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The super connectivity of augmented cubes ☆

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Abstract

The augmented cube AQ_n, proposed by Choudum and Sunitha [S.A. Choudum, V. Sunitha, Augmented cubes, Networks 40 (2) (2002) 71–84], is a $(2n-1)$-regular $(2n-1)$-connected graph ($n \neq 3$). This paper determines that the super connectivity of AQ_n is $4n-8$ for $n \geqslant 6$ and the super edge-connectivity is $4n-4$ for $n \geqslant 5$. That is, for $n \geqslant 6$ (respectively, $n \geqslant 5$), at least $4n-8$ vertices (respectively, $4n-4$ edges) of AQ_n are removed to get a disconnected graph that contains no isolated vertices. When the augmented cube is used to model the topological structure of a large-scale parallel processing system, these results can provide more accurate measurements for reliability and fault tolerance of the system.

Keywords: Interconnection networks; Augmented cube; Super connectivity; Super edge-connectivity

1. Introduction

An interconnection network is usually represented by an undirected simple graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are the vertex set and the edge set, respectively, of G. In this paper, we use a graph and a network interchangeably. For graph terminology and notation not defined here we follow [15].

It is well known that interconnection networks play an important role in parallel computing/communication systems. The connectivity $\kappa(G)$ or the edge-connectivity $\lambda(G)$ of a graph G is an important measurement for fault-tolerance of the network, and the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is. It is well known that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum degree of G. As more refined indices than connectivity and edge-connectivity, super connectivity and super edge-connectivity were proposed in [1,3].

A subset $S \subset V(G)$ (respectively, $F \subset E(G)$) is called a super vertex-cut (respectively, super edge-cut) if $G - S$ (respectively, $G - F$) is not connected and every component contains at least two vertices. In general, super vertex-cuts or super edge-cuts do not always exist. The super connectivity $\kappa'(G)$ (respectively, super edge-connectivity $\lambda'(G)$) is the minimum cardinality over all super vertex-cuts (respectively, super edge-cuts) in G if any, and, by convention, is $+\infty$ otherwise. The super connectivity has been studied for many networks,

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Table 1
Results of some networks

Networks	Supper connectivity	Supper edge-connectivity
Q_n	$2n-2$ ($n \geqslant 3$)	$2n-2$ ($n \geqslant 3$)
CQ_n	$2n-2$ ($n \geqslant 3$)	$2n-2$ ($n \geqslant 3$)
MQ_n	$2n-2$ ($n \geqslant 3$)	$2n-2$ ($n \geqslant 3$)
AQ_n	$4n-8$ ($n \geqslant 6$)	$4n-4$ ($n \geqslant 5$)

such as [3,5,7,8,10–14,16,17]. Some of the results are listed in Table 1.

It is well known that the hypercube is one of the most popular interconnection networks for parallel computer/communication system. As an enhancement on the hypercube Q_n, the augmented cube AQ_n, proposed by Choudum and Sunitha [2], not only retains some of the favorable properties of Q_n but also possesses some embedding properties that Q_n does not (see, for example, [6,9]). In this paper, we prove that $\kappa'(AQ_n) = 4n-8$ for $n \geqslant 6$ and $\lambda'(AQ_n) = 4n-4$ for $n \geqslant 5$.

For a graph $G = (V, E)$ and a subset $S \subset V(G)$, we set $N_G(S) = \{X \in V(G) \setminus S\colon \exists U \in S$ such that $(U, X) \in E(G)\}$. Let $N_G[S] = N_G(S) \cup S$. If $S = \{U\}$, we write $N_G(U)$ and $N_G[U]$ instead of $N_G(S)$ and $N_G[S]$, respectively. We will write $N(S)$ (respectively, $N[S]$) instead of $N_G(S)$ (respectively, $N_G[S]$) if there is no ambiguity. The minimum edge-degree of G is $\xi(G) = \min\{d(U) + d(V) - 2\colon (U, V) \in E(G)\}$, $d(U) = |N(U)|$ standing for the degree of a vertex U.

The following of this paper is organized as follows. Section 2 gives the definition of augmented cube and its properties. The main results are given in Section 3. Finally, we conclude our paper in Section 4.

2. Augmented cube and its properties

The n-dimensional augmented cube AQ_n ($n \geqslant 1$) can be defined recursively as follows.

Definition 1. AQ_1 is a complete graph K_2 with the vertex set $\{0, 1\}$. For $n \geqslant 2$, AQ_n is obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and adding $2 \times 2^{n-1}$ edges between the two as follows:

Let $V(AQ^0_{n-1}) = \{0u_{n-1}\dots u_2u_1\colon u_i = 0 \text{ or } 1\}$ and $V(AQ^1_{n-1}) = \{1u_{n-1}\dots u_2u_1\colon u_i = 0 \text{ or } 1\}$. A vertex $U = 0u_{n-1}\dots u_2u_1$ of AQ^0_{n-1} is joined to a vertex $W = 1w_{n-1}\dots w_2w_1$ of AQ^1_{n-1} if and only if either

(i) $u_i = w_i$ for $1 \leqslant i \leqslant n-1$; or
(ii) $u_i = \bar{w}_i$ for $1 \leqslant i \leqslant n-1$.

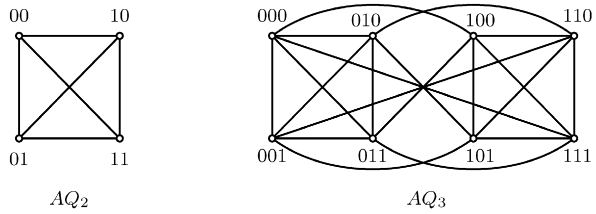

Fig. 1. Two augmented cubes AQ_2 and AQ_3.

The augmented cubes AQ_2 and AQ_3 are shown in Fig. 1.

According to Definition 1 of augmented cubes, we write this recursive construction of AQ_n symbolically as $AQ_n = L \oplus R$, where $L \cong AQ^0_{n-1}$ and $R \cong AQ^1_{n-1}$. We call the edges between L and R crossed edges. Clearly every vertex of AQ_n is incident with two crossed edges.

For an n-bit binary string $U = u_nu_{n-1}\dots u_1$, we use U_i (respectively, $\bar{U}_i$) to denote the binary string $u_n \dots \bar{u}_i \dots u_1$ (respectively, $u_n \dots \bar{u}_i \dots \bar{u}_1$) which differs with U in the ith bit position (respectively, from the first to the ith bit positions). It is clear that $U_1 = \bar{U}_1$. We use U_1 rather than $\bar{U}_1$.

An alternative definition of AQ_n is given in the following.

Definition 2. The augmented cube AQ_n of dimension n has 2^n vertices. Each vertex is labeled by a unique n-bit binary string as its address. Two vertices $U = u_nu_{n-1}\dots u_1$ and $W = w_nw_{n-1}\dots w_1$ are joined iff either

(i) There exists an integer i, $1 \leqslant i \leqslant n$, such that $W = U_i$; in this case, the edge is called a hypercube edge of dimension i, denoted by (U, U_i), or
(ii) There exists an integer i, $2 \leqslant i \leqslant n$, such that $W = \bar{U}_i$; in this case, the edge is called a complement edge of dimension i, denoted by $(U, \bar{U}_i)$.

It has been shown that AQ_n ($n \neq 3$) is a $(2n-1)$-regular $(2n-1)$-connected graph in [2]. The following two properties are derived directly from Definition 2.

Property 1. *If (U, U_i) is a hypercube edge of dimension i ($2 \leqslant i \leqslant n$), then*

$$N_{AQ_n}(U) \cap N_{AQ_n}(U_i) = \begin{cases} \{\bar{U}_i, \bar{U}_{i-1}\} & \text{if } i > 1, \\ \{\bar{U}_2, U_2\} & \text{if } i = 1, \end{cases}$$

that is, U and U_i have exactly two common neighbors in AQ_n and $|N_{AQ_n}(\{U, U_i\})| = 4n-6$.

Property 2. *If $(U, \bar{U}_i)$ is a complement edge of dimension i $(2 \leqslant i \leqslant n-1)$, then $N_{AQ_n}(U) \cap N_{AQ_n}(\bar{U}_i) = \{U_i, U_{i+1}, \bar{U}_{i-1}, \bar{U}_{i+1}\}$, that is, U and $\bar{U}_i$ have exactly four common neighbors in AQ_n and $|N_{AQ_n}(\{U, \bar{U}_i\})| = 4n-8$. If $(U, \bar{U}_n)$ is a complement edge of dimension n, then $N_{AQ_n}(U) \cap N_{AQ_n}(\bar{U}_n) = \{\bar{U}_{n-1}, U_n\}$, that is, U and $\bar{U}_n$ have exactly two common neighbors in AQ_n and $|N_{AQ_n}(\{U, \bar{U}_n\})| = 4n-6$.*

Note that AQ_n can expressed as $AQ_n = L \oplus R$, where $L \cong AQ^0_{n-1}$ and $R \cong AQ^1_{n-1}$. We can also obtain the following property from Definition 1.

Property 3. *If two vertices U and W in L (respectively, R) have common neighbors in R (respectively, L), then $W = \bar{U}_{n-1}$ and they have exactly two common neighbors U_n and $\bar{U}_n$ in R.*

With the above properties, we can obtain the following property which is useful to us.

Property 4. *Any two vertices in AQ_n have at most four common neighbors for $n \geqslant 3$.*

Proof. We prove the conclusion by induction on n. It is trivially true for AQ_3 (see Fig. 1). Suppose that the result is true for AQ_{n-1} with $n \geqslant 4$. We will prove the result is true for AQ_n according to the location of the two vertices.

Case 1. Both vertices U and W are in L or R. Without loss of generality, we may assume they are in L.

If U and W have no common neighbor in R, by the induction hypothesis, they have at most four neighbors in L. The conclusion is true.

If U and W have common neighbors in R, by Property 3, $W = \bar{U}_{n-1}$ and they have exactly two common neighbors in R. Then (U, W) is a complement edge of dimension $(n-1)$ in $L \cong AQ^0_{n-1}$. By Property 2, they have exactly two common neighbors in L. Thus, U and W have exactly four common neighbors in $AQ_n = L \oplus R$.

Case 2. One of the two vertices is in L, the other is in R. Without loss of generality, we may assume $U \in L$ and $W \in R$. Since U (respectively, W) has exactly two neighbors in R (respectively, L), they have at most four common neighbors in AQ_n. □

3. Main results

Lemma 1. *$\kappa'(AQ_n) \leqslant 4n-8$ for $n \geqslant 6$.*

Proof. Let $S = \{U, \bar{U}_i\}$ $(2 \leqslant i \leqslant n-1)$. By Property 2, we have $|N(S)| = 4n-8$. We will prove that $N(S)$ is a super vertex-cut, which means $\kappa'(AQ_n) \leqslant 4n-8$. To the end, we need to prove that $AQ_n - N[S]$ has no isolated vertex.

Let W be a vertex in $AQ_n - N[S]$. By Property 4, U (respectively, $\bar{U}_i$) and W have at most four common neighbors. Hence, $|N(W) \cap N[S]| \leqslant 10$. Since $n \geqslant 6$, we have $|N(W)| \geqslant 11$. Thus, W has at least one neighbor in $AQ_n - N[S]$. The lemma follows. □

Theorem 1. *$\kappa'(AQ_n) = 4n-8$ for $n \geqslant 6$.*

Proof. By Lemma 1, we only need to prove $\kappa'(AQ_n) \geqslant 4n-8$. Let S be an arbitrary set of vertices in AQ_n such that $|S| \leqslant 4n-9$ and there are no isolated vertices in $AQ_n - S$. We will prove that $AQ_n - S$ is connected.

Note that $AQ_n = L \oplus R$ where $L \cong AQ^0_{n-1}$ and $R \cong AQ^1_{n-1}$. For convenience, let $S_L = S \cap L$ and $S_R = S \cap R$. Without loss of generality, we may suppose that $|S_L| \geqslant |S_R|$. Then $|S_R| \leqslant \lfloor (4n-9)/2 \rfloor = 2n-5$.

Since $R \cong AQ^1_{n-1}$ and $\kappa(R) = 2(n-1)-1 = 2n-3$, we have $R - S_R$ is connected. It remains to be shown that any vertex in $L - S_L$ is connected via a path to a vertex in $R - S_R$. Let U be an arbitrary vertex in $L - S_L$. Its neighbors in R are U_n and $\bar{U}_n$. If $\{U_n, \bar{U}_n\} \not\subset S_R$, we are done. So assume that $\{U_n, \bar{U}_n\} \subset S_R$ below. Consider the following two cases.

Case 1. $\bar{U}_{n-1} \notin S_L$. By Property 2, we have $|N_L(\{U, \bar{U}_{n-1}\})| = 4n-10$. Let $X = \{X^i\colon X^i \in N_L(U) \setminus \{\bar{U}_{n-1}\}\}$, $Y = \{Y^j\colon Y^j \in N_L(\bar{U}_{n-1}) \setminus N_L[U]\}$, and $S' = S - \{U_n, \bar{U}_n\}$. It is not difficult to see that $(N_R(X) \cup N_R(Y)) \cap \{U_n, \bar{U}_n\} = \emptyset$, $|X| = 2n-4$, $|Y| = 2n-6$ and $|S'| \leqslant 4n-11$. For each vertex X^i, $1 \leqslant i \leqslant 2n-4$ (respectively, Y^j, $1 \leqslant j \leqslant 2n-6$), let $P_i = (U, X^i, X^i_n)$ (respectively, $P_j = (\bar{U}_{n-1}, Y^j, Y^j_n)$) be a path joining U (respectively, $\bar{U}_{n-1}$) to a vertex in R. Note that these paths are vertex disjoint except for U (respectively, $\bar{U}_{n-1}$) and $|\{P_i\colon 1 \leqslant i \leqslant 2n-4\}| + |\{P_j\colon 1 \leqslant j \leqslant 2n-6\}| = 4n-10$. Since $|S'| \leqslant 4n-11$ and each vertex in S' can correspond to at most one such path, there must exist a path P_i or P_j such that $V(P_i) \cap S' = \emptyset$ or $V(P_j) \cap S' = \emptyset$. This implies that in $AQ_n - S$, vertex U is connected via a path to a vertex in $R - S_R$ (see Fig. 2(a)).

Case 2. $\bar{U}_{n-1} \in S_L$. There must exist a neighbor W of U in $L - S_L$ since there are no isolated vertices in $AQ_n - S$. The two neighbors of W in R is W_n and $\bar{W}_n$. If $\{W_n, \bar{W}_n\} \not\subset S_R$, we are done. So assume that $\{W_n, \bar{W}_n\} \subset S_R$. If $\bar{W}_{n-1} \notin S_L$, we can obtain a path joining W or $\bar{W}_{n-1}$ to a vertex in $R - S_R$ by using a method similar to the one used in Case 1. Thus, U is connected via a path to a vertex in $R - F_R$. Hence, assume $\bar{W}_{n-1} \in S_L$ below.

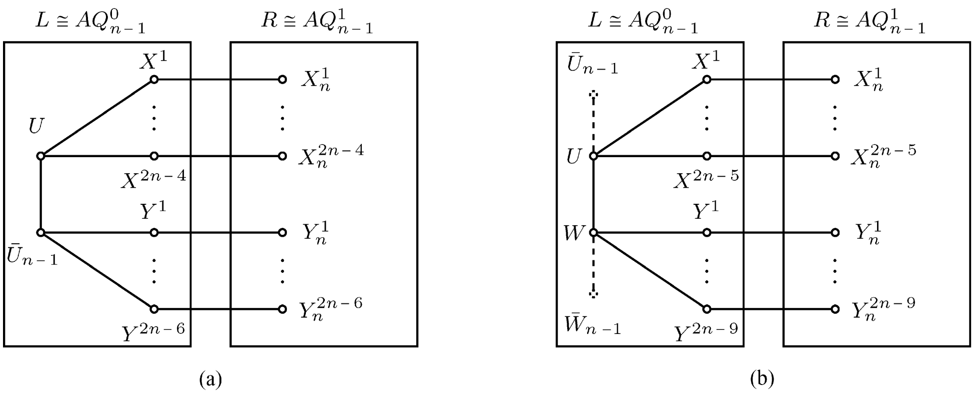

Fig. 2. Illustrations for the proof of Theorem 1.

By Property 4, we have $|\{N_L(\{U,W\}) \setminus \{\bar{U}_{n-1}, \bar{W}_{n-1}\}| \geqslant 4n-14$. Let $X = \{X^i\colon\ X^i \in N_L(U) \setminus \{\bar{U}_{n-1}, W\}\}$, $Y = \{Y^j\colon Y^j \in N_L(W) \setminus (N_L[U] \cup \bar{W}_{n-1})\}$, and $S' = S - \{U_n, \bar{U}_n, W_n, \bar{W}_n, \bar{U}_{n-1}, \bar{W}_{n-1}\}$. It is not difficult to see that $(N_R(X) \cup N_R(Y)) \cap \{U_n, \bar{U}_n, W_n, \bar{W}_n\} = \emptyset$, $|X| = 2n-5$, $|Y| \geqslant 2n-9$ and $|S'| = 4n-15$. For each vertex X^i, $1 \leqslant i \leqslant 2n-5$ (respectively, Y^j, $1 \leqslant j \leqslant 2n-9$), let $P_i = (U, X^i, X_n^i)$ (respectively, $P_j = (W, Y^j, Y_n^j)$) be a path joining U (respectively, W) to a vertex in R. Note that these paths are vertex disjoint except for U (respectively, W) and $|\{P_i\colon 1 \leqslant i \leqslant 2n-5\}| + |\{P_j\colon 1 \leqslant j \leqslant 2n-9\}| = 4n-14$. Since $|S'| = 4n-15$ and each vertex in S' can correspond to at most one such path, there must exist a path P_i or P_j such that $V(P_i) \cap S' = \emptyset$ or $V(P_j) \cap S' = \emptyset$. This implies that in $AQ_n - S$, vertex U is connected via a path to a vertex in $R - S_R$ (see Fig. 2(b)).

We have proved that $AQ_n - S$ is connected, which means $\kappa'(AQ_n) \geqslant 4n-8$ for $n \geqslant 6$. The theorem follows. □

The following result, which can be found in [4], is useful in the proof of Theorem 2.

Lemma 2. $\lambda'(G) \leqslant \xi(G)$ *for any graph G with order at least* 4 *and not a star.*

Theorem 2. $\lambda'(AQ_n) = 4n-4$ *for* $n \geqslant 5$.

Proof. By Lemma 2, we only need to prove $\lambda'(AQ_n) \geqslant 4n-4$ for $n \geqslant 5$.

Let F be an arbitrary subset of edges in AQ_n such that $|F| \leqslant 4n-5$ and there are no isolated vertices in $AQ_n - F$. We will prove that $AQ_n - F$ is connected.

For convenience, let $F_L = F \cap L$ and $F_R = F \cap R$. Without loss of generality, we may suppose that $|F_L| \geqslant |F_R|$. Then $|F_R| \leqslant \lfloor (4n-5)/2 \rfloor = 2n-3$.

Since $R \cong AQ^1_{n-1}$ ($(n-1) \geqslant 4$) and R is $(2n-1)$-regular $(2n-1)$-connected graph, we conclude that R is $(2n-1)$-edge-connected. That is $R - F_R$ is connected. It remains to be shown that any vertex in L is connected via a path to a vertex in R. Let U be an arbitrary vertex in L. If the two crossed edges $\{(U, U_n), (U, \bar{U}_n)\} \not\subset F$, we are done. So assume that $\{(U, U_n), (U, \bar{U}_n)\} \subset F$ below.

Since there are no isolated vertices in $AQ_n - F$, there is an edge (U, W) incident with U in L such that $(U, W) \notin F_L$. If the two crossed edges $\{(W, W_n), (W, \bar{W}_n)\} \not\subset F$, we are done. So assume that $\{(W, W_n), (W, \bar{W}_n)\} \subset F$ below.

Let $E_1 = \{(U, U^i)\colon\ (U, U^i) \in E(L) \setminus \{(U, W)\}\}$, $E_2 = \{(W, W^j)\colon\ (W, W^j) \in E(L) \setminus \{(U, W)\}\}$, and $F' = F - \{(U, U_n), (U, \bar{U}_n), (W, W_n), (W, \bar{W}_n)\}$. It is not difficult to see that $|E_1| = |E_2| = 2n-4$, $E_1 \cap E_2 = \emptyset$ and $|F'| \leqslant 4n-9$. Let $P_i = (U, U^i, U_n^i)$ (respectively, $P_j = (W, W^i, W_n^i)$) be a path joining U (respectively, W) to a vertex in R. Note that these paths are edge disjoint and $|\{P_i\colon 1 \leqslant i \leqslant 2n-4\}| + |\{P_j\colon 1 \leqslant j \leqslant 2n-4\}| = 4n-8$. Since $|F'| \leqslant 4n-9$ and each edge in F' can correspond to at most one such path, there must exist a path P_i or P_j such that $E(P_i) \cap F' = \emptyset$ or $E(P_j) \cap F' = \emptyset$. This implies that in $AQ_n - F$, vertex U is connected via a path to a vertex in R.

We proved that $AQ_n - F$ is connected, which means $\lambda'(AQ_n) \geqslant 4n-4$ for $n \geqslant 5$. The theorem follows. □

4. Conclusions

In this paper, we concentrate on two stronger measures of network reliability called super connectivity $\kappa'(G)$ and super edge-connectivity $\lambda'(G)$ which not only compensate for shortcoming but also generalize the classical connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$. For the augmented cube AQ_n, an enhancement on the hypercube Q_n, we proved that $\kappa'(AQ_n) = 4n-8$

for $n \geqslant 6$ and $\lambda'(AQ_n) = 4n - 4$ for $n \geqslant 5$. The two results show that the augmented cube is robust when it is used to model the topological structure of a large-scale parallel processing system.

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