第38卷第5期 中国科学技术大学学报 Vol. 38, No. 5
2008年5月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA May 2008

Article ID: 0253-2778(2008)05-0456-04

Forwarding indices of 3-connected and 3-regular graphs

ZHOU Min-jie[1], XU Min[2], XU Jun-ming[1]

(1. *Department of Mathematics, University of Science and Technology of China, Hefei* 230026, *China*;
2. *School of Mathematical Sciences, Beijing Normal University, Beijing* 100875, *China*)

Abstract: A routing R of a connected graph G of order n is a collection of $n(n-1)$ paths connecting every ordered pair of vertices of G. The vertex-forwarding index $\xi(G,R)$ (resp. the edge-forwarding index $\pi(G,R)$) of G with respect to R is defined to be the maximum number of paths of R passing through any vertex (resp. any edge) of G. The vertex-forwarding index $\xi(G)$ (resp. the edge-forwarding index $\pi(G)$) of G is defined to be the minimum $\xi(G,R)$ (resp. $\pi(G,R)$) over all routings R of G. For a k-regular k-connected graph G, it was shown by Fernandez de la Vega and Manoussakis [Discrete Applied Mathematics, 1989, 23(2): 103-123] that $\xi(G)\leqslant(n-1)\lceil(n-k-1)/k\rceil$ and $\pi(G)\leqslant n\lceil(n-k-1)/k\rceil$, and conjectured that $\xi(G)\leqslant\lceil(n-k)(n-k-1)/k\rceil$. The upper bounds as $\xi(G)\leqslant(n-1)\lceil(n-k-1)/k\rceil-(n-k-1)$ and $\pi(G)\leqslant n\lceil(n-k-1)/k\rceil-(n-k)$ were improved, and the conjecture for $k=3$ was proved.

Key words: forwarding index; vertex-forwarding index; edge-forwarding index; routing; k-regular k-connected graphs

CLC number: O157.5; TP302.1 **Document code**: A

AMS Subject Classification (2000): Primary 05C35; Secondary 05C38

3 正则 3 连通图的转发指数

周敏杰[1], 徐　敏[2], 徐俊明[1]

(1. 中国科学技术大学数学系，安徽合肥 230026; 2. 北京师范大学数学科学学院，北京 100875)

摘要: n 阶连通图 G 的路由选择 R 是由连接 G 的每个有向顶点对的 $n(n-1)$ 条路组成. R 经过 G 的每个顶点(每条边)的路的最大条数称为 G 关于 R 的点转发指数 $\xi(G,R)$(边转发指数 $\pi(G,R)$). 对 G 的所有路由选择 R, $\xi(G,R)(\pi(G,R))$的最小值称为 G 的点转发指数 $\xi(G)$(边转发指数 $\pi(G)$). 对于 k 正则 k 连通图 G, Fernandez de la Vega 和 Manoussakis [Discrete Applied Mathematics, 1989, 23(2): 103-123]证明 $\xi(G)\leqslant(n-1)\cdot\lceil(n-k-1)/k\rceil$ 和 $\pi(G)\leqslant n\lceil(n-k-1)/k\rceil$, 并且猜想 $\xi(G)\leqslant\lceil(n-k)(n-k-1)/k\rceil$. 我们分别改进了 $\xi(G)\leqslant(n-1)\lceil(n-k-1)/k\rceil-(n-k-1)$ 和 $\pi(G)\leqslant n\lceil(n-k-1)/k\rceil-(n-k)$, 并且证明了猜想对 $k=3$ 的情形.

关键词: 转发指数; 点转发指数; 边转发指数; 路由选择; k 正则 k 连通图

Received: 2006-10-25; **Revised**: 2007-02-01

Foundation item: Supported by NNSF of China (No. 10671191 and No. 10301031).

Biography: ZHOU Min-jie, male, born 1978, master. Research field: graph theory.

Corresponding author: XU Jun-ming, Prof. E-mail: xujm@ustc.edu.cn

0 Introduction

We follow Refs. [1, 2] for the graph-theoretical and combinatory network terminologies and notations not defined here. Throughout this paper, a graph $G=(V, E)$ always means a connected simple graph with order n (without loops and multiple edges), where $V=V(G)$ is the vertex set and $E=E(G)$ is the edge set. The diameter of G, $d(G)$, is the maximum length over all shortest paths between any two vertices in G. The connectivity of G, $\kappa(G)$, is the minimum cardinality over all vertex-separating sets in G if G is not a complete graph K_n, otherwise $\kappa(K_n)=n-1$. A graph G is said to be k-connected if $\kappa(G)\geqslant k$. A routing R in G is a set of $n(n-1)$ fixed paths for all ordered pairs (x, y) of vertices of G. The path $R(x, y)$ specified by R carries the data transmitted from the source x to the destination y. It is possible that the fixed paths specified by a given routing R passing through some verteices or edges are too many, which means that the routing R loads the verteices or the edges too much. The load of any vertex or edge is limited by the capacity of the vertex or edge, for otherwise it would affect the efficiency of transmission, even result in malfunction of the network. In order to measure the load of a vertex or an edge, Refs. [3, 4] proposed the notion of the forwarding index.

Let G be a graph with a given routing R, x a vertex of G and e an edge of G. The load of x with respect to R, denoted by $\xi(G,R,x)$ [resp. the load of e with respect to R, denoted by $\pi(G,R,e)$], is defined as the number of paths specified by R passing through x [resp. e]. The vertex-forwarding index and the edge-forwarding index of G with respect to R are, respectively, defined as

$$\xi(G,R)=\max\{\xi(G,R,x): x\in V(G)\}$$

and

$$\pi(G,R)=\max\{\pi(G,R,e): e\in E(G)\}.$$

The vertex-forwarding index and the edge-forwarding index of G are, respectively, defined as

$$\xi(G)=\min\{\xi(G,R): R \text{ is a routing of } G\}$$

and

$$\pi(G)=\min\{\pi(G,R): R \text{ is a routing of } G\}.$$

The original study of forwarding indices is motivated by the problem of maximizing network capacity, see Ref. [2]. Minimizing the forwarding indices of a routing will result in maximizing the network capacity. Thus, it becomes very significant to determine the vertex and edge-forwarding indices of a given graph.

Many authors are interested in the forwarding indices of a graph (see, for example, Refs. [3~12]). Moreover, Ref. [13] showed that for any graph determining the forwarding index problem is NP-complete even if the diameter of the graph is two. It is still of interest to determine the exact value of the forwarding indices with some graph-theoretical parameters. For example, the upper bound of forwarding indices of graphs concerning with given connectivity. There are some results about connectivity constraints.

Theorem 0.1[3,6] If G is a connected graph of order n, then $\xi(G)\leqslant(n-2)(n-2)$ and $\pi(G)\leqslant\lfloor n^2/2\rfloor$. The two bounds are the best possible in view of the star $K_{1,n-1}$ and the complete bipartite graph $K_{n/2,n/2}$, respectively.

Theorem 0.2[4,6] If G is a 2-connected graph of order n, then $\xi(G)\leqslant(n-2)(n-3)/2$ and $\pi(G)\leqslant\lfloor n^2/4\rfloor$. The two bounds are the best possible in view of $K_{2,n-2}$ and a cycle C_n, respectively.

Theorem 0.3[5] If G is a k-connected graph of order n with $k\geqslant 1$, then $\xi(G)\leqslant(n-1)\lceil(n-k-1)/k\rceil$ and $\pi(G)\leqslant n\lceil(n-k-1)/k\rceil$.

Conjecture 0.4[5] If G is a k-connected graph of order n with $n\geqslant 2k\geqslant 2$, then

$$\xi(G)\leqslant\lceil(n-k)(n-k-1)/k\rceil$$

and $\pi(G)\leqslant\lceil n^2/2k\rceil$.

Theorem 0.1 and Theorem 0.2 show that this conjecture is true for $k=1$ and 2.

1 Main results

In this paper, we improve the result of Theorem 0.3 and show that Conjecture 0.4 is true for $k=3$ by proving the following two theorems.

Theorem 1.1 If G is a k-connected graph of order n with the maximum degree Δ, then

$$\xi(G) \leqslant (n-1)\lceil (n-k-1)/k \rceil - (n-\Delta-1)$$

and

$$\pi(G) \leqslant n\lceil (n-k-1)/k \rceil - (n-\Delta).$$

Corollary 1.2 If G is a k-regular and k-connected graph, then

$$\xi(G) \leqslant (n-1)\lceil (n-k-1)/k \rceil - (n-k-1)$$

and

$$\pi(G) \leqslant n\lceil (n-k-1)/k \rceil - (n-k).$$

Theorem 1.3 If G is a 3-regular and 3-connected graph of order $n \geqslant 4$. then

$$\xi(G) \leqslant \lceil (n-3)(n-4)/3 \rceil.$$

2 Proofs of our results

In this section, we give the proofs of our results. We first show some lemmas used in the proofs.

Lemma 2.1 If G is a 3-regular and 3-connected graph of order n at least 6, then $V(G)$ can be partitioned into $n/2$ independent sets of order two.

Proof Let G be a 3-regular 3-connected graph of order $n \geqslant 6$. First note that $|E(G)| = 3n/2$ and so n is even. The conclusion is true for $n=6$ since a graph of order 6 that is 3-regular 3-connected is either $G=K_{3,3}$ or $K_2 \times K_3$. Assume $n \geqslant 8$ below. Since G is 3-regular 3-connected and $n \geqslant 8$, for arbitrary four vertices, there is an independent set of order two. So we can partition $V(G)$ into $n/2-1$ independent sets $V_1=\{v_1, v_2\}, V_2=\{v_3, v_4\}, \cdots, V_{n/2-1}=\{v_{n-3}, v_{n-2}\}$ and the last two vertices v_{n-1} and v_n. If $v_{n-1}v_n \notin E(G)$, then let $V_{n/2}=\{v_{n-1}, v_n\}$, and so the result is true. So assume $v_{n-1}v_n \in E(G)$ below. Since $n \geqslant 8$, there exists an independent set in $\{V_1, V_2, \cdots, V_{n/2-1}\}$, say V_1, such that there exists at most one edge between $\{v_1, v_2\}$ and $\{v_{n-1}, v_n\}$. If such an edge does not exist, or assume without loss of generality $v_1 v_n \in E(G)$. We redefine $V_1=\{v_1, v_{n-1}\}, V_2=\{v_3, v_4\}, \cdots, V_{n/2-1}=\{v_{n-3}, v_{n-2}\}, V_n=\{v_2, v_n\}$ which is a desired partition, and so the lemma follows. □

Lemma 2.2 If G is a 3-regular and 3-connected graph of order n at least 12, then for each vertex $x \in V(G)$ there exists a vertex $y \in V(G)$ such that the distance between them, $d(x,y)$, is equal to three.

Proof Since G is 3-regular and 3-connected, for each vertex $x \in V(G)$, there are at most nine vertices, each of which has distance at most two with x. Since $n \geqslant 12$, there exists at least one vertex that has distance three with x, and so the lemma follows. □

Lemma 2.3 (Menger's theorem)[1, Theorem 4.5] If G is a k-connected graph, then there are k vertex-disjoint (resp. edge-disjoint) paths between any two vertices x and y in G.

Proof of Theorem 1.1 Let G be a k-connected graph of order n. We first show that

$$\xi(G) \leqslant (n-1)\lceil (n-k-1)/k \rceil - (n-\Delta-1).$$

To this end, let $m=\lceil (n-1)/k \rceil$. We define a routing R in G as follows. For any vertex x of G, partition $V(G)\setminus\{x\}$ into m subsets $A_1, A_2, \cdots, A_m$ such that $|A_1|=|A_2|=\cdots=|A_{m-1}|=k$ and $|A_m|=n-1-k(m-1)$, where $A_1 \subseteq N_G(x)$. Since G is k-connected, by Lemma 2.3 we can choose k vertex-disjoint paths (as short as possible) from x to each vertex of the subset $A_j, j=1,2,\cdots, m-1$ and $n-1-k(m-1)$ paths from x to each vertex of A_m. Let R_x be the set as defined above paths from x to each vertex of others in G. Do this construction for each vertex of G and obtain a routing R in G, that is, $R=\bigcup_{x \in V(G)} R_x$.

We now estimate the load of any vertex z with respect to R. If $z \in A_1$, then at most one of the paths from x to $A_j (j \neq 1)$ in R_x passes through z, and so at most $m-1$ paths in R_x pass through z. If $z \in A_i (i \neq 1)$, then all of paths from x to A_1 in R_x do not pass through z since $A_1 \subseteq N_G(x)$, at most one of paths from x to $A_j (j \neq 1, i)$ in R_x passes through z, and so at most $m-2$ paths in R_x pass through z. Thus, summing up all x yields the load of each vertex z with respect to R,

$$\xi(G,R,z) \leqslant \Delta(m-1)+(n-1-\Delta)(m-2) = (n-1)\lceil (n-k-1)/k \rceil - (n-\Delta-1).$$

Similarly, we can prove that

$$\pi(G) \leqslant n\lceil (n-k-1)/k \rceil - (n-\Delta)$$

by replacing k vertex-disjoint paths with k edge-disjoint paths, and omitted here for details. □

Proof of Theorem 1.3 Let G be a 3-regular and 3-connected graph of order n. Then n is even at least 4. Clearly, $G=K_4$ for $n=4$ and $G=K_{3,3}$ or $G=K_2\times K_3$ for $n=6$. $\xi(K_4)=0$ and $\xi(K_{3,3})=\xi(K_2\times K_3)=2$. Thus we can assume $n\geqslant 8$ below. We prove the theorem by considering the following three cases, respectively.

Case 1 $n=3m-2$. In this case, $\lceil (n-4)/3 \rceil = (n-4)/3$. By Corollary 1.2 for $k=3$, we have

$$\begin{aligned}\xi(G) \leqslant & (n-1)(n-4)/3-(n-4) = \\ & (n-4)(n-4)/3 < \\ & (n-3)(n-4)/3.\end{aligned}$$

Case 2 $n=3m-1$. In this case $m=\lceil (n-1)/3 \rceil$ is odd and $m\geqslant 3$. By Lemma 2.1, we can partition $V(G)$ into $n/2$ independent sets of order two. For a given vertex x in G, let $\mathscr{I}_x=\{I_1, I_2, \cdots, I_{n/2}\}$ be such a partition of $V(G)$, where $I_1=\{x,y\}$. We define the set of paths R_x to obtain a routing R in G as the proof of Theorem 1.1 for $k=3$, where $A_1=N_G(x)$ and $A_m=\{y\}$ such that $\{x,y\}=I_1\in \mathscr{I}_x$. Moreover, the path $R(x,y)$ from x to y and the path $R(y,x)$ from y to x in R are internally disjoint. Such two paths exist since G is 3-connected.

We now estimate the load of any vertex z with respect to R. We first do not consider paths between x and y in R. If $z\in A_1$, then at most one of the paths from x to A_j $(2\leqslant j\leqslant m-1)$ in R_x passes through z, and so at most $m-2$ paths in R_x pass through z. If $z\in A_i$ $(2\leqslant i\leqslant m)$, then all paths from x to A_1 in R_x do not pass through z since $A_1=N_G(x)$, and at most one of the paths from x to A_j $(2\leqslant j\leqslant m-1, j\neq i)$ in R_x passes through z. Thus, when $z\in A_i$ $(2\leqslant i\leqslant m-1)$, at most $m-3$ paths in R_x pass through z; when $z\in A_m=\{y\}$, at most $m-2$ paths in R_x pass through z. We now consider paths between x and y in R. Since $R(x,y)$ and $R(y,x)$ are internally disjoint, at most one of them passes through z wherever z is. There are exactly $n/2$ independent sets of order two in G, which offer the load of the vertex z at most $n/2$. Thus, summing up all x yields the load of any vertex z with respect to R,

$$\begin{aligned}&\xi(G,R,z) \leqslant \\ &\quad 3(m-2)+(n-5)(m-3)+(m-2)+n/2 = \\ &\quad 4(m-2)+(3m-6)(m-3)+(3m-1)/2 = \\ &\quad 3m^2-\frac{19}{2}m+\frac{19}{2} \leqslant \\ &\quad 3m^2-9m+7 = \\ &\quad \lceil (n-3)(n-4)/3 \rceil.\end{aligned}$$

Case 3 $n=3m$. In this case $m=\lceil (n-1)/3 \rceil$ is even and $m\geqslant 3$, which implies $n\geqslant 10$. If $n=10$, then G is Petersen graph and $\xi(G)=6$. Now assume $n\geqslant 12$, that is, $m\geqslant 4$. For a vertex x in G, we define the set of paths R_x to obtain a routing R in G as the proof of Theorem 1.1 for $k=3$, where $A_1=N_G(x)$ and $A_m=\{x_1,x_2\}$.

If $z\in A_1$, then at most $m-1$ paths in R_x pass through z stated as the proof of Theorem 1.1. If $d(x,z)=3$ (z may be such a vertex by Lemma 2.2), then $z\in A_i$ for some $i(\neq 1)$, and any path from x to z must pass through A_1 and A_j for some $j(\neq 1,i)$, which implies that at most $m-3$ paths in R_x pass through z. Otherwise at most $m-2$ paths in R_x pass through z. Thus, summing up all x and noting that there is at least one vertex x in G such that $d(x,z)=3$, we have the load of any vertex z with respect to R,

$$\begin{aligned}&\xi(G,R,z) \leqslant \\ &\quad 3(m-1)+(m-3)+(3m-5)(m-2) = \\ &\quad 3m^2-7m+4 = \\ &\quad \lceil (n-3)(n-4)/3 \rceil.\end{aligned}$$

We complete the proof. □

References

[1] Xu Jun-ming. Theory and Application of Graphs [M]. Dordrecht/Boston/ London: Kluwer Academic Publishers, 2003.

[2] Xu Jun-ming. Topological Structure and Analysis of Interconnection Networks [M]. Dordrecht/ Boston/ London: Kluwer Academic Publishers, 2001.

[3] Chung F R K, Coffman E G Jr, Reiman M I, et al.

(下转第 495 页)

到了把得到的方程变为常系数方程的条件，这样就得到了显式解存在的一个充分条件．虽然是在$\alpha=\beta=\gamma=0$这个条件下得到了结论，但是当这个条件不满足时，我们同样可以根据相同的方法来解决这个问题，不过要复杂些．

参考文献（References）

[1] Davis M，Lischka F. Convertible bonds with market risk and credit risk[R]. Tokyo：Tokyo-Mitssubishi International Plc，1999.

[2] Hull J. Options，Futures，and Other Derivatives[M]. Forth ed.. New York：Prentice Hall，1999.

[3] Bluman G W，Kumei S. Symmetries and Differential Equations[M]. Berlin：Springer-Verlag，1989.

[4] Jarrow R A，Turnbull S M. Pricing derivatives on financial securities subject to credit risk[J]. The Journal of Finance，1995，50(1)：53-85.

（上接第 459 页）

The forwardingindex of communication networks [J]. IEEE Transaction on Information Theory，1987，33(2)：224-232.

[4] Heydemann M C，Meyer J C，Sotteau D. On the forwarding index of networks [J]. Discrete Applied Mathematics，1989，23(2)：103-123.

[5] Fernandez de la Vega W，Manoussakis Y. The forwarding index of communication networks with given connectivity [J]. Discrete Applied Mathematics，1992，37-38：147-155.

[6] Heydemann M C，Meyer J C，Opatrny J，et al. Forwarding indices of k-connected graphs [J]. Discrete Applied Mathematics，1992，37-38：287-296.

[7] Heydemann M C，Meyer J C，Sotteau D. On the forwarding index problem for small graphs [J]. Ars Combinatoria，1988，25：253-266.

[8] Hou Xin-min，Xu Min，Xu Jun-ming. Forwarding indices of folded cubes [J]. Discrete Applied Mathematics，2005，145(3)：490-492.

[9] Xu Min，Hou Xin-min，Xu Jun-ming. The proof of a conjecture of Bouabdallah and Sotteau [J]. Networks，2004，44(4)：292-296.

[10] Xu Min，Xu Jun-ming. The forwarding index of augmented cubes [J]. Information Processing Letters，2007，101(5)：185-189.

[11] Xu Min，Xu Jun-ming，Hou Xin-min. On edge-forwarding index of graphs with degree restriction [J]. Journal of University of Science and Technology of China，2005，35(6)：732-737.

[12] Xu Jun-ming，Xu Min，Hou Xin-min. Forwarding indices of Cartesian product graphs [J]. Taiwanese Journal of Mathematics，2006，10(5)：1 305-1 315.

[13] Saad R. Complexity of the forwarding index problem [J]. SIAM Journal of Discrete Mathematics，1993，6(3)：418-427.