; & \exp\left(-\tfrac{46}{93m^2+2m}\left(m - \tfrac{(93m+2)}{8m\times 92}\right)^2 \ln q\right) \\
\leq\; & \exp\left(-\tfrac{46m^2}{93m^2+2m}\left(1 - \tfrac{(93m+2)}{8m^2\times 92}\right)^2 \ln q\right) \\
\leq\; & \exp\left(-\tfrac{46}{95}\left(1 - \tfrac{95}{736}\right)^2 \ln q\right) \\
\leq\; & \left(\tfrac{1}{q}\right)^{0.367}.
\end{aligned}$$

Since the event that a vertex v is picked into X is independent of the event that v is a weakly dominated vertex. Hence, the probability that a vertex is in X and is weakly dominated is,

$$\begin{aligned}
& P\left[v \in X;\ |N_1^X(v)| < \frac{1}{8m^2}\ln q\right] \\
&= P[v \in X] \cdot P\left[|N_1^X(v)| < \frac{1}{8m^2}\ln q\right] \\
&\leq p\left(\frac{1}{q}\right)^{0.367}.
\end{aligned}$$

Thus, we have

$$E[|\mathscr{D}|] \leq np\left(\frac{1}{q}\right)^{0.367} = n\frac{\ln q}{q^{1.367}}.$$

By Markov's inequality,

$$P\left[|\mathscr{D}| > 19n\frac{\ln q}{q^{1.34}}\right] < \frac{1}{19q^{0.027}} < \frac{1}{19 \times 93^{0.027}} < 0.047$$

as required. ∎

From Claim 3, Claim 4 and Claim 5, we find that all of these events that

$$\begin{aligned} \alpha &\leq n\frac{\ln q}{q} + n\frac{0.5\sqrt{\ln q}}{q} \\ \beta &\leq 17\frac{n}{q} \\ |\mathscr{D}| &\leq 19n\frac{\ln q}{q^{1.34}} \end{aligned}$$

could happen simultaneously with positive probability, that is,

$$1 - 0.892 - 0.059 - 0.047 = 0.002 > 0.$$

Now we choose a set X satisfying all of these events simultaneously. Every component of X that contains no weakly dominated vertex has size at least $\frac{1}{8m^2}\ln q$, and $\mathscr{D}$ has at most $|\mathscr{D}|$ components. Thus, we have the number of components in $G[X]$ satisfies,

$$\mu \leq \frac{\alpha}{\frac{1}{8m^2}\ln q} + 19n\frac{\ln q}{q^{1.34}} \ .$$

Since $f(\delta) = \frac{\ln q}{q^{0.34}}$ is a decreasing function for $\delta \geq 72\lfloor\frac{k}{m}\rfloor + 20km \geq 92$, we obtain

$$\frac{\ln q}{q^{0.34}} \leq \frac{\ln(93m+2-t)}{(93m+2-t)^{0.34}} \leq \frac{\ln(95-t)}{(95-t)^{0.34}} \leq \frac{\ln(93)}{(93)^{0.34}} < 1,$$

that is $19n\frac{\ln q}{q^{1.34}} < 19\frac{n}{q}$. Now we take

$$\alpha \leq n\frac{\ln q}{q} + n\frac{0.5\sqrt{\ln q}}{q}$$

to the inequality above, we have

$$\begin{aligned} \mu &< n\frac{8m^2}{q} + n\frac{4m^2}{q}\frac{1}{\sqrt{\ln q}} + \frac{19n}{q} \\ &< n\frac{8m^2}{q} + n\frac{4m^2}{q} \times \frac{1}{2} + n\frac{19}{q} \\ &= n\frac{10m^2+19}{q}, \end{aligned}$$

where

$$\frac{1}{\sqrt{\ln q}} \leq \frac{1}{\sqrt{\ln(93m+2-t)}} \leq \frac{1}{\sqrt{\ln(95-t)}} < \frac{1}{\sqrt{\ln(93)}} < \frac{1}{2}.$$

Finally, we have

$$\alpha + (2k+1)\beta + 2k\mu - 3k < n\frac{72k + 20km^2 + 17 + 0.5\sqrt{\ln q} + \ln q}{q}.$$

So, the inequality (2) is proved and the theorem follows. ∎

Remark 1 For k=1,

$$\gamma_1^c(G) < n\frac{109 + 0.5\sqrt{\ln(\delta+1)} + \ln(\delta+1)}{(\delta+1)}.$$

It improves the bound in [3], that is,

$$\gamma_1^c(G) < n\frac{145 + 0.5\sqrt{\ln(\delta+1)} + \ln(\delta+1)}{(\delta+1)}.$$

Remark 2 Since $X \cup Y$ is also a k-dominating set of G, and $E[\alpha] + E[\beta] \le n\frac{1+\ln q}{q}$, there is at least one choice of $X \subseteq V(G)$ such that $\gamma_k(G) \le |X \cup Y| \le n\frac{1+\ln q}{q}$, where $q = m(\delta+1) + 2 - t,\ m = \lceil \frac{k}{3} \rceil$, and $t = 3\lceil \frac{k}{3} \rceil - k$. It improves the well-known result of Lovász [11], that is,

$$\gamma_1(G) \le n\frac{1 + \ln(\delta+1)}{\delta+1}.$$

Theorem 6 For any nontrivial connected graph G with order n and minimum degree δ,

$$\gamma_k^c(G) \le (1 + o_\delta(1))n\frac{\ln q}{q},$$

where $q = m(\delta+1) + 2 - t,\ m = \lceil \frac{k}{3} \rceil$, and $t = 3\lceil \frac{k}{3} \rceil - k$.

Proof By Theorem 5, we have

$$\gamma_k^c(G) < n\frac{\ln q}{q}\left(1 + \frac{72k + 20km^2 + 17}{\ln q} + \frac{0.5}{\sqrt{\ln q}}\right).$$

We get the theorem as

$$\lim_{\delta\to\infty}\left(\frac{72k + 20km^2 + 17}{\ln q} + \frac{0.5}{\sqrt{\ln q}}\right) = 0.$$

Remark 3 Theorem 6 generalizes the result of Caro *et al* [3] for $k = 1$, that is,

$$\gamma_1^c(G) \le (1 + o_\delta(1))n\frac{\ln(\delta+1)}{\delta+1}.$$

For δ is sufficiently large, we also find that the upper bound for $\gamma_k^c(G)$ behaves like the upper bound for $\gamma_k(G)$.

References

[1] Bollobás, B., *Random Graphs.* Cambridge University Press, 2001.

[2] Bondy, J. A. and Murty, U. S. R., *Graph Theory with Applications.* New York: North Holland, 1976.

[3] Caro, Y. West, D. B. and Yuster, R., Connected domination and spanning trees with many leaves. *SIAM J. Discrete Math.*, **13**(2) (2000), 202-211.

[4] Chang, G. J., k-domination and graph covering problems. Ph.D. Thesis, School of OR and IE, Cornell University, Ithaca, NY(1982).

[5] Chang, G. J. and Nemhauser, G. L., The k-domination and k-stability problems on sun-free chordal graphs. *SIAM J. Algebraic Discrete Methods*, **5** (1984), 332-345.

[6] Hattingh, J. H. and Henning, M. A., The ratio of the distance irredundance and domination numbers of a graph. *J. Graph Theory*, **18** (1994), 1-9.

[7] Henning, M. A., Oellermann, O. R. and Swart, H. C., Bounds on distance domination parameters. *J. Combin. Inform. Systems Sci.*, **16** (1991), 11-18.

[8] Henning, M. A., Oellermann, O. R. and Swart, H. C., Relations between distance domination parameters. *Math. Pannon.*, **5**(1) (1994), 69-79.

[9] Henning, M. A., Oellermann, O. R. and Swart, H. C., The diversity of domination. *Discrete Math.*, **161** (1996), 161-173.

[10] Li, S.-G., On connected k-domination numbers of graphs. *Discrete Math.*, **274** (2004), 303-310.

[11] Lovász, L., On the ratio of optimal and integral fractional covers. *Discrete Math.*, **13** (1975), 383-390.

[12] Rautenbach, D. and Volkmann, L., On $\alpha_r\gamma_s(k)$-perfect graphs. *Discrete Math.*, **270** (2003), 241-250.

[13] Xu, J.-M., *Theory and Application of Graphs.* Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[14] Xu, J.-M., Tian, F. and Huang, J., Distance irredundance and connected domination numbers of a graph, *a manuscript submitted to Discrete Mathematics.*