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# The bondage numbers and efficient dominations of vertex-transitive graphs☆

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**Abstract**

The bondage number of a graph $G$ is the minimum number of edges whose removal results in a graph with larger domination number. A dominating set $D$ is called an efficient dominating set of $G$ if $|N^-[v] \cap D| = 1$ for every vertex $v \in V(G)$. In this paper we establish a tight lower bound for the bondage number of a vertex-transitive graph. We also obtain upper bounds for regular graphs by investigating the relation between the bondage number and the efficient domination. As applications, we determine the bondage number for some circulant graphs and tori by characterizing the existence of efficient dominating sets in these graphs.


*MSC:* 05C69; 05C20

*Keywords:* Bondage number; Efficient dominating set; Vertex-transitive graph

## 1. Introduction

In this paper we consider nonempty undirected graphs as well as digraphs. Usually both of them are included if we only say a "graph" without "directed" or "undirected". For the terminology and notation not given here, the reader is referred to [17]. Let $G = (V, E)$ be a simple graph with vertex-set $V = V(G)$ and edge-set $E = E(G)$. We use $(u, v)$ to denote an edge from $u$ to $v$. Note that $(u, v) \neq (v, u)$ when $G$ is a digraph, while $(u, v) = (v, u)$ when $G$ is undirected. The order of $G$ is the number of vertices in $G$, and denoted by $n(G) = |V|$. For a vertex $u$ in $G$, let $N^+(u) = \{v \in V : (u, v) \in E\}$, $N^-(u) = \{v \in V : (v, u) \in E\}$ and $N^+[u] = N^+(u) \cup \{u\}$, $N^-[u] = N^-(u) \cup \{u\}$. Note that $N^+(u) = N^-(u) = N(u)$ and $N^+[u] = N^-[u] = N[u]$ if $G$ is undirected. The distance between two vertices $u$ and $v$ is denoted by $d(u, v)$.

Let $G$ be a digraph. Given two vertices $u, v \in V(G)$ (maybe identical), *$u$ dominates $v$* if $v \in N^+[u]$. A subset $D$ of $V(G)$ is called a *dominating set* if every vertex of $G$ is dominated by at least one vertex in $D$. The minimum cardinality over all dominating sets in $G$ is called the *domination number*, and denoted by $\gamma(G)$. We call a dominating set a *$\gamma$-set*, if its cardinality is $\gamma(G)$. The *bondage number* of $G$, denoted by $b(G)$, is the cardinality of a smallest set of edges whose removal from $G$ results in a graph with domination number larger than $\gamma(G)$. Clearly, these concepts are also valid for an undirected graph.

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As a measure of the vulnerability of networks under link failures, the bondage number has attracted much attention. There are many research articles on this parameter for undirected graphs, such as [5,8,9,11,13], while only [5,12] have been known for digraphs. In this paper we deal with the bondage number for both undirected graphs and digraphs. We will establish a tight lower bound for the vertex-transitive graphs.

In order to obtain upper bounds, we need to consider the efficient domination. A dominating set $D$ is called an *efficient dominating set* of $G$ if $|N^-[v] \cap D| = 1$ for every vertex $v \in V(G)$. This concept is a measure of the efficiency of domination in graphs. Bange et al. [2] have proved that it is an NP-complete problem to determine whether a given graph has an efficient dominating set. In addition, it has been shown by Clark [6] that for a wide range of $p$, almost every random undirected graph $G \in \mathscr{G}(n, p)$ has no efficient dominating set. This means that undirected graphs possessing an efficient dominating set are rare. However, it is easy to show that every undirected graph has an orientation with an efficient dominating set [1]. Barkauskas and Host [3] showed that determining whether an arbitrary oriented graph has an efficient dominating set is NP-complete. Even so, the existence of efficient dominating sets for some graphs has been examined (see, for example, [7,14,15]).

Although bondage number is not a concept based directly upon the efficient domination number, it does have relation to efficient domination for the regular graphs. By investigating this relation we are able to establish some upper bounds. To show the applications of our bounds, we determine the bondage number for some circulant graphs and tori by characterizing the existence of efficient dominating sets in these graphs.

The rest of the paper is organized as follows. In Section 2 we present some preliminaries. We give our main results in Section 3, and their applications for circulant graphs, tori and cubes in Sections 4–6, respectively. In the last section we summarize our research on the vertex-transitive graphs.

## 2. Preliminaries

First recall two upper bounds for the bondage number of undirected graphs.

**Lemma 2.1** (*Hartnell and Rall [11]*). *If $G$ is an undirected graph, then $b(G) \leqslant d(x) + d(y) - 1 - |N(x) \cap N(y)|$ for any two adjacent vertices $x$ and $y$ in $G$.*

**Lemma 2.2** (*Carlson and Develin [5]*). *Let $G$ be a digraph and $(u, v) \in E(G)$. Then $b(G) \leqslant d(v) + d^-(u) - |N^-(u) \cap N^-(v)|$ where $d(v) = d^+(v) + d^-(v)$.*

Next we consider the efficient domination in regular graphs. From the definition, it is clear that a dominating set $D$ is efficient if and only if $\mathscr{N}^+[D] = \{N^+[v]: v \in D\}$ is a partition of $V(G)$.

**Lemma 2.3.** *Let $G$ be a $k$-regular graph. Then $\gamma(G) \geqslant n(G)/(k+1)$, with the equality if and only if $G$ has an efficient dominating set. In addition, if $G$ has an efficient dominating set, then every efficient dominating set must be a $\gamma$-set, and vice versa.*

**Proof.** Since $G$ is $k$-regular, then $|N^+[v]| = k + 1$ for each $v \in V(G)$. Hence $\gamma(G) \geqslant \lceil n(G)/(k+1) \rceil$. It is easy to observe that the equality holds if and only if there exists a dominating set $D$ such that $\mathscr{N}^+[D]$ is a partition of $V(G)$, equivalently, $D$ is an efficient dominating set.

Now suppose that $G$ has an efficient dominating set, i.e., $\gamma(G) = n(G)/(k+1)$. Then a dominating set $D$ is a $\gamma$-set if and only if $|D| = n(G)/(k+1)$. On the other hand, $D$ is efficient if and only if $|D| = n(G)/(k+1)$. The lemma follows. □

We now introduce a parameter to bound $b(G)$ below. Let $e$ be an edge and $D$ a dominating set in $G$. We say *$e$ supports $D$* if $e \in (D, \bar{D}) = \{(u, v) \in E(G) : u \in D, v \notin D\}$. Denote by $s(G)$ the minimum number of edges which support all $\gamma$-sets in $G$.

**Lemma 2.4.** *$b(G) \geqslant s(G)$, with the equality if $G$ is regular and has an efficient dominating set.*

**Proof.** Assume $E' \subseteq E(G)$ with $|E'| < s(G)$. Then $E'$ cannot support all $\gamma$-sets in $G$. Let $D$ be a $\gamma$-set not supported by $E'$. We prove by contradiction that $D$ is still a dominating set in $G - E'$.

Suppose to the contrary that there exists a vertex $v \in V(G)\backslash D$ such that $D$ cannot dominate it in $G - E'$. Since $D$ is a dominating set in $G$, there exists a vertex $u \in D$ which dominates $v$ in $G$. Hence $(u, v) \in E(G)$ supports $D$, which implies that $(u, v) \notin E'$. It follows that $u \in D$ dominates $v$ in $G - E'$, a contradiction. Thus, $\gamma(G - E') = \gamma(G)$ for any set $E' \subseteq E(G)$ with $|E'| < s(G)$, and so $b(G) \geqslant s(G)$.

Now let $G$ be a regular graph with an efficient dominating set, and $E'$ a set of $s(G)$ edges which supports all $\gamma$-sets. We show that any $\gamma$-set $D$ is not a dominating set in $H = G - E'$. Since $E'$ supports $D$, then there exists an edge $(u, v) \in E'$ such that $u \in D$ and $v \notin D$. Hence $v$ is not dominated by $u$ in $H$. By Lemma 2.3, $D$ is efficient, which implies that $D$ dominates $v$ only by $u$. Thus, $D$ cannot dominates $v$ in $H$. It follows that $\gamma(H) > \gamma(G)$ and $b(G) \leqslant |E'| = s(G)$. The result follows. □

Next we present the bondage number of cycles, from which we will immediately see the tightness of some bounds established in Section 3.

**Proposition 2.5** (*Fink et al. [9]*). *Let $C_n$ be the undirected cycle of length $n \geqslant 2$. Then*

$$\gamma(C_n) = \lceil n/3 \rceil \quad \text{and} \quad b(C_n) = \begin{cases} 3 & \text{if } n \equiv 1 \pmod 3, \\ 2 & \text{otherwise}. \end{cases}$$

**Proposition 2.6** (*Huang and Xu [12]*). *Let $\overrightarrow{C}_n$ be a directed cycle of length $n \geqslant 2$. Then*

$$\gamma(\overrightarrow{C}_n) = \lceil n/2 \rceil \quad \text{and} \quad b(\overrightarrow{C}_n) = \begin{cases} 3 & \text{if } n \text{ is odd}, \\ 2 & \text{if } n \text{ is even}. \end{cases}$$

We conclude the section with a brief recall of vertex-transitive graphs. A graph $G$ is called to be *vertex-transitive* if its automorphism group $Aut(G)$ acts transitively on its vertex-set $V(G)$. A vertex-transitive graph is regular. A *Cayley graph* $G = C(\Gamma, S)$ is a digraph $G$ with vertex-set $V(G) = \Gamma$ being a finite group and edge-set $E(G) = \{(x, y) : x^{-1}y \in S\}$ where $S \subseteq \Gamma$ and the identity element $e \notin S$. If $G = C(\Gamma, S)$ is a *symmetric digraph*, i.e., $(x, y) \in E(G)$ if and only if $(y, x) \in E(G)$, then we usually consider $G$ as an undirected graph. A Cayley graph is vertex-transitive, and the cartesian product $G_1 \times G_2$ of two Cayley graphs $G_1$ and $G_2$ remains a Cayley graph, with the regularity equal to the sum of the regularities of $G_1$ and $G_2$. (See Sections 2.2 and 2.3 in [16].)

## 3. Main results

In this section we present our main results about the bondage number of vertex-transitive graphs. First we establish a lower bound whose tightness can be easily observed from Propositions 2.5 and 2.6.

**Theorem 3.1.** *Let $G$ be a vertex-transitive graph. Then*

$$b(G) \geqslant \begin{cases} \lceil n(G)/2\gamma(G) \rceil & \text{if } G \text{ is undirected}, \\ \lceil n(G)/\gamma(G) \rceil & \text{if } G \text{ is directed}. \end{cases}$$

**Proof.** Assume $V(G) = \{v_1, \ldots, v_n\}$. Let $\mathscr{D}_i$ be the family of all $\gamma$-sets that contain $v_i$ in $G$. We first show that $|\mathscr{D}_i| = |\mathscr{D}_j|$ for any $i$ and $j$. Since $G$ is vertex-transitive, there exists an automorphism $\phi$ of $G$ such that $\phi(v_i) = v_j$. Clearly $\phi(D_i) \neq \phi(D_i')$ for any distinct $D_i, D_i' \in \mathscr{D}_i$. On the other hand, for any $D_j \in \mathscr{D}_j$, it holds that $\phi^{-1}(D_j) \in \mathscr{D}_i$ and $\phi(\phi^{-1}(D_j)) = D_j$. Thus, $\phi$ is a bijection from $\mathscr{D}_i$ to $\mathscr{D}_j$, and so $|\mathscr{D}_i| = |\mathscr{D}_j| = s$ for any $i, j \in \{1, 2, \ldots, n\}$.

Note that $\bigcup_{i=1}^{n} \mathscr{D}_i$ contains all $\gamma$-sets of $G$ and every $\gamma$-set appears $\gamma(G)$ times in it. Hence there are exactly $n(G)s/\gamma(G)$ $\gamma$-sets in $G$.

If $G$ is undirected, then an edge $(v_i, v_j)$ may only support those $\gamma$-sets in $\mathscr{D}_i$ and $\mathscr{D}_j$ whose number is at most $2s$. Hence it needs at least $(n(G)s/\gamma(G))/2s$ edges to support $\bigcup_{i=1}^{n} \mathscr{D}_i$. It follows from Lemma 2.4 that $b(G) \geqslant s(G) \geqslant \lceil n(G)/2\gamma(G) \rceil$.

If $G$ is directed, then an edge $(v_i, v_j)$ only supports those $\gamma$-sets in $\mathscr{D}_i$. Hence $b(G) \geqslant s(G) \geqslant \lceil n(G)s/(\gamma(G)s) \rceil = \lceil n(G)/\gamma(G) \rceil$. The theorem follows. □

Next we will establish an upper bound of $b(G)$. To this aim, we introduce the following terminology, which generalizes the concept of edge cover of a graph $G$. For $V' \subseteq V(G)$ and $E' \subseteq E(G)$, we say $E'$ *covers* $V'$ and call $E'$ an *edge cover* for $V'$ if there exists an edge $(u, v) \in E'$ for any vertex $u \in V'$. For any $v \in V(G)$, denote by $\beta'[v]$ the minimum cardinality over all edge covers for $N^-[v]$. It is easy to see that $\lceil (k+1)/2 \rceil \leqslant \beta'[v] \leqslant k$ when $G$ is undirected and $\beta'[v] = k + 1$ when $G$ is directed. The following upper bound is tight by Propositions 2.5 and 2.6.

**Theorem 3.2.** *If $G$ is a $k$-regular graph and $n(G) = \gamma(G)(k+1)$, then $b(G) \leqslant \beta'[v]$ for any $v \in V(G)$.*

**Proof.** For any $v \in V(G)$, let $E'$ be the smallest set of edges that covers $N^-[v]$. To dominate $v$, any $\gamma$-set $D$ in $G$ must contain some vertex $w$ in $N^-[v]$. Since $E'$ covers $N^-[v]$, then $w$ dominates at most $k$ vertices in $H = G - E'$. Hence $D$ dominates at most

$$(|D| - 1)(k+1) + k < \gamma(G)(k+1) = n(G) = n(H)$$

vertices, which implies that $D$ is not a dominating set in $H$. Thus $\gamma(H) > \gamma(G)$ and $b(G) \leqslant |E'| = \beta'[v]$. □

Theorem 3.2 holds subject to the condition $n(G) = \gamma(G)(k+1)$. Even if this condition is not satisfied, we can obtain another upper bound for $b(G)$, provided that $n(G)$ is close to $\gamma(G)(k+1)$, i.e., $\gamma(G)$ is close to its lower bound $n(G)/(k+1)$.

**Theorem 3.3.** *If $G$ is a $k$-regular graph, then*

$$b(G) \leqslant \begin{cases} k & \text{if } G \text{ is undirected and } n(G) \geqslant \gamma(G)(k+1) - k + 1, \\ k+1+l & \text{if } G \text{ is directed and } n(G) \geqslant \gamma(G)(k+1) - l, \quad 0 \leqslant l \leqslant k-1. \end{cases}$$

**Proof.** First assume $G$ is undirected. For any $v \in V(G)$ let $E' = \{(u, v) \in E(G) : u \in N(v)\}$. Then any minimum dominating set $D$ in $H = G - E'$ must contain $v$. But $v$ dominates only itself in $H$. If $|D| = \gamma(G)$, then $D$ dominates at most

$$(|D| - 1)(k+1) + 1 = \gamma(G)(k+1) - k < n(G) = n(H)$$

vertices in $H$, a contradiction. Hence $\gamma(H) = |D| > \gamma(G)$ and $b(G) \leqslant |E'| = k$.

Now assume $G$ is a digraph. For $v \in V(G)$ let $N^+(v) = \{w_1, \ldots, w_k\}$ and $E' = \{(u, v) : u \in N^-(v)\} \cup \{(v, w_i) : 1 \leqslant i \leqslant l+1\}$, where $0 \leqslant l \leqslant k-1$. Then any minimum dominating set $D$ in $H = G - E'$ must contain $v$. But $v$ dominates only $k - l$ vertices in $H$. If $|D| = \gamma(G)$, then in $H$, $D$ dominates at most

$$(|D| - 1)(k+1) + k - l = \gamma(G)(k+1) - l - 1 < n(G) = n(H)$$

vertices, a contradiction. Hence $\gamma(H) = |D| > \gamma(G)$ and $b(G) \leqslant |E'| = k + 1 + l$. □

In view of Lemma 2.3, the following corollary is merely a simple combination of Theorems 3.1 and 3.2. We will use this corollary to obtain results for special classes of graphs.

**Corollary 3.4.** *Let $G$ be a vertex-transitive graph of degree $k$. If $G$ has an efficient dominating set, then*

$$\begin{cases} \left\lceil \dfrac{k+1}{2} \right\rceil \leqslant b(G) \leqslant k & \text{if } G \text{ is undirected}, \\ b(G) = k+1 & \text{if } G \text{ is directed}. \end{cases}$$

**Corollary 3.5.** *If $G$ is an undirected vertex-transitive cubic graph with girth $g(G) \leqslant 5$ and $n(G) = 4\gamma(G)$, then $b(G) = 2$.*

**Proof.** Since $G$ is a cubic graph of order $n(G) = 4\gamma(G)$, then by Lemma 2.3, any $\gamma$-set in $G$ is efficient. By Corollary 3.4, $2 \leqslant b(G) \leqslant 3$. Thus, we only need to show $b(G) \leqslant 2$. Let $D$ be an efficient dominating set in $G$. By the proof of

Theorem 3.1, there are $n(G)s/\gamma(G) = 4s$ distinct efficient dominating sets in $G$, provided that a vertex of $G$ belongs to $s$ distinct efficient dominating sets.

If $g = 3$, then there exists a cycle $(u_1, u_2, u_3)$ of length 3. Suppose that $v_1$ is the neighbor of $u_1$ such that $v_1$ is not in the cycle, then $E' = \{(u_1, v_1), (u_2, u_3)\}$ covers $N[u_1]$. By Theorem 3.2, $b(G) \leqslant \beta'[u_1] \leqslant |E'| = 2$.

If $g = 4$ or 5, then there exists a cycle $(u_1, u_2, u_3, u_4)$ or $(u_1, u_2, u_3, u_4, u_5)$. For any $1 \leqslant i < j \leqslant 4$, it is easy to observe that $d(u_i, u_j) \leqslant 2$. Note that two distinct vertices $u$, $v$ in $D$ satisfy $d(u, v) \geqslant 3$, since $N[u] \cap N[v] = \emptyset$. Hence there exists no efficient dominating set containing both $u_i$ and $u_j$. Suppose that $\mathscr{D}_i$ is the family of efficient dominating sets containing $u_i$ for $i = 1, 2, 3, 4$. Then $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$. It follows that $E' = \{(u_1, u_2), (u_3, u_4)\}$ supports exactly $4s$ efficient dominating sets, i.e., all sets in $\bigcup_{i=1}^{4} \mathscr{D}_i$. Since there are only $4s$ distinct efficient dominating sets in $G$, then by Lemma 2.4, $b(G) = s(G) \leqslant |E'| = 2$. □

**Remark.** The above proof leads to a byproduct. In the case of $g = 5$ we have $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$ for $i = 1, 2, \ldots, 5$. Then $G$ has at least $5s$ efficient dominating sets. But there are only $n(G)s/\gamma(G) = 4s$ distinct efficient dominating sets in $G$. This contradiction implies that an undirected vertex-transitive cubic graph with girth five has no efficient dominating set. But a similar argument for $g(G) = 3, 4$ or $g(G) \geqslant 6$ could not give any contradiction. This is consistent with the result that CCC($n$), a vertex-transitive cubic graph with girth $n$ if $3 \leqslant n \leqslant 8$, or girth 8 if $n \geqslant 9$, has efficient dominating sets for all $n \geqslant 3$ except $n = 5$. (See Section 6.)

## 4. Circulant graphs

Circulant graphs are an important class of topological structures of interconnection networks. It has been widely used for decades in the telecommunication network, VLSI design and distribute computation. There are many good properties in circular graphs, such as their symmetry, fault-tolerance, routing capabilities, and so on.

A *circulant graph* $\overrightarrow{C}(n; S)$ of order $n$ is a Cayley graph $C(Z_n, S)$, where $Z_n = \{0, 1, \ldots, n-1\}$ is the addition group of order $n$ and $S$ is a nonempty subset of $Z_n$ without the identity element. It is well known that $\overrightarrow{C}(n; S)$ is a vertex-transitive digraph of degree $|S|$. If $S^{-1} = S$, then $\overrightarrow{C}(n; S)$ is symmetric and we view it undirected.

### *4.1. Double loop network*

First we consider the *double loop network* whose topological structure is the circulant graph $\overrightarrow{C}(n; \{1, s\}) = \overrightarrow{C}(n; 1, s)$ or $C(n; \{\pm 1, \pm s\}) = C(n; 1, s)$, where $2 \leqslant s \leqslant n-2$.

**Lemma 4.1.** *Let $G = \overrightarrow{C}(n; 1, s)$. Then $\lceil n/3 \rceil \leqslant \gamma(G) \leqslant \lceil n/2 \rceil$, and $G$ has an efficient dominating set if and only if $3 \mid n$ and $s \equiv 2 \pmod 3$. In addition, all efficient dominating sets in $G$ have the form $D_i = \{v \in V(G): v \equiv i \pmod 3\}$.*

**Proof.** Since $G$ is 2-regular, we have $\gamma(G) \geqslant \lceil n/3 \rceil$ by Lemma 2.3. On the other hand, $\gamma(G) \leqslant \gamma(\overrightarrow{C}_n) = \lceil n/2 \rceil$, since $\overrightarrow{C}_n$ is a spanning subgraph of $G$.

Now consider efficient dominating sets in $G$. It is easy to verify that $D_i = \{i, 3+i, \ldots, 3t+i\}$ is an efficient dominating set, provided $3|n$ and $s \equiv 2 \pmod 3$. Conversely, if $G$ has an efficient dominating set $D$, then by Lemma 2.3, we have $n = 3(t+1)$ for some integer $t$. We show that there exists some integer $i$ such that $D = D_i$.

Suppose to the contrary that there exists an efficient dominating set $D$ such that $D \neq D_i$ for any $i$. Note that $D_i$'s are all those subsets of $V(G)$ in which any two successive vertices have difference 3. Thus, we assume $D$ contains 0 and $v \in \{1, 2\}$, without loss of generality. Clearly $v \neq 1$, since $1 \in N^+[0] = \{0, 1, s\}$. Hence $v = 2$ and $N^+[v] = \{2, 3, s+2\}$. Since $1, s \in N^+[0]$, then $s+1$ belongs to $D$ in order to dominate itself. It follows that $s+2 \in N^+[2] \cap N^+[s+1]$, a contradiction.

Thus, $D = D_i$ for some $i$ and $V(G) = \bigcup_{v \in D_i} N^+[v] = D_i \cup D_{i+1} \cup D_{i+s}$. Since $D_p = D_q$ if and only if $p \equiv q \pmod 3$, we have $D_{i+s} = D_{i+2}$ and $s \equiv 2 \pmod 3$. □

Lemma 4.1 establishes an upper bound for the domination number of $G = \overrightarrow{C}(n; 1, s)$. From Theorem 3.1 together with Lemma 2.2, we immediately have $2 \leqslant b(G) \leqslant 6$. Furthermore, by Lemma 4.1 and Corollary 3.4 we can determine the bondage number of $\overrightarrow{C}(n; 1, s)$ that has an efficient dominating set.

**Theorem 4.2.** *Let* $G = \overrightarrow{C}(n; 1, s)$. *If* $3 \mid n$ *and* $s \equiv 2 \pmod 3$, *then* $b(G) = 3$.

Next we consider the undirected double loop network $C(n; 1, s)$ analogously. If $n$ is even and $s = n/2$, then $C(n; 1, s)$ is a Harary graph $H^1_{3,n}$ which will be considered in the next subsection. Thus, we assume $s \neq n/2$, and then $C(n; 1, s)$ is 4-regular.

**Lemma 4.3.** *Let* $G = C(n; 1, s)$ *with* $s \neq n/2$. *Then* $\lceil n/5 \rceil \leqslant \gamma(G) \leqslant \lceil n/3 \rceil$, *and* $G$ *has an efficient dominating set if and only if* $5 \mid n$ *and* $s \equiv \pm 2 \pmod 5$. *In addition, all efficient dominating sets in* $G$ *have the form* $D_i = \{v \in V(G) : v \equiv i \pmod 5\}$.

**Proof.** Since $G$ is 4-regular, then $\gamma(G) \geqslant \lceil n/5 \rceil$ by Lemma 2.3. Note that $C_n$ is a spanning subgraph of $G$ with domination number $\gamma(C_n) = \lceil n/3 \rceil$. Hence $\gamma(G) \leqslant \lceil n/3 \rceil$.

Now consider the efficient domination in $G$. It is easy to observe that $D_i = \{i, 5+i, \ldots, 5t+i\}$ is an efficient dominating set in $G$, provided $5 \mid n$ and $s \equiv \pm 2 \pmod 5$. Conversely, if $G$ has an efficient dominating set $D$, then by Lemma 2.3, $n = 5(t+1)$ for some integer $t$. We show for any $s$ that $D = D_i$ for some integer $i$.

Suppose to the contrary that there exists an efficient dominating set $D$ such that $D \neq D_i$ for any $i$. Note that $D_i$'s are all those subsets of $V(G)$ in which two successive vertices have difference 5. Thus, assume $D$ contains 0 and $v \in \{1, 2, 3, 4\}$, without loss of generality. Then we deduce a contradiction.

Clearly $v \neq 1, 2$, for $1 \in N[0] = \{0, \pm 1, \pm s\}$. If $v = 3$ then $N[3] = \{2, 3, 4, s \pm 3\}$. To dominate $s+1$, the only choice for $D$ is $2s+1$, since $1, s \in N[0]$ and $s+1, s+2$ dominate $s \in N[0]$, $s+3 \in N[3]$, respectively; analogously, $2s+2$ belongs to $D$ to dominate $s+2$. But $2s+1 \in N[2s+2]$, a contradiction.

If $v = 4$ then $N[4] = \{3, 4, 5, 4 \pm s\}$. First we show $s+2 \in D$. Otherwise, $D$ has to contain $2s+1$ to dominate $s+1$. Then $2s+2$ also belongs to $D$ to dominate $s+2$, which contradicts that $2s+1 \in N[2s+2]$. Hence $s+2 \in D$ and $N[s+2] = \{2, s+1, s+2, s+3, 2s+2\}$. Next we show that $2-s \in D$. Otherwise, $D$ has to contain $3-2s$ to dominate $3-s$; analogously we have that $2-2s$ also belongs to $D$ to dominate $2-s$, which contradicts to $3-2s \in N[2-2s]$. Hence $2-s \in D$. But $2 \in N[s+2] \cap N[2-s]$, a contradiction.

Thus, $D = D_i$ for some integer $i$ and $V(G) = N[D_i] = D_i \cup D_{i+1} \cup D_{i-1} \cup D_{i+s} \cup D_{i-s}$. Note that $D_p = D_q$ if and only if $p \equiv q \pmod 5$. Hence $D_s \cup D_{-s} = D_{i+2} \cup D_{i-2}$ and $s \equiv \pm 2 \pmod 5$. □

By Theorem 3.1 and Lemma 2.1 we obtain $2 \leqslant b(G) \leqslant 7$ for $G = C(n; 1, s)$. Furthermore, we can determine the bondage number of $C(n; 1, s)$ if it has an efficient dominating set.

**Theorem 4.4.** *Let* $G = C(n; 1, s)$. *If* $5 \mid n$ *and* $s \equiv \pm 2 \pmod 5$, *then* $b(G) = 3$.

**Proof.** Corollary 3.4 and Lemma 4.3 yield $3 \leqslant b(G) \leqslant 4$. We only need to prove $b(G) \leqslant 3$. By Lemma 4.3, $D_i$'s are all efficient dominating sets in $G$, and each $D_i$ contains vertices $v \in \{0, 1, 2, 3, 4\}$. Then $E' = \{(0, 1), (1, 2), (3, 4)\}$ supports all $D_i$'s. By Lemma 2.4, $b(G) = s(G) \leqslant |E'| = 3$. □

*4.2. Harary graphs*

Now we consider another class of circulant graph $C(n; S)$, where $|S|$ need not be two. The Harary graph $H_{k,n}$ has been discussed in [4], which is an undirected graph of order $n$ and connectivity $k$ with minimum number of edges. Given $k < n$, place $n$ vertices $0, 1, \ldots, n-1$ around a circle. If $k$ is even, $H_{k,n}$ is constructed by making each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle. If $k$ is odd and $n$ is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $(k-1)/2$ vertices in each direction and to the diametrically opposite vertex around the circle. When $k$ and $n$ are both odd, construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $(i, i+(n-1)/2)$ for $0 \leqslant i \leqslant (n-1)/2$.

The Harary graphs constructed in the three cases above are said of type-0, type-1, type-2, and denoted by $H^0_{k,n}$, $H^1_{k,n}$, $H^2_{k,n}$, respectively. It is easy to observe that both $H^0_{k,n}$ and $H^1_{k,n}$ are circulant graphs of degree $k$, while $H^2_{k,n}$ is not regular, for the vertex $(n-1)/2$ is of degree $k+1$ and others of $k$. Thus, we only consider the Harary graphs of type-0 and type-1.

**Lemma 4.5.** $\gamma(H^0_{k,n}) = \lceil n/(k+1)\rceil$.

**Proof.** Clearly, $\gamma(H^0_{k,n}) \geqslant \lceil n/(k+1)\rceil$, since $H^0_{k,n}$ is $k$-regular. According to the definition of $H^0_{k,n}$,

$$D = \left\{ j(k+1)\colon j = 0, 1, \ldots, \left\lceil \frac{n}{k+1} \right\rceil - 1 \right\}$$

is a domination set in $H^0_{k,n}$. Thus, $\gamma(H^0_{k,n}) \leqslant |D| = \lceil n/(k+1)\rceil$. □

**Theorem 4.6.**

$$\left\lceil n \Big/ \left( 2 \left\lceil \frac{n}{k+1} \right\rceil \right) \right\rceil \leqslant b(H^0_{k,n}) \leqslant \begin{cases} k+1 & \text{if } n \equiv 1 \pmod{k+1}, \\ k & \text{if } n \not\equiv 1 \pmod{k+1}. \end{cases}$$

*In addition,* $b(H^0_{k,n}) = \lceil (k+1)/2 \rceil$ *if* $(k+1)|n$.

**Proof.** By Lemma 4.5 and Theorem 3.1, $b(H^0_{k,n}) \geqslant \lceil n/(2\lceil n/(k+1)\rceil)\rceil$. Since two adjacent vertices in $H^0_{k,n}$ have $k-2$ common neighbors, it follows from Lemma 2.1 that $b(H^0_{k,n}) \leqslant k + k - 1 - (k-2) = k+1$.

Suppose $n = p(k+1)+q$, $1 \leqslant q \leqslant k+1$. Then $\gamma(H^0_{k,n}) = \lceil n/(k+1)\rceil = p+1$, and $n \geqslant (p+1)(k+1) - k + 1$ if and only if $q \geqslant 2$. It follows from Theorem 3.3 that $b(H^0_{k,n}) \leqslant k$ if $q \neq 1$.

Now consider $(k+1)|n$. Then $\gamma(H^0_{k,n}) = n/(k+1)$, i.e. $H^0_{k,n}$ has an efficient dominating set. Corollary 3.4 yields that $\lceil (k+1)/2 \rceil \leqslant b(H^0_{k,n}) \leqslant \beta'[v]$ for any $v \in V(H^0_{k,n})$. Let $v = k/2$. Then $N[v] = \{0, 1, \ldots, k\}$ and

$$E' = \{(0,1), (2,3), \ldots, (k-2, k-1), (k-1, k)\}$$

covers $N[v]$. Thus, $b(H^0_{k,n}) \leqslant \beta'[v] \leqslant |E'| = \lceil (k+1)/2 \rceil$. The theorem follows. □

**Lemma 4.7.** *Let* $G = H^1_{k,n}$. *Then* $G$ *has an efficient dominating set if and only if* $n = (k+1)p$ *for an odd* $p$, *and all efficient dominating sets in* $G$ *have the form* $D_i = \{v \in V(G)\colon v \equiv i \pmod{k+1}\}$.

**Proof.** Define $u \sim v$ if $u \equiv v \pmod{k+1}$ for any $u, v \in Z_n$. It is clear that "$\sim$" is an equivalent relation and the equivalent class is $D_i = \{v \in V(G)\colon v \equiv i \pmod{k+1}\}$. If $n = (k+1)p$ for an odd $p$, then $D_{n/2} = D_{(k+1)/2}$, and so

$$\begin{aligned} \bigcup_{v \in D_0} N[v] &= \left( \bigcup_{j=-(k-1)/2}^{(k-1)/2} D_j \right) \bigcup D_{n/2} \\ &= \left( \bigcup_{j=0}^{(k-1)/2} D_j \right) \bigcup \left( \bigcup_{j=(k+3)/2}^{k} D_j \right) \bigcup D_{(k+1)/2} \\ &= V(G). \end{aligned}$$

Thus, $D_0$ is a dominating set in $G$. Since $|D_0| = n/(k+1)$, then $D_0$ is efficient.

Conversely, if $G$ has an efficient dominating set $D$, then by Lemma 2.3, $n = (k+1)p$ for some integer $i$. First we show $D = D_i$ for some integer $i$. Suppose to the contrary that $D \neq D_i$, and we assume $D$ contains two vertices 0 and $v \in \{1, \ldots, k\}$, without loss of generality. Clearly, $(k+1)/2 \in N[v]$. We show that there exists another vertex in $D$ which dominates $(k+1)/2$, which leads to a contradiction.

Consider the vertex $w = n/2 + 1$, and suppose that $D$ dominates $w$ by $u$. Clearly $u \neq 1 = w - n/2$, since $1 \in N[0]$. Hence $w - (k-1)/2 \leqslant u \leqslant w + (k-1)/2$. If $w - (k-1)/2 \leqslant u \leqslant w + (k-3)/2$, then $n/2 \in N[u]$, which contradicts $n/2 \in N[0]$. Hence $u = w + (k-1)/2$. It follows that $(k+1)/2 \in N[u] \cup N[v]$, a contradiction.

Therefore $D = D_i$ for any efficient dominating set $D$ in $G$. By this property we prove that if $G$ has an efficient dominating set $D$, then $p$ is odd. Assume $D = D_0$, without loss of generality. If $p$ is even, then $n/2 \in D_0$. It follows that both 0 and $n/2$ in $D_0$ dominate $n/2$, which implies that $D_0$ is inefficient. That contradiction completes the proof of the lemma. □

**Theorem 4.8.** *If $n=(k+1)p$ for an odd $p$, then $b(H^1_{k,n})=(k+1)/2$.*

**Proof.** By Lemma 4.7 and Corollary 3.4, we have $(k+1)/2 \leqslant b(H^1_{k,n}) \leqslant \beta'[v]$ for any vertex $v$. Denote an edge $(-i,-j)$ by $-(i,j)$ and let

$$E' = \begin{cases} \left\{\left(0,\frac{n}{2}\right), \pm\left(\frac{k-1}{2},\frac{k-3}{2}\right),\ldots,\pm(2,1)\right\} & \text{if } \frac{k-1}{2} \text{ is even,} \\ \left\{\left(0,\frac{n}{2}\right), \pm\left(\frac{k-1}{2},\frac{k-3}{2}\right),\ldots,\pm(3,2),(1,-1)\right\} & \text{if } \frac{k-1}{2} \text{ is odd.} \end{cases}$$

Clearly $E'$ covers $N[0]=\{0,\pm1,\pm2,\ldots,\pm(k-1)/2,n/2\}$. Hence $b(H^1_{k,n}) \leqslant \beta'[v] \leqslant |E'| = (k+1)/2$. The result follows. □

We now consider the directed circulant graph $\overrightarrow{C}(n;S)$ with $S=\{1,2,\ldots,k\}$.

**Lemma 4.9.** *Let $G=\overrightarrow{C}(n;S)$ with $S=\{1,2,\ldots,k\}$. Then $\gamma(G)=\lceil n/(k+1)\rceil$.*

**Proof.** Clearly $\gamma(G) \geqslant \lceil n/(k+1)\rceil$, since $G$ is $k$-regular. On the other hand, $D=\{0,k+1,\ldots,(\lceil n/(k+1)\rceil-1)(k+1)\}$ is a dominating set in $G$, with $|D|=\lceil n/(k+1)\rceil$. Hence $\gamma(G)=\lceil n/(k+1)\rceil$. □

**Theorem 4.10.** *Let $G=\overrightarrow{C}(n;S)$ with $S=\{1,2,\ldots,k\}$ and $n=p(k+1)+q$, where $0 \leqslant q \leqslant k$. Then*

$$\lceil n/(\lceil n/(k+1)\rceil)\rceil \leqslant b(G) \leqslant 2k+2-q$$

*if $q \geqslant 1$ and $b(G)=k+1$ if $q=0$.*

**Proof.** Theorem 3.1 and Lemma 2.2 immediately yield that

$$\lceil n/(\lceil n/(k+1)\rceil)\rceil \leqslant b(G) \leqslant 2k+k-(k-1)=2k+1.$$

Assume $n=p(k+1)+q$, where $0 \leqslant q \leqslant k$. Then $\gamma(G)=p+1$ and $n(G)=(p+1)(k+1)-(k+1-q)$. If $q \geqslant 2$ then $l=k+1-q \leqslant k-1$ and it follows from Theorem 3.3 that $b(G) \leqslant k+1+l=2k+2-q$. If $q=1$ then $b(G) \leqslant 2k+1=2k+2-q$. If $q=0$ then $\gamma(G)=n/(k+1)$ and $b(G)=k+1$ by Lemma 2.3 and Corollary 3.4. □

**Remark.** Given an even $n$, the circulant graph $G=\overrightarrow{C}(n;S)$ with $S=\{1,2,\ldots,k,n/2\}$ has no efficient dominating set. In fact, if $D$ is a dominating set in $G$ containing 0, then $N^+[0]=\{0\}\cup S$. Note that $D$ contains a vertex in $N^-[n/2+1]$. It is easy to observe that $D$ is inefficient.

## 5. Tori

The $s \times t$ *torus* is the cartesian product $C_s \times C_t$ of two undirected cycles. Tori are Cayley graphs, since the cycles are Cayley graphs and the product of Cayley graphs is still a Cayley graph. Kang et al. [13] showed $b(G)=4$ for $G=C_n \times C_4$ with $n \geqslant 4$. In this section, we show $b(G)=3$ if $G=C_s \times C_t$ with both $s$ and $t$ being multiples of 5.

First we present some results on the efficient domination in the torus $C_s \times C_t$, due to Gu et al. [10]. Since $C_n = C(Z_n,\{\pm1\})$, then $C_s \times C_t = C(Z_s \times Z_t, \{(0,\pm1),(\pm1,0)\})$.

**Lemma 5.1** (*Gu et al. [10]*). *Suppose that $D$ is an efficient dominating set in $C_s \times C_t$ with $s \geqslant 3$ and $t \geqslant 3$. If $(x,y) \in D$, then $(x,y+5) \in D$.*

**Lemma 5.2** (*Gu et al. [10]*). *Let $s \geqslant 3$ and $t \geqslant 3$ be integers. Then the torus $C_s \times C_t$ has an efficient dominating set if and only if both $s$ and $t$ are multiples of* 5.

Combining Lemma 5.2 with Corollary 3.4 yields $3 \leqslant b(C_s \times C_t) \leqslant 4$ if both $s$ and $t$ are multiples of 5. In order to prove $b(C_s \times C_t)=3$, we need to investigate the structure of the efficient dominating sets in $C_s \times C_t$.

**Lemma 5.3.** *Let $D$ be an efficient dominating set containing the vertex $(0, 0)$ in $C_s \times C_t$. Then $D = \bigcup_{i=0}^{4} D_{ij(i)}$ where $j(i) \equiv 2i$ or $3i \pmod 5$, and $D_{ij(i)} = \{(x, y): x \equiv i \pmod 5$ and $y \equiv j(i) \pmod 5\}$.*

**Proof.** Suppose that $s = 5p$ and $t = 5q$ for natural numbers $p$ and $q$, and $D$ is an efficient dominating set in $C_s \times C_t$. Lemma 5.1 also implies that $(x + 5, y) \in D$ if $(x, y) \in D$, since the cartesian product is commutative. Hence $D$ is uniquely determined by its vertices with both coordinates in the interval $[0, 4]$. Define $(x, y) \sim (x', y')$ if $x \equiv x' \pmod 5$ and $y \equiv y' \pmod 5$. Clearly "$\sim$" is an equivalent relation on $Z_s \times Z_t$ and the equivalent class is $D_{ij} = \{(x, y): (x, y) \sim (i, j)\}$. Then $D$ is the union of $D_{ij}$'s, where both $i$ and $j$ belong to the interval $[0, 4]$. Furthermore, $D$ must be the union of five distinct $D_{ij}$'s, since $|D| = n(C_s \times C_t)/5 = 5pq$ and $|D_{ij}| = pq$.

First we show $D = \bigcup_{i=1}^{4} D_{ij(i)}$. Otherwise, there exists an $x \in [0, 4]$ such that $D_{xy_1} \subset D$ and $D_{xy_2} \subset D$ for distinct $y_1, y_2 \in [0, 4]$. Assume $0 = x = y_1 < y_2$, without loss of generality. Then $y_2 \neq 1$, since $(0, 1) \in N[(0, 0)]$.

If $y_2 = 2$, then $(0, 1) \in N[(0, 0)] \cap N[(0, 2)]$, a contradiction.

If $y_2 = 3$, then $(0, -2) \in D$ by Lemma 5.1. But $(0, -1) \in N[(0, 0)] \cap N[(0, -2)]$, a contradiction.

If $y_2 = 4$, then $(0, -1) \in D$ by Lemma 5.1. But $(0, -1) \in N[(0, 0)]$, a contradiction.

Therefore $D = \bigcup_{i=1}^{4} D_{ij(i)}$. We show that $j(i) \equiv 2i \pmod 5$ or $j(i) \equiv 3i \pmod 5$, if $(0, 0) \in D$. Clearly, that is valid for $i = 0$. Assume $j(i) \in [0, 4]$, without loss of generality. In order to dominate $(0, 2)$, we have to select $(1, 2)$ or $(4, 2)$.

If $(1, 2) \in D$, then $(2, 4)$ belongs to $D$ to dominate $(1, 4)$, since $(0, 4) \notin D$. It follows that $(3, 1)$ and $(4, 3)$ belong to $D$ to dominate $(2, 1)$ and $(3, 3)$, respectively.

If $(4, 2) \in D$, then $(3, 4)$ belongs to $D$ to dominate $(4, 4)$, since $(0, 4) \notin D$. It follows that $(2, 1)$ and $(1, 3)$ belong to $D$ to dominate $(3, 1)$ and $(2, 3)$.

Therefore $j(i) \equiv 2i \pmod 5$ or $j(i) \equiv 3i \pmod 5$, and the lemma follows. $\square$

Since $C_s \times C_t$ is vertex-transitive, Lemma 5.3 has determined all efficient dominating sets in $C_s \times C_t$. In fact, from the proof of Theorem 3.1 we know that there are exactly 10 distinct efficient dominating sets, since $(0, 0)$ belongs exactly to two of such sets. Then we can determine the bondage number of $C_s \times C_t$.

**Theorem 5.4.** *Let $G = C_s \times C_t$. Then $b(G) = 3$ if both $s$ and $t$ are multiples of 5.*

**Proof.** We only need to show $b(G) \leqslant 3$. Denote the two distinct efficient dominating sets containing $(x, y)$ by $D_1(xy)$ and $D_2(xy)$. Lemma 5.3 has determined $D_1(00)$ and $D_2(00)$. Let $\phi((x, y)) = (x, y + 1)$ where the addition is taken module $t$. It is clear that $\phi$ is an automorphism of $C_s \times C_t$. Then $D_i(0j) = \phi^j(D_i(00))$ for $i = 1, 2$ and $j = 0, 1, 2, 3, 4$. We show that these 10 set are all efficient dominating sets in $G$.

It is easy to verify that the vertex $(0, j)$ only appears in $D_1(0j)$ and $D_2(0j)$ for any $j = 0, 1, 2, 3, 4$. Clearly $D_1(0j) \neq D_2(0j)$ for $j = 0, 1, 2, 3, 4$. Hence these 10 sets are pairwise distinct, which implies that they are all efficient dominating sets in $G$. Let $E' = \{((0, j), (0, j + 1)): j = 0, 1, 3\}$. Then $E'$ supports $D_i(0j)$ for $i = 0, 1$ and $j = 0, 1, 2, 3, 4$. By Lemma 2.4, $b(G) = s(G) \leqslant |E'| = 3$. The theorem follows. $\square$

The definition of torus can be generalized to $t$-dimensional torus, the cartesian product of $t$ undirected cycles. It is a Cayley graph of degree $2t$.

**Lemma 5.5** (Gu et al. [10]). *For any positive integers $k_1, k_2, \ldots, k_t$, the $t$-dimensional torus $G = C_{(2t+1)k_1} \times C_{(2t+1)k_2} \times \cdots \times C_{(2t+1)k_t}$ has an efficient dominating set.*

Then we can bound the bondage number of $t$-dimensional torus by Corollary 3.4.

**Theorem 5.6.** *Let $G = C_{(2t+1)k_1} \times C_{(2t+1)k_2} \times \cdots \times C_{(2t+1)k_t}$ where $k_1, k_2, \ldots, k_t$ are positive integers. Then $t + 1 \leqslant b(G) \leqslant 2t$.*

**Remark.** In this theorem $t$ may take the value of all positive integers. If $t = 1$ then $G = C_n$ with $3|n$ and we have $b(G) = 2$, which is identical with Proposition 2.5.

Now we consider the directed torus $\vec{C}_s \times \vec{C}_t$ analogously. It is clear that $\vec{C}_s \times \vec{C}_t = C(Z_s \times Z_t, \{(0,1),(1,0)\})$ is 2-regular.

**Lemma 5.7.** *Let $G = \vec{C}_s \times \vec{C}_t$ with $s \geqslant 3$ and $t \geqslant 3$. If $D$ is an efficient dominating set in $G$ and $(x, y) \in D$, then $(x+1, y+1)$, $(x+2, y+2)$, $(x, y+3)$ and $(x+3, y)$ also belong to $D$.*

**Proof.** Let $D$ be an efficient dominating set in $G$ and assume $(x, y) \in D$. Then $N^+[(x, y)] = \{(x, y), (x, y+1), (x+1, y)\}$. Hence $(x, y+1) \notin D$ and $(x+1, y) \notin D$, which implies that $(x+1, y+1)$ belongs to $D$ to dominate itself. Similarly we have $(x+2, y+2) \in D$. Suppose to the contrary that $(x, y+3) \notin D$. Then $(x+1, y+3)$ belongs to $D$ to dominate itself, since $(x+1, y+2) \in N^+[(x+1, y+1)]$. But $(x+1, y+3)$ also dominates $(x+2, y+3) \in N^+[(x+2, y+2)]$, which contradicts $(x+2, y+2) \in D$. Hence $(x, y+3) \in D$. By the commutation of Cartesian product, $(x+3, y) \in D$. □

**Lemma 5.8.** *Let $G = \vec{C}_s \times \vec{C}_t$. Then $G$ has an efficient dominating set if and only if both $s$ and $t$ are multiples of 3. In addition, there exists only one efficient dominating set containing the vertex $(0, 0)$.*

**Proof.** Suppose that $D$ is an efficient dominating set containing $(0, 0)$. Then $3|n(G) = st$. Assume $s = 3p$, without loss of generality. Define $(x, y) \sim (x', y')$ if $x \equiv i \pmod 3$ and $y \equiv j \pmod 3$. Then "$\sim$" is an equivalent relation on $Z_s \times Z_t$ and the equivalent class is $D_{ij} = \{(x, y): x \equiv i \pmod 3, y \equiv j \pmod 3\}$. By Lemma 5.7, $\bigcup_{i=0}^{2} D_{ii} \subseteq D$. Since $|D| = n(G)/3 = pt$ and $|D_{ii}| = \sum_{i=0}^{2} p\lceil (t-i)/3 \rceil$, then $t \geqslant \sum_{i=0}^{2} \lceil (t-i)/3 \rceil$. If $t = 3q+1$ or $3q+2$ then $t \geqslant \sum_{i=0}^{2} \lceil (t-i)/3 \rceil \geqslant q+1+q+q > t$, a contradiction. Hence $t = 3q$ and $|D_{ii}| = pq = |D|/3$. Note that $D_{00}$, $D_{11}$ and $D_{22}$ are distinct equivalent classes. Thus $D = \bigcup_{i=0}^{2} D_{ii}$.

Conversely, if $s = 3p$ and $t = 3q$ for natural numbers $p, q$, it is easy to verify that $D = \bigcup_{i=0}^{2} D_{ii}$ is an efficient dominating set in $G$. □

Combining Lemma 5.8 and Corollary 3.4 we can determine the bondage number of the directed torus immediately.

**Theorem 5.9.** *Let $G = \vec{C}_s \times \vec{C}_t$. Then $b(G) = 3$ if both $s$ and $t$ are multiples of 3.*

At last we generalize the directed torus to $t$-dimensional directed torus.

**Lemma 5.10.** *For any positive integers $k_1, k_2, \ldots, k_t$, the $t$-dimensional directed torus $G = \vec{C}_{(t+1)k_1} \times \vec{C}_{(t+1)k_2} \times \cdots \times \vec{C}_{(t+1)k_t}$ has an efficient dominating set.*

**Proof.** Given two vertices $x = (x_1, \ldots, x_t)$ and $y = (y_1, \ldots, y_t)$, define $x \sim y$ if $x_p \equiv y_p \pmod{t+1}$. Then "$\sim$" is an equivalent relation on $Z_{(t+1)k_1} \times \cdots \times Z_{(t+1)k_t}$ and the equivalent class of any nonnegative integers $i_1, \ldots, i_t$ is

$$D_{i_1 \ldots i_t} = \{(x_1, \ldots, x_t): x_p \equiv i_p \pmod{t+1}, 1 \leqslant p \leqslant t\}.$$

We show that

$$D = \bigcup_{i_1=0}^{t} \cdots \bigcup_{i_{t-1}=0}^{t} D_{i_1 \ldots i_t}$$

is an efficient dominating set, where $i_t = i_t(i_1, \ldots, i_{t-1}) = \sum_{p=1}^{t-1} p i_p$. Note that all these $D_{i_1 \ldots i_t}$'s are pairwise distinct. Then

$$|D| = (t+1)^{t-1} |D_{i_1 \ldots i_t}| = (t+1)^{t-1} \prod_{p=1}^{t} k_p.$$

On the other hand, $G$ is $t$ regular and

$$n(G) = (t+1)^t \prod_{p=1}^{t} k_p = (t+1)|D|.$$

Thus, we only need to show that $N^+[x] \cap N^+[y] = \emptyset$ for any distinct $x, y \in D$.

Let $x = (x_1, \ldots, x_t)$ and $y = (y_1, \ldots, y_t)$ be distinct vertices in $D$. By the definition of $D$, we obtain

$$x_t - \sum_{p=1}^{t-1} px_p \equiv 0 \equiv y_t - \sum_{p=1}^{t-1} py_p \pmod{t+1},$$

i.e.,

$$\sum_{p=1}^{t-1} pa_p \equiv a_t \pmod{t+1}, \tag{1}$$

where $a_p = x_p - y_p$ for $p = 1, 2, \ldots, t$. We proceed by contradiction.

If $x \in N^+[y]$, then, by the definition of $G$, there exists an integer $q \in \{1, \ldots, t\}$ such that $a_q = 1$ and $a_p = 0$ for any $p \neq q$. Clearly $q \neq t$, since the equality (1) yields that $a_t \neq 1$ if $a_p = 0$ for all $p \in \{1, \ldots, t-1\}$. Hence $q \in \{1, \ldots, t-1\}$ and $a_t = \sum_{p=1}^{t-1} px_p \equiv q \pmod{t+1}$ by (1), which contradicts $a_t = 0$. Thus, $x \notin N^+[y]$, and analogously we have $y \notin N^+[x]$.

If there exists a vertex $z \in N^+[x] \cap N^+[y]$ which is other than $x$ and $y$, then, by the definition of $G$, there exist distinct integers $q, r \in \{1, \ldots, t\}$ such that $a_q = 1$, $a_r = -1$ and $a_p = 0$ for any $p \neq q, r$. Clearly $q \neq t$, since the equality (1) yields that $a_t \neq 1$ if $a_r = -1$ for $1 \leqslant r \leqslant t-1$ and $a_p = 0$ for any $p \neq q, r$. Similarly we have $r \neq t$. Hence $q, r \in \{1, \ldots, t-1\}$ and $a_t \equiv q - r \not\equiv 0 \pmod{t+1}$ by (1), which contradicts $a_t = 0$.

Therefore it holds that $N^+[x] \cap N^+[y] = \emptyset$, and the lemma follows. □

**Theorem 5.11.** *Let* $G = \overrightarrow{C}_{(t+1)k_1} \times \overrightarrow{C}_{(t+1)k_2} \times \cdots \times \overrightarrow{C}_{(t+1)k_t}$ *for any positive integers* $k_1, k_2, \ldots, k_t$. *Then* $b(G) = t+1$.

**Proof.** By Lemma 5.10 and Corollary 3.4. □

## 6. Other applications

Corollary 3.4 reveals the relationship between bondage number and efficient domination for vertex-transitive graphs. The efficient domination has important applications in many areas, such as error-correcting codes, and receives much attention in the late years. For various classes of graphs, a large amount of which are vertex-transitive, the existence of efficient dominating sets has already been proved. Therefore, we can determine at least the interval of $b(G)$ for such a graph $G$.

The hypercube $Q_n$ is the Cayley graph $C(\Gamma, S)$ where $\Gamma = Z_2 \times \cdots \times Z_2 = (Z_2)^n$ and $S = \{100\ldots0, 010\ldots0, \ldots, 00\ldots01\}$. Lee [14] showed that $Q_n$ has an efficient dominating set if and only if $n = 2^m - 1$ for a positive integer $m$. Then we obtain the following result by Corollary 3.4.

**Proposition 6.1.** *If* $n = 2^m - 1$ *for a natural number m, then* $2^{m-1} \leqslant b(Q_n) \leqslant 2^m - 1$.

The cube-connected cycle is an important derivative networks of the hypercube. The *n-dimensional cube-connected cycle*, denoted by CCC($n$), is constructed from the $n$-dimensional hypercube $Q_n$ by replacing each vertex $v \in V(Q_n)$ with an undirected cycle $C_n$ of length $n$ and linking the $i$th vertex of the $C_n$ to the $i$th neighbor of $v$. It is easy to observe that CCC($n$) is 3-regular; indeed, it is a Cayley graph. Assume $n \geqslant 3$ below. Van Wieren et al. [15] proved that CCC($n$) has an efficient dominating set if and only if $n \neq 5$. Then we derive the following result from Corollaries 3.4 and 3.5.

**Proposition 6.2.** *Let* $G = \mathrm{CCC}(n)$ *be the n-dimensional cube-connected cycles with* $n \geqslant 3$ *and* $n \neq 5$. *Then* $\gamma(G) = 2^{n-2}n$ *and* $2 \leqslant b(G) \leqslant 3$. *In addition,* $b(\mathrm{CCC}(3)) = b(\mathrm{CCC}(4)) = 2$.

## 7. Conclusion

In this paper we mainly consider the vertex-transitive graphs. Generally speaking, to determine the exact value of the bondage number is often difficult. Thus much work focus on its bounds. So far, many upper bounds about the bondage number of undirected graphs have been found, among which Lemma 2.1 is an essential one. According to its corollary that $b(G) \leqslant \Delta(G) + \delta(G) - 1$ and a similar one for the directed case, $b(G) \leqslant \Delta(G) + \delta^-(G)$, the regular graphs are among the worst case.

However, our research on vertex-transitive graphs shows that the results are sometimes better. As long as the existence of efficient domination has been proved for a vertex-transitive graph $G$, we immediately know that $b(G)$ is bounded above by its regularity if $G$ is undirected, or $b(G)$ is equal to its regularity plus one if $G$ is directed. Furthermore, Theorem 3.3 shows that a vertex-transitive graph with more efficiency of its domination will be more vulnerable under link failure.

On the other hand, we are also able to establish a lower bound. Then by these bounds, we can determine the exact value of bondage number for some particular graphs, if we had known the structure of the efficient dominating sets. In Sections 4 and 5 we apply such results to circulant graphs and tori, respectively; some other examples arise in Section 6. Of course, we cannot enumerate all possible applications, which are worthy of further research.

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