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THE FORWARDING INDEX OF THE CIRCULANT NETWORKS*

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Abstract: In this paper, we study the forwarding index of circulant graph. By using the character of Cayley graph, we establish the tight upper and the lower bounds of forwarding index for circulant graph. As applications, we determine the forwarding index for some wellknown graphs.

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1 Introduction

In general, we use a graph to model an interconnection network which consists of hardware and/or software entities that are interconnected to facilitate efficient computation and communications (see[4]).

Since a directed network is naturally modelled by a digraph, we will speak about digraph, path, vertex and arc instead of directed network, route, node and line in this paper.

A routing R of a connected (di)graph G of order n is a set of $n(n-1)$ elementary paths $R(u,v)$ specified for all (ordered) pairs u,v of vertices of G. A routing R is said to be minimal if all the paths $R(u,v)$ of R are shortest paths from u to v, denoted by R_m. To measure the efficiency of a routing deterministically, Chung, Coffman, Reiman and Simon [3] introduced the concept of the forwarding index of a routing.

The load of a vertex v (resp. an edge e) in a given routing R of $G=(V,E)$, denoted by $\xi(G,R,v)$ (resp. $\pi(G,R,e)$), is the number of paths of R going through v (resp. e), where v is not an end vertex. The parameters

$$\xi(G,R) = \max_{v\in V(G)} \xi(G,R,v) \text{ and } \pi(G,R) = \max_{e\in E(G)} \pi(G,R,e)$$

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are defined as the vertex forwarding index and the edge forwarding index of G with respect to R, respectively; and the parameters

$$\xi(G) = \min_R \xi(G,R) \text{ and } \pi(G) = \min_R \pi(G,R)$$

are defined as the vertex forwarding index and the edge forwarding index of G, respectively. Similary, we can define the parameters

$$\xi_m(G) = \min_{R_m} \xi(G,R_m) \text{ and } \pi_m(G) = \min_{R_m} \pi(G,R_m)$$

Clearly, $\xi(G) \leqslant \xi_m(G)$ and $\pi(G) \leqslant \pi_m(G)$. The equality however does not always hold. The original research of the forwarding indices is motivated by the problem of the maximizing network capacity. To minimizing the forwarding indices of a routing will result in maximizing the network capacity. Thus, the forwarding index problem has been studied widely by various researchers (see, for example, [2]~[7]).

Although, determining the forwarding index problem has been shown to be NP-complete by Saad [7], the exact values of the forwarding index of many important classes of graphs have been determined (see, for example, [2,5]).

The topological structure of a circulant network, is a circulant digraph, which is originally proposed by Elspas and Turner, denoted by $G(n; s_1, s_2, \cdots, s_k)$ or $G(n; S)$, consists of the vertex set $V = \{0, 1, \cdots, n-1\}$ and the edge set

$$E = \{(i,j): j - i \equiv s \pmod n, s \in S\},$$

where $S = \{s_1, s_2, \cdots, s_k\} \subseteq \{1, 2, \cdots, n-1\}$ and $s_1 < s_2 < \cdots < s_k$.

For a circulant digraph, the difference of an arc (i,j) is defined as $j-i \pmod n$. Let us notice that if an arc (i,j) has difference d then all the arcs $((i+k) \pmod n, (j+k) \pmod n)$ have difference d, and they are only arcs of $G(n;S)$ having this difference. If every vertex i of $G(n;S)$ can be expressed by $e_{i1}s_1 + e_{i2}s_2 + \cdots + e_{ik}s_k \pmod n$, where e_{ij} is a nonngative integer for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant k$, we call $E = \{e_{ij}: 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant k\}$ an expression of n with respect to S. Obviously, $G(n;S)$ is strongly connected if and only if there exist an expression E of n with respect to S. In this paper, we will give the following results.

Theorem 1 Let G be a strongly circulant digraph $G(n;S)$ with $S = \{s_1, s_2, \cdots, s_k\} \subseteq \{1, 2, \cdots, n-1\}$ and $s_1 < s_2 < \cdots < s_k$. Then

(1) $$\xi(G(n;S)) = \min_E \{\sum_{i=1}^{n-1} (e_{i1} + e_{i2} + \cdots + e_{ik})\} - (n-1);$$

(2) $$\min_E \{\sum_{i=1}^{n-1} (e_{i1} + e_{i2} + \cdots + e_{ik})/k\} \leqslant \pi(G(n;S)) \leqslant \min_E \{\max_{i \in \{1,2,\cdots,k\}} \{e_{1i} + e_{2i} + \cdots + e_{(n-1)i}\}\},$$

where $E = \{e_{ij}: 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant k\}$ is the expression of n with respect to S.

The proofs of the results are in Section 3. In Section 2, we will give some definitions and also recall some known results to by used in our proofs. In Section 4, as applications of these results, we will determine the vertex-forwarding index and the edge-forwarding index of some well-known graphs.

2 Definitions and Lemmas

Let Γ be a non-trivial finite group, S be a non-empty subset of Γ without the identity element e of Γ. Define a digraph G as follows.

$$V(G) = \Gamma; (x,y) \in E(G) \Leftrightarrow x^{-1}y \in S, \text{for any } x,y \in \Gamma.$$

Such defined a digraph, proposed by Cayley [1], is called a Cayley graph of the group Γ with respect to S, denoted by $G_\Gamma(S)$.

Lemma 1[8] Let G be a circulant digraph $G(n;S)$ with the set $S=\{s_1, s_2, \cdots, s_k\}$. Then

(a) G is a Cayley graph $C_{Z_n}(S)$ of the additive group Z_n of residue classes modulo n with respect to S, and, hence, is vertex-transitive and k-regular;

(b) G is strongly connected if and only if G is connected;

(c) G is strongly connected if and only if $g.c.d.(n; s_1, \cdots, s_k)=1$.

Lemma 2[6]) Let D be a strong digraph of order n. Then

(a) $(1/n)\sum_x \sum_{y\neq x}(d(x,y)-1) \leqslant \xi(D) \leqslant \xi_m(D) \leqslant (n-1)(n-2)$, and

(b) the equalities $(1/n)\sum_x \sum_{y\neq x}(d(x,y)-1)\xi(D) = \xi_m(D)$ hold if and only if there exists a routing of shortest paths in D that loads every edge equally.

Lemma 3[6] Let D be a strong digraph of order n. Then

(a) $(1/\mid E(D)\mid)\sum_{x,y\in V(D)} d(x,y) \leqslant \pi(D) \leqslant \pi_m(D) \leqslant (n-1)(n-2)+1$, and

(b) the equalities $(1/\mid E(D)\mid)\sum_{x,y\in V(D))} d(x,y)\pi(D) = \pi_m(D)$ hold if and only if there exists a routing of shortest paths in D that loads every edge equally.

Lemma 4[4]) If $D=(V,E)$ is a Cayley graph of order n, then, for any vertex u in V,

$$\xi(G) = \xi_m(G) = \sum_{v\in V, v\neq u} d(u,v) - (n-1).$$

3 Main Results

In this section, our aim is to give our main results on the vertex-forwarding index and the edge-forwarding index of the circulant digraphs.

By Lemma 1, $C(n;S)$ is a Cayley graph of the additive group Z_n of residue classes modulo n with respect to S. This means that the vertices of $C(n;S)$ are the elements of Z_n, two of them i and j being joined by an arc (i,j) if and only if $j-i\in S$. Let us denote by Γ the subgroup of autmorphisms of $C(n;S)$ defined by

$$\Gamma = \{\phi_k \in \mathrm{Aut}(G): k \in Z_n\},$$

with $\phi_k(i)=i+k \pmod n$ for any vertex i of $C(n;S)$. Obviously $\phi_k^{-1}(i)=\phi_{-k}(i)=i-k \pmod n$ for any vertex i of $C(n;S)$. If i and j are any two vertices of $C(n;S)$, there exist one and only one automorphism ϕ_k of Γ such that $\phi_k(i)=j$ and it is given by $k=j-i \pmod n$.

For $1\leqslant i\leqslant n-1$ and $i=e_{i1}s_1+e_{i2s_2}+\cdots+e_{ik}s_k$, let

$$R_E(0,i)=(0,s_1,2s_1,\cdots,e_{i1}s_1,e_{i1}s_1+s_2,e_{i1}s_1+2s_2,\cdots,e_{i1}s_1+e_{i2}s_2,\cdots,e_{i1}s_1+e_{i2}s_2+\cdots+e_{i(k-1)}s_{k-1}+s_k,e_{i1}s_1+e_{i2}s_2+\cdots+e_{i(k-1)}s_{k-1}+2s_k,\cdots,e_{i1}s_1+e_{i2}s_2+\cdots+e_{i(k-1)}s_{k-1}+e_{ik}s_k).$$

We can define a routing R_E as follows.

$$R_E(i,j)=\phi_i(R_E(0,\phi_{-i}(j)))\ \text{for}\ i,j\in V,$$

where ϕ_i is the unique element of Γ such that $i=\phi_i(0)$.

Lemma 5 Let G be a circulant digraph $G(n;S)$ with the set $S=\{s_1,s_2,\cdots,s_k\}$ and $s_1<s_2<\cdots<s_k$. Then

$$\xi(G(n;S))=\min_E\{\sum_{i=1}^{n-1}(e_{i1}+e_{i2}+\cdots+e_{ik})\}-(n-1),$$

where $E=\{e_{ij}:1\leqslant i\leqslant n-1,1\leqslant j\leqslant k\}$ is the expression of n with respect to S.

Proof For a vertex i in $V(G)$, let $d_i=d(0,i)$. By Lemma 1, $G(n;S)$ is a Cayley digraph, and so by Lemma 4 we have

$$\xi(G(n;S))=\sum_{i=1}^{n-1}d_i-(n-1).$$

Let $P_i=(0,v_1,v_2,\cdots,v_{d_i}(=i))$ be a shortest path between 0 and i, and d_{ij} the number of edges in P_i with difference s_j for $1\leqslant i\leqslant n-1,1\leqslant j\leqslant k$. Then $D=\{d_{ij}:1\leqslant i\leqslant n-1,1\leqslant j\leqslant k\}$ is an expression of n with respect to S and $\sum_{j=1}^{k}d_{ij}=d_i$. On the othere hand, for any expression E, we have $d_{i1}+d_{i2}+\cdots+d_{ik}\leqslant e_{i1}+e_{i2}+\cdots+e_{ik}$. Then

$$\xi(G(n;S))=\sum_{i=1}^{n-1}(d_{i1}+d_{i2}+\cdots+d_{ik})-(n-1)$$

$$=\min_E\{\sum_{i=1}^{n-1}(e_{i1}+e_{i2}+\cdots+e_{ik})\}-(n-1),$$

as required.

Lemma 6 For any expression E of n with respect to S, two arcs (i,j) and (i',j') of $G(n;S)$ have the same load in the routing R_E if they have the same difference.

Proof Let $\phi_{i'-i}$ be the unique element of Γ such that $\phi_{i'-i}(i)=i'$. Let $\phi_{j'-j}$ be the unique element of Γ such that $\phi_{j'-j}=j'$. Since (i,j) and (i',j') have the same difference, we have $\phi_{i'-i}=\phi_{j'-j}$. Then every path $R_E(x,y)$ going through (i,j) is transformed by $\phi_{i'-i}$ into a path of R_E, $R_E(\phi_{i'-i}(x),\phi_{i'-i}(y))$, going through (i',j'). Indeed, let ϕ_x be the element of Γ such that $\phi_x(0)=x$. We have $\phi_{i'-i}\cdot\phi_x\in\Gamma$ and $\phi_{i'-i}\cdot\phi_x(0)=\phi_{i'-i}(x)$ and so, from the definition of $R_E(\phi_{i'-i}(x),\phi_{i'-i}(y))$, we have

$$R_E(\phi_{i'-i}(x),\phi_{i'-i}(y))=\phi_{i'-i}\cdot\phi_x(R_E(0,(\phi_{i'-i}\cdot\phi_x)^{-1}(\phi_{i'-i}(y))))$$

$$=\phi_{i'-i}\cdot\phi_x(R_E(0,\phi_x^{-1}(y)))=\phi_{i'-i}(R_E(x,y)).$$

Moreover, as $\phi_{i'-i}$ is an automorphism, if $x\neq x'$ or $y\neq y'$, $R_E(\phi_{i'-i}(x),\phi_{i'-i}(y))$ is distinct from $R_E(\phi_{i'-i}(x'),\phi_{i'-i}(y'))$. Therefore,

$$\pi(C(n;S),R_E,(i,j))\leqslant\pi(C(n;S),R_E,(i',j')$$

and by symmetry, $\pi(C(n;S),R_E,(i,j))=\pi(C(n;S),R_E,(ii',j'))$.

We get the result.

Lemma 7 For the strongly connected circulant digraph $G(n;S)$ with $S=\{s_1,s_2,\cdots,s_k\}$ and $s_1<s_2<\cdots<s_k$. Then

$$\pi(G(n;S)) \leqslant \min_E\{\max_{i\in\{1,2,\cdots,k\}}\{e_{1i}+e_{2i}+\cdots+e_{(n-1)i}\}\},$$

where $E=\{e_{ij}:1\leqslant i\leqslant n-1,1\leqslant j\leqslant k\}$ is the expression of n with respect to S.

Proof For any expression E, we can define a routing R_E. Use B_j to denote the sum of the edges used in R_E with difference s_j for $1\leqslant j\leqslant k$. Then $B_j = n\times(\sum_{i=1}^{n-1}e_{ij})$. Since there exist only n edges with difference s_j in $G(n;S)$, by Lemma 6, we can get the result.

Combining Lemma 5, Lemma 7 with Lemma 3, we obtain the following results immediately.

Theorem 1 Let G be a strongly circulant digraph $G(n;S)$ with $S=\{s_1,s_2,\cdots,s_k\}\subseteq\{1,2,\cdots,n-1\}$ and $s_1<s_2<\cdots<s_k$. Then

(1) $\xi(G(n;S)) = \min_E\{\sum_{i=1}^{n-1}(e_{i1}+e_{i2}+\cdots+e_{ik})\}-(n-1)$;

(2) $\min_E\{\sum_{i=1}^{n-1}(e_{i1}+e_{i2}+\cdots+e_{ik})/k\} \leqslant \pi(G(n;S))$
$\leqslant \min_E\{\max_{i\in\{1,2,\cdots,k\}}\{e_{1i}+e_{2i}+\cdots+e_{(n-1)i}\}\},$

where $E=\{e_{ij}:1\leqslant i\leqslant n-1,1\leqslant j\leqslant k\}$ is the expression of n with respect to S.

4 Applocations

Example 1 The circulant digraph $G(n;S)$ with $n=d^m$, $S=\{1,d,\cdots,d^{m-1}\}$ and $d\geqslant 2$. For the vertex i, $1\leqslant i\leqslant d^m-1$, there exist an expression $A=\{a_{ij}:1\leqslant i\leqslant d^m-1,1\leqslant j\leqslant m\}$ of d^m with respect to S such that $i=a_{i1}+a_{i2}d+\cdots+a_{im}d^{m-1}$ for $0\leqslant a_{ij}\leqslant d-1$. We can check easily that

$$\xi(G(d^m;S))=\min_E\{\sum_{i=1}^{d^m-1}(e_{i1}+e_{i2}+\cdots+e_{ik})\}-(d^m-1)$$
$$=\sum_{i=1}^{d^m-1}(a_{i1}+a_{i2}+\cdots+a_{im})\}-(d^m-1).$$

Let $V_i=\{j:d^{i-1}\leqslant j\leqslant d^i-1\}$, where $1\leqslant i\leqslant m$. Use A_i denote the sum of the distance between 0 and j in R_A where $j\in V_i$, $1\leqslant i\leqslant m$.

We partition the set V_i into $d-1$ subsets

$$V_{i1}=\{d^{i-1},d^{i-1}+1,\cdots,d^{i-1}+(d^{i-1}-1)\},$$
$$V_{i2}=\{2d^{i-1},2d^{i-1}+1,\cdots,2d^{i-1}+(d^{i-1}-1)\},$$
$$\cdots,$$
$$V_{i(d-1)}=\{(d-1)d^{i-1},(d-1)d^{i-1}+1,\cdots,(d-1)d^{i-1}+(d^{i-1}-1)\}.$$

But the sum of the distance from 0 to all the vertices in V_{ij} is $jd^{i-1}+A_1+A_2+\cdots+A_{i-1}$, where $1\leqslant i\leqslant m$, $1\leqslant j\leqslant d-1$. So

$$A_1=1+2+\cdots+(d-1)=d(d-1)/2;$$
$$A_2=(d+A_1)+(2d+A_1)+\cdots+[(d-1)d+A_1]=(d-1)A_1+dA_1;$$

$$A_3 = [d^2 + (A_1 + A_2)] + [2d^2 + (A_1 + A_2)] + \cdots + [(d-1)d^2 + (A_1 + A_2)]$$
$$= (d-1)(A_1 + A_2) + d^2 A_1;$$
$$\cdots$$
$$A_m = [d^{m-1} + (A_1 + A_2 + \cdots + A_{m-1})] + [2d^{m-1} + (A_1 + A_2 + \cdots + A_{m-1})] +$$
$$\cdots + [(d-1)d^{m-1} + (A_1 + A_2 + \cdots + A_{m-1})]$$
$$= (d-1)(A_1 + A_2 + \cdots + A_{m-1}) + d^{m-1} A_1$$

Then

$$A_1 + A_2 + \cdots + A_m$$
$$= (A_1 + A_2 + \cdots + A_{m-1}) + [(d-1)(A_1 + A_2 + \cdots + A_{m-1}) + d^{m-1} A_1]$$
$$= (A_1 + A_2 + \cdots + A_{m-1})d + d^{m-1} A_1$$
$$\cdots$$
$$= A_1 d^{m-1} + (m-1) d^{m-1} A_1$$
$$= m d^{m-1} A_1 = (d-1) d^m m/2.$$

So we have $\xi(G(d^m;S)) = (d-1)d^m m/2 - (d^m - 1)$ and $\pi(G(d^m;S)) \geqslant (d-1)d^m/2$.

On the other hand, we need to show $\pi(G(d^m;S)) \leqslant (d-1)d^m/2$.

Suppose that the routing R_A is decided by A. By Theorem 1,

$$\pi(G(d^m;S)) \leqslant \max_{i\in\{1,2,\cdots,m\}} (a_{1i} + a_{2i} + \cdots + a_{(d^m-1)i}).$$

Let R_A^0 denote the paths with start vertex 0 and the end vertex $V(G) - \{0\}$ in R_A. For every $i \in \{0, 1, \cdots, m-1\}$, we use B_i to denote the sum of the edges with difference d^i in the paths of R_A^0. We partition the set $V(G)$ into $d^{m-(i+1)} \cdot d = d^{m-i}$ subsets where

$$b_{kj} = \{(k-1)d^{i+1} + (j-1)d^i, (k-1)d^{i+1}$$
$$+ (j-1)d^i + 1, \cdots, (k-1)d^{i+1} + (j-1)d^i + (d^i - 1)\}$$

and $1 \leqslant k \leqslant d^{m-(i+1)}, 1 \leqslant j \leqslant d$.

But the sum of the edges with difference d^i for the paths in R_A^0 with end vertex in each b_{kj} is $(j-1)d^i$ for $1 \leqslant k \leqslant d^{m-(i+1)}, 1 \leqslant j \leqslant d$.

Then

$$B_0 = d^{m-1} \cdot d^0 [1 + 2 + \cdots + (d-1)] = (d-1)d^m/2;$$
$$B_1 = d^{m-2} \cdot d^1 [1 + 2 + \cdots + (d-1)] = (d-1)d^m/2;$$
$$\cdots$$
$$B_{m-1} = d^0 \cdot d^{m-1} [1 + 2 + \cdots + (d-1)] = (d-1)d^m/2.$$

By Theorem 1, we have $\pi(G(d^m;S)) \leqslant (d-1)d^m/2$. Then $\pi(G(d^m;S)) = (d-1)d^m/2$.

Example 2 The directed cycle $\dot{C}_n = G(n;1)$ is a digraph consisting of the vertex set $V(C_n) = \{0, 1, \cdots, n-1\}$ and the edge set

$$E(C_n) = \{(i,j) : j - i \equiv 1 \pmod{n}\}.$$

Then

$$\xi(C_n) = [1 + 2 + \cdots + (n-1)] - (n-1) = (n-1)(n-2)/2,$$

and

$$\pi(C_n) = [1 + 2 + \cdots + (n-1)] = n(n-1)/2.$$

Example 3 (Heydemann et al[5]) Suppose $C_n = G(n; 1, n-1)$ is the cycle of length

$n, n \geqslant 3$. Then

(1) $\xi_m(C_n) = \xi(C_n) = [\frac{1}{4}(n-2)^2]$,

(2) $\pi_m(C_n) = \pi(C_n) = [\frac{1}{4}n^2]$.

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循环图的转发指数

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摘要：本文研究了循环网络的转发指数. 利用循环图是 Cayley 图的性质，获得循环图的点转发指数和边转发指数紧的上界和下界. 并确定了一些网络的转发指数.

关键词：路由选择；点转发指数；边转发指数；循环网络

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