

# **$(n, 2n)$ -Dominating Numbers of Undirected Toroidal Mesh $C(3, 3, \dots, 3)$**

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**Abstract** The  $(d, k)$ -dominating number is a new measure to characterize reliability of resource-sharing in fault tolerant networks. This paper obtains that the  $(n, 2n)$ -dominating number of the  $n$ -dimensional undirected toroidal mesh  $C(3, 3, \dots, 3)$  is equal to 3 ( $n \geq 3$ ).

**Keywords** reliability; wide-diameter; undirected toroidal mesh;  $(d, k)$ -dominating number.

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## 1. Introduction

In this paper, we quote from [1] the terminology and notations and use the graphs to represent networks. Let  $G = (V, E)$  be a  $k$ -connected graph (simple undirected). By Menger's theorem, we know graph  $G$  contains at least  $k$  internally disjoint  $(x, y)$ -paths for any two distinct vertices  $x$  and  $y$  in  $G$ .

In order to characterize the reliability of transmission delay in a network, Hsu and Lyuu<sup>[2]</sup>, Fladrin and Li<sup>[3]</sup> independently introduced wide-diameter as follows:

**Definition 1** Let  $G$  be a  $k$ -connected graph. The distance with width  $k$  from vertices  $x$  to  $y$ , denoted by  $d_k(G; x, y)$ , is the minimum number  $d$  for which there are  $k$  internally disjoint  $(x, y)$ -paths in  $G$  of length at most  $d$ . The diameter with width  $k$ , denoted by  $d_k(G)$ , is the minimum number  $d$  for which there are  $k$  internally disjoint  $(x, y)$ -paths in  $G$  of length at most  $d$  for any distinct vertices  $x$  and  $y$  in  $G$ .

In a real-time processing network, Li and Xu<sup>[4]</sup> defined a new parameter  $(d, k)$ -dominating number, which, in some sense, can more accurately characterize the reliability of resources-sharing in a fault-tolerant network.

**Definition 2** Let  $d$  ( $\geq 1$ ) be an integer,  $G$  be a  $k$  ( $\geq 1$ )-connected graph,  $\emptyset \neq S \subset V(G)$ , and  $y \in V(G - S)$ . A path from  $y$  to some vertex in  $S$  is called a  $(y, S)$ -path. For a given integer  $d$ , if there are  $k$  internally disjoint  $(y, S)$ -paths in  $G$  of length at most  $d$ , then we say

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that  $S$  can  $(d, k)$ -dominate  $y$ . If  $S$  can  $(d, k)$ -dominate every vertex in  $G - S$ , then  $S$  is called a  $(d, k)$ -dominating set of  $G$ . We use the symbol  $S_{d,k}(G)$  to denote a set of all  $(d, k)$ -dominating sets in  $G$ . The parameter

$$\gamma_{d,k}(G) = \min\{|S| : S \in S_{d,k}(G)\}$$

is called the  $(d, k)$ -dominating number of  $G$ . A  $(d, k)$ -dominating set  $S$  of  $G$  is called minimum if  $|S| = \gamma_{d,k}(G)$ .

Clearly,  $(1, 1)$ -dominating set and  $(1, 1)$ -dominating number are usually dominating set and dominating number. Thus,  $(d, k)$ -dominating set and  $(1, 1)$ -dominating number are direct generalizations of the dominating set and dominating number. However, the problem determining the dominating number of a graph is NP-complete [5], and, hence, the problem finding  $(d, k)$ -dominating number is NP-complete too. Thus, it is very important to find the  $(d, k)$ -dominating number of some well-known networks for their wide applications. Of course, we have  $\gamma_{d,k}(G) = 1$  for  $d \geq d_k(G)$  and  $\gamma_{d,k}(G) \geq 2$  for  $d < d_k(G)$ . For such graphs  $G$ , it is of interest to determine values of  $\gamma_{d,k}(G)$  for  $d < d_k(G)$ .

**Definition 3** We define the  $n$ -dimensional toroidal mesh  $C(d_1, d_2, \dots, d_n)$  with  $V = \{(x_1, x_2, \dots, x_n) | x_i \in \{0, 1, \dots, d_i - 1\}, i = 1, 2, \dots, n\}$  as the set of vertices. The vertex  $(x_1, x_2, \dots, x_n)$  is adjacent to the vertex  $(y_1, y_2, \dots, y_n)$  if and only if there exists  $i \in \{1, \dots, n\}$  such that

$$\begin{cases} x_j = y_j, & j \neq i, \\ x_i - y_i = 1 \text{ or } d_i - 1 \pmod{d_i}, & j = i. \end{cases}$$

The  $n$ -dimensional undirected toroidal mesh  $C(d_1, d_2, \dots, d_n)$  is  $2n$ -regular and is vertex-transitive. Its connectivity is  $2n$ . It was proved in [6] that the diameter of  $d(C(d_1, d_2, \dots, d_n))$  is  $\sum_{i=1}^n \lfloor \frac{d_i}{2} \rfloor$ , and the wide diameter  $d_{2n}(C(d_1, d_2, \dots, d_n))$  is  $\sum_{i=1}^n \lfloor \frac{d_i}{2} \rfloor + 1$ . The toroidal mesh is widely used in network's theory (see [7, 8, 9]). Lü Changhong and Zhang Keming<sup>[10]</sup> proved the  $(d, 2n)$ -dominating number of  $n$ -dimensional undirected toroidal mesh  $C(d_1, d_2, \dots, d_n)$  ( $\neq C(3, 3, \dots, 3)$ ) is 2 ( $n \geq 3, d_i \geq 3, i \in \{1, \dots, n\}$ ) for  $d = d(C(d_1, d_2, \dots, d_n))$ .

In this paper, we denote by  $C_n(3)$  the  $n$ -dimensional undirected toroidal mesh  $C(3, 3, \dots, 3)$ , the diameter  $d(C_n(3))$  of which is  $n$ . We will prove that the  $(n, 2n)$ -dominating number of  $C_n(3)$  is 3.

## 2. Main results

**Theorem** Let  $G = C_n(3)$ . Then we have  $\gamma_{n,2n}(G) = 3$ .

**Proof** First we prove  $\gamma_{n,2n}(G) > 2$ . It is easy to verify  $\gamma_{n,2n}(G) \geq 2$  for  $d(G) = n$  and  $d_{2n}(G) = n + 1$ . If  $\gamma_{n,2n}(G) = 2$ , we suppose  $S = \{u, v\}$  is one of the  $(n, 2n)$ -dominating sets by vertex-transitivity, where  $u = (0, 0, \dots, 0)$ ,  $v = (x_1, x_2, \dots, x_n)$ ,  $x_i \in \{0, 1, 2\}$ ,  $i = 1, 2, \dots, n$ , and  $v \neq u$ . By vertex-transitivity, we only consider three cases as follows:

**Case 1**  $v = (x_1, \dots, x_k, 0, \dots, 0)$ , where  $x_1 \neq 0, \dots, x_k \neq 0, 1 \leq k \leq n - 2$ .

Let  $y = (3 - x_1, \dots, 3 - x_k, 1, \dots, 1) \in V(G) - S$ . Then we have  $3 - x_i \neq x_i$  and  $x_i \neq 0$  when  $i = 1, 2, \dots, n - 2$ . Since  $G$  is  $2n$ -regular, there is one path which goes through vertex

$(3 - x_1, \dots, 3 - x_k, 2, 1, \dots, 1)$  of  $2n$  internally disjoint  $(y, S)$ -paths. Let

$P_1 : y = (3 - x_1, \dots, 3 - x_k, 1, \dots, 1) \rightarrow (3 - x_1, \dots, 3 - x_k, 2, 1, \dots, 1) \rightarrow \dots \rightarrow u = (0, \dots, 0)$   
or  $v = (x_1, \dots, x_k, 0, \dots, 0)$ .

Of course we have  $|P_1| \geq n + 1$ , so  $S$  cannot  $(n, 2n)$ -dominate the vertex  $y$ .

**Case 2**  $v = (x_1, \dots, x_{n-1}, 0)$ , where  $x_1 \neq 0, \dots, x_{n-1} \neq 0$ .

Let  $y = (3 - x_1, \dots, 3 - x_{n-1}, 1) \in V(G) - S$ . Then we have  $3 - x_i \neq x_i$  and  $x_i \neq 0$  when  $i = 1, 2, \dots, n - 1$ . And there is one path which goes through vertex  $(3 - x_1, \dots, 3 - x_{n-1}, 2)$  of  $2n$  internally disjoint  $(y, S)$ -paths. Let

$P_2 : y = (3 - x_1, \dots, 3 - x_{n-1}, 1) \rightarrow (3 - x_1, \dots, 3 - x_{n-1}, 2) \rightarrow \dots \rightarrow u = (0, \dots, 0)$  or  
 $v = (x_1, \dots, x_{n-1}, 0)$ .

We have  $|P_2| \geq n + 1$ ,  $S$  cannot  $(n, 2n)$ -dominate the vertex  $y$ .

**Case 3**  $v = (x_1, \dots, x_n)$ , where  $x_1 \neq 0, \dots, x_n \neq 0$ .

Let  $y = (x_1, 3 - x_2, \dots, 3 - x_n) \in V(G) - S$ . Then we have  $3 - x_i \neq x_i$  and  $x_i \neq 0$  when  $i = 1, 2, \dots, n$ . And there is one path which goes through vertex  $(3 - x_1, \dots, 3 - x_n)$  of  $2n$  internally disjoint  $(y, S)$ -paths. Let

$P_3 : y = (x_1, 3 - x_2, \dots, 3 - x_n) \rightarrow (3 - x_1, \dots, 3 - x_n) \rightarrow \dots \rightarrow u = (0, \dots, 0)$  or  $v = (x_1, \dots, x_n)$ .

We have  $|P_3| \geq n + 1$ ,  $S$  cannot  $(n, 2n)$ -dominate the vertex  $y$ .

Thus  $S$  is not an  $(n, 2n)$ -dominating set of  $G$ , so we have  $\gamma_{n, 2n}(G) > 2$  by vertex-transitivity.

Next we prove  $\gamma_{n, 2n}(G) \leq 3$ . Let  $S = \{u, v, w\} \subset V(G)$ , where  $u = (0, \dots, 0)$ ,  $v = (1, \dots, 1)$ ,  $w = (2, \dots, 2)$ . Now we prove  $S$  can  $(n, 2n)$ -dominate any vertex  $x \in V(G) - S$ . We only consider the following cases:

**Case 1**  $x = (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k})$ , where  $k \geq 1$  and  $n - k \geq 1$ .

**Subcase 1a**  $k > 1$ , and  $n - k > 1$ .

We can construct the  $2n$  internally disjoint  $(x, S)$ -paths, denoted by  $P_i$  ( $1 \leq i \leq 2n$ ):

$$\begin{aligned} P_1 : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{0, 1, \dots, 1}^{n-k-1}) \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{0, 0, 1, \dots, 1}^{n-k-2}) \\ &\rightarrow \dots \rightarrow (0, \dots, 0, 1) \rightarrow u = (0, \dots, 0); \\ P_2 : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{1, 0, 1, \dots, 1}^{n-k-2}) \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{1, 0, 0, 1, \dots, 1}^{n-k-3}) \\ &\rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{1, 0, \dots, 0}^{n-k-1}) \rightarrow u = (0, \dots, 0); \\ &\vdots \\ P_{n-k} : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1, 0}^{n-k-1}) \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{0, 1, \dots, 1, 0}^{n-k-2}) \\ &\rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{0, 0, 1, \dots, 1, 0}^{n-k-3}) \rightarrow \dots \rightarrow u = (0, \dots, 0); \end{aligned}$$

$$\begin{aligned}
 P_{n-k+1} : x &= (\overbrace{0, \dots, 0}^k, 1, \dots, 1) \rightarrow (\overbrace{1, 0, \dots, 0}^{k-1}, 1, \dots, 1) \rightarrow (\overbrace{1, 1, 0, \dots, 0}^{k-2}, 1, \dots, 1) \\
 &\rightarrow \dots \rightarrow (\overbrace{1, \dots, 1}^{k-1}, 0, 1, \dots, 1) \rightarrow v = (1, \dots, 1); \\
 P_{n-k+2} : x &= (\overbrace{0, \dots, 0}^k, 1, \dots, 1) \rightarrow (0, \overbrace{1, 0, \dots, 0}^{k-2}, 1, \dots, 1) \\
 &\rightarrow (0, 1, 1, \overbrace{0, \dots, 0}^{k-3}, 1, \dots, 1) \rightarrow \dots \rightarrow (0, 1, \dots, 1) \rightarrow v = (1, \dots, 1); \\
 &\vdots \\
 P_n : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^{k-1}, \overbrace{1, 1, \dots, 1}^{n-k}) \rightarrow (\overbrace{1, 0, \dots, 0}^{k-2}, 1, 1, \dots, 1) \\
 &\rightarrow (1, 1, \overbrace{0, \dots, 0}^{k-3}, 1, 1, \dots, 1) \rightarrow \dots \rightarrow (\overbrace{1, \dots, 1}^{k-1}, 0, 1, 1, \dots, 1) \\
 &\rightarrow v = (1, \dots, 1); \\
 P_{n+1} : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^k, 2, \overbrace{1, \dots, 1}^{n-k-1}) \rightarrow (\overbrace{0, \dots, 0}^k, 2, 2, \overbrace{1, \dots, 1}^{n-k-2}) \\
 &\rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^k, 2, \dots, 2) \rightarrow (2, \overbrace{0, \dots, 0}^{k-1}, 2, \dots, 2) \\
 &\rightarrow (2, 2, \overbrace{0, \dots, 0}^{k-2}, 2, \dots, 2) \rightarrow \dots \rightarrow w = (2, \dots, 2); \\
 P_{n+2} : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^k, 1, 2, \overbrace{1, \dots, 1}^{n-k-2}) \\
 &\rightarrow (\overbrace{0, \dots, 0}^k, 1, 2, 2, \overbrace{1, \dots, 1}^{n-k-3}) \rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^k, 1, 2, \dots, 2) \\
 &\rightarrow (2, \overbrace{0, \dots, 0}^{k-1}, 1, 2, \dots, 2) \rightarrow (2, 2, \overbrace{0, \dots, 0}^{k-2}, 1, 2, \dots, 2) \rightarrow \dots \\
 &\rightarrow (\overbrace{2, \dots, 2}^k, 1, 2, \dots, 2) \rightarrow w = (2, \dots, 2); \\
 &\vdots \\
 P_{2n-k} : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k-1}, 2) \rightarrow (2, \overbrace{0, \dots, 0}^{k-1}, 1, \dots, 1, 2) \\
 &\rightarrow (2, 2, \overbrace{0, \dots, 0}^{k-2}, 1, \dots, 1, 2) \rightarrow \dots \rightarrow (\overbrace{2, \dots, 2}^k, 1, \dots, 1, 2) \\
 &\rightarrow (\overbrace{2, \dots, 2}^k, 2, 1, \dots, 1, 2) \rightarrow \dots \rightarrow (\overbrace{2, \dots, 2}^k, 2, \dots, 2, 1, 2) \\
 &\rightarrow w = (2, \dots, 2); \\
 P_{2n-k+1} : x &= (\overbrace{0, \dots, 0}^k, 1, \dots, 1) \rightarrow (2, \overbrace{0, \dots, 0}^{k-1}, 1, \dots, 1) \rightarrow (2, 2, \overbrace{0, \dots, 0}^{k-2}, 1, \dots, 1) \\
 &\rightarrow \dots \rightarrow (\overbrace{2, \dots, 2}^k, 1, \dots, 1) \rightarrow (\overbrace{2, \dots, 2}^k, 2, 1, \dots, 1) \rightarrow \dots \\
 &\rightarrow w = (2, \dots, 2);
 \end{aligned}$$

$$\begin{aligned}
P_{2n-k+2} : x &= (\overbrace{0, \dots, 0}^k, 1, \dots, 1) \rightarrow (0, 2, \overbrace{0, \dots, 0}^{k-2}, 1, \dots, 1) \\
&\rightarrow (0, 2, 2, \overbrace{0, \dots, 0}^{k-3}, 1, \dots, 1) \rightarrow \cdots \rightarrow (0, \overbrace{2, \dots, 2}^{k-1}, 1, \dots, 1) \\
&\rightarrow (0, \overbrace{2, \dots, 2}^{k-1}, 2, 1, \dots, 1) \rightarrow \cdots \rightarrow (0, \overbrace{2, \dots, 2}^{k-1}, 2, \dots, 2) \rightarrow w = (2, \dots, 2); \\
&\vdots \\
P_{2n} : x &= (\overbrace{0, \dots, 0}^k, \overbrace{1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^{k-1}, \overbrace{2, 1, \dots, 1}^{n-k}) \rightarrow (\overbrace{0, \dots, 0}^{k-1}, \overbrace{2, 2, 1, \dots, 1}^{n-k-1}) \\
&\rightarrow \cdots \rightarrow (\overbrace{0, \dots, 0}^{k-1}, 2, \dots, 2) \rightarrow (2, \overbrace{0, \dots, 0}^{k-2}, 2, \dots, 2) \\
&\rightarrow (2, \overbrace{2, 0, \dots, 0}^{k-3}, 2, \dots, 2) \rightarrow \cdots \rightarrow w = (2, \dots, 2).
\end{aligned}$$

We can easily get  $|P_i| \leq n$ ,  $i = 1, \dots, 2n$ .

**Subcase 1b**  $x = (0, 1, \dots, 1)$  for  $k = 1$ , or  $x = (0, \dots, 0, 1)$  for  $n - k = 1$ .

One can carry out the proof in the same way as in Subcase 1a.

**Case 2**  $x = (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2})$ , where  $k_1 \geq 1, k_2 \geq 1, n - k_1 - k_2 \geq 1$ .

**Subcase 2a**  $k_1 > 1, k_2 > 1$ , and  $n - k_1 - k_2 > 1$ .

We can construct the  $2n$  internally disjoint  $(x, S)$ -paths, denoted by  $P_i$  ( $1 \leq i \leq 2n$ ):

$$\begin{aligned}
P_1 : x &\rightarrow (1, \overbrace{0, \dots, 0}^{k_1-1}, 1, \dots, 1, 2, \dots, 2) \rightarrow (1, 1, \overbrace{0, \dots, 0}^{k_1-2}, 1, \dots, 1, 2, \dots, 2) \\
&\rightarrow \cdots \rightarrow (\overbrace{1, \dots, 1}^{k_1+k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \rightarrow (\overbrace{1, \dots, 1}^{k_1+k_2+1}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}) \rightarrow \cdots \\
&\rightarrow (\overbrace{1, \dots, 1}^{n-1}, 2) \rightarrow v = (1, \dots, 1); \\
P_2 : x &\rightarrow (0, 1, \overbrace{0, \dots, 0}^{k_1-2}, 1, \dots, 1, 2, \dots, 2) \rightarrow (0, 1, 1, \overbrace{0, \dots, 0}^{k_1-3}, 1, \dots, 1, 2, \dots, 2) \\
&\rightarrow \cdots \rightarrow (0, \overbrace{1, \dots, 1}^{k_1+k_2-1}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \rightarrow (0, \overbrace{1, \dots, 1}^{k_1+k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}) \rightarrow \cdots \\
&\rightarrow (0, \overbrace{1, \dots, 1}^{n-1}, 2) \rightarrow (0, 1, \dots, 1) \rightarrow v = (1, \dots, 1); \\
&\vdots \\
P_{k_1} : x &\rightarrow (\overbrace{0, \dots, 0}^{k_1-1}, \overbrace{1, \dots, 1}^{k_2+1}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \rightarrow (\overbrace{0, \dots, 0}^{k_1-1}, \overbrace{1, \dots, 1}^{k_2+2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}) \\
&\rightarrow \cdots \rightarrow (\overbrace{0, \dots, 0}^{k_1-1}, 1, \dots, 1) \rightarrow (1, \overbrace{0, \dots, 0}^{k_1-2}, 1, \dots, 1) \rightarrow \cdots \\
&\rightarrow (1, 1, \overbrace{0, \dots, 0}^{k_1-3}, 1, \dots, 1) \rightarrow \cdots \rightarrow v = (1, \dots, 1);
\end{aligned}$$

$$\begin{aligned}
 P_{k_1+1} : x &\rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{2, 1, \dots, 1}^{k_2-1}, 2, \dots, 2) \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{2, 2, 1, \dots, 1}^{k_2-2}, 2, \dots, 2) \\
 &\rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^{k_1}, 2, \dots, 2) \rightarrow (2, \overbrace{0, \dots, 0}^{k_1-1}, 2, \dots, 2) \\
 &\rightarrow (2, 2, \overbrace{0, \dots, 0}^{k_1-2}, 2, \dots, 2) \rightarrow \dots \rightarrow w = (2, \dots, 2); \\
 &\vdots \\
 P_{k_1+k_2} : x &\rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2-1}, \overbrace{2, \dots, 2}^{n-k_1-k_2+1}) \rightarrow (2, \overbrace{0, \dots, 0}^{k_1-1}, \overbrace{1, \dots, 1}^{k_2-1}, 2, \dots, 2) \\
 &\rightarrow (2, 2, \overbrace{0, \dots, 0}^{k_1-2}, \overbrace{1, \dots, 1}^{k_2-1}, 2, \dots, 2) \rightarrow \dots \rightarrow (2, \dots, 2, \overbrace{1, \dots, 1}^{k_1}, 2, \dots, 2) \\
 &\rightarrow (2, \dots, 2, \overbrace{1, \dots, 1}^{k_1+1}, 2, \dots, 2) \rightarrow \dots \rightarrow w = (2, \dots, 2); \\
 P_{k_1+k_2+1} : x &\rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}) \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2}, 0, 0, \overbrace{2, \dots, 2}^{n-k_1-k_2-2}) \\
 &\rightarrow \dots \rightarrow (0, \dots, 0, \overbrace{1, \dots, 1}^{k_1+1}, \overbrace{0, \dots, 0}^{k_2-1}) \rightarrow (0, \dots, 0, \overbrace{1, \dots, 1}^{k_1+1}, 0, \dots, 0) \\
 &\rightarrow \dots \rightarrow u = (0, \dots, 0); \\
 &\vdots \\
 P_n : x &\rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}, 0) \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{0, 1, \dots, 1}^{k_2-1}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}, 0) \\
 &\rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^{k_1+k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}, 0) \rightarrow (\overbrace{0, \dots, 0}^{k_1+k_2+1}, \overbrace{2, \dots, 2}^{n-k_1-k_2-2}, 0) \rightarrow \dots \\
 &\rightarrow u = (0, \dots, 0); \\
 P_{n+1} : x &\rightarrow (2, \overbrace{0, \dots, 0}^{k_1-1}, 1, \dots, 1, 2, \dots, 2) \rightarrow (2, 2, \overbrace{0, \dots, 0}^{k_1-2}, 1, \dots, 1, 2, \dots, 2) \\
 &\rightarrow \dots \rightarrow (2, \dots, 2, \overbrace{1, \dots, 1}^{k_1}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \rightarrow (2, \dots, 2, \overbrace{1, \dots, 1}^{k_1+1}, \overbrace{2, \dots, 2}^{k_2-1}) \\
 &\rightarrow \dots \rightarrow (2, \dots, 2, \overbrace{1, \dots, 1}^{k_1+k_2-1}, 2, \dots, 2) \rightarrow w = (2, \dots, 2); \\
 &\vdots \\
 P_{n+k_1} : x &\rightarrow (\overbrace{0, \dots, 0}^{k_1-1}, \overbrace{2, 1, \dots, 1}^{k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \rightarrow (\overbrace{0, \dots, 0}^{k_1-1}, \overbrace{2, 2, 1, \dots, 1}^{k_2-1}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \\
 &\rightarrow (\overbrace{0, \dots, 0}^{k_1-1}, \overbrace{2, \dots, 2}^{n-k_1+1}) \rightarrow (2, \overbrace{0, \dots, 0}^{k_1-2}, \overbrace{2, \dots, 2}^{n-k_1+1}) \rightarrow \dots \rightarrow w = (2, \dots, 2); \\
 P_{n+k_1+1} : x &\rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{0, 1, \dots, 1}^{k_2-1}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{0, 0, 1, \dots, 1}^{k_2-2}, \overbrace{2, \dots, 2}^{n-k_1-k_2}) \\
 &\rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^{k_1+k_2}, 2, \dots, 2) \rightarrow (\overbrace{0, \dots, 0}^{k_1+k_2+1}, 2, \dots, 2) \rightarrow \dots \\
 &\rightarrow u = (0, \dots, 0);
 \end{aligned}$$

$$\begin{aligned}
& \vdots \\
P_{n+k_1+k_2} : x & \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2-1}, \overbrace{0, 2, \dots, 2}^{n-k_1-k_2}) \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2-1}, \overbrace{0, 0, 2, \dots, 2}^{n-k_1-k_2-1}) \\
& \rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2-1}, \overbrace{0, \dots, 0}^{k_2-2}) \rightarrow (\overbrace{0, \dots, 0}^{k_1+1}, \overbrace{1, \dots, 1}^{k_2-2}, \overbrace{0, \dots, 0}^{n-k_1-k_2-1}) \\
& \rightarrow \dots \rightarrow u = (0, \dots, 0); \\
P_{n+k_1+k_2+1} : x & \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2+1}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}) \rightarrow \dots \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{n-k_1}) \\
& \rightarrow (1, \overbrace{0, \dots, 0}^{k_1-1}, \overbrace{1, \dots, 1}^{n-k_1}) \rightarrow \dots \rightarrow v = (1, \dots, 1); \\
& \vdots \\
P_{2n} : x & \rightarrow (\overbrace{0, \dots, 0}^{k_1}, \overbrace{1, \dots, 1}^{k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}, 1) \\
& \rightarrow (1, \overbrace{0, \dots, 0}^{k_1-1}, \overbrace{1, \dots, 1}^{k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}, 1) \\
& \rightarrow \dots \rightarrow (\overbrace{1, \dots, 1}^{k_1+k_2}, \overbrace{2, \dots, 2}^{n-k_1-k_2-1}, 1) \rightarrow (\overbrace{1, \dots, 1}^{k_1+k_2+1}, \overbrace{2, \dots, 2}^{n-k_1-k_2-2}, 1) \\
& \rightarrow \dots \rightarrow v = (1, \dots, 1).
\end{aligned}$$

We have  $|P_i| \leq n$ ,  $i = 1, \dots, 2n$ .

**Subcase 2b**  $k_1 = 1$ ,  $k_2 = 1$  or  $n - k_1 - k_2 = 1$ .

One can carry out the proof in the same way as in Subcase 2a.

Therefore we obtain  $\gamma_{n,2n}(G) \leq 3$  from above.

Thus we can see  $\gamma_{n,2n}(G) = 3$ . The proof of the Theorem is completed.  $\square$

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