Restricted Edge-Connectivity of de Bruijn Digraphs*

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Abstract

The restricted edge-connectivity of a graph is an important parameter to measure fault-tolerance of interconnection networks. This paper determines that the restricted edge-connectivity of the de Bruijn digraph $B(d,n)$ is equal to $2d-2$ for $d \geq 2$ and $n \geq 2$ except $B(2,2)$. As consequences, the super edge-connectedness of $B(d,n)$ is obtained immediately.

Keywords: Connectivity, Restricted edge-connectivity, Super edge-connected, de Bruijn digraphs, Kautz digraphs

AMS Subject Classification: 05C40

1 Introduction

It is well-known that when the underlying topology of an interconnection network is modelled by a graph or digraph G, the edge-connectivity $\lambda(G)$ of G is an important measurement for fault-tolerance of the network. This paper considers the de Bruijn digraph $B(d,n)$. It has been shown that $\lambda(B(d,n)) = d-1$ and $\lambda(K(d,n)) = d$ (see, for example, [9]). A connected graph G is said to be *super edge-connected* if every minimum edge-cut isolates a vertex of G [1]. Soneoka [8] showed that the $B(d,n)$ is super edge-connected for any $d \geq 2$ and $n \geq 1$, and Fàbrega and Fiol [4] proved that $K(d,n)$ is super edge-connected for any $d \geq 3$ and $n \geq 2$.

A quite natural question is how many edges must be removed to disconnect a graph such that every connected component of the resulting graph

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contains no isolated vertex. To measure this type of edge-connectivity, Esfahanian and Hakimi [2, 3] introduced the concept of the restricted edge-connectivity of a graph. The definition given here is slightly different from the original definition. The *restricted edge-connectivity* of a graph G, denoted by $\lambda'(G)$, is the minimum number λ' for which G has a λ'-edge cut F such that every connected component of $G-F$ has at least two vertices. They solved the existence of $\lambda'(G)$ for a given graph by proving that if G is neither $K_{1,n}$ nor K_3, then $\lambda(G) \le \lambda'(G) \le \xi(G)$, where $\xi(G)$ is the minimum edge-degree of G. Clearly, if $\lambda'(G) > \lambda(G)$ then G is super edge-connected. Since then one has paid much attention to the concept and determined the restricted edge-connectivity for many well-known graphs. In particular, λ' has been completely determined for the Kautz digraph $K(d,n)$, the undirected de Bruijn graph $UB(d,n)$ and Kautz graph $UK(d,n)$ (see, for example, [5, 6, 7, 10, 11]). In this paper, we determine λ' for de Bruijn digraph $B(d,n)$.

Theorem For any de Bruijn digraph $B(d,n)$ with $n \ge 1$ and $d \ge 2$,

$$\lambda'(B(d,n)) = \begin{cases} \text{not exist,} & \text{for } n=1 \text{ and } 2 \le d \le 3, \text{or } n=d=2; \\ 2d-4, & \text{for } n=1 \text{ and } d \ge 4; \\ 2d-2, & \text{otherwise.} \end{cases}$$

The proof of the theorem is in Section 3. Our way presented in this paper can prove the result for the Kautz digraph $K(d,n)$ in [5]. However, the methods used in [5] do not work for the de Bruijn digraph $B(d,n)$.

2 Some Lemmas

The de Bruijn digraph $B(d,n)$ has the vertex-set

$$V = \{x_1x_2\cdots x_n : \ x_i \in \{0,1,\cdots,d-1\}, \ i=1,2,\cdots,n\},$$

and the edge-set E, where for $x,y \in V$, if $x = x_1x_2\cdots x_n$, then

$$(x,y) \in E \Leftrightarrow y = x_2x_3\cdots x_n\alpha, \quad \alpha \in \{0,1,\cdots,d-1\}.$$

Clearly, $B(d,1)$ is a complete digraph of order d plus a self-loop at every vertex. It has been shown that $B(d,n)$ is d-regular and $(d-1)$-connected. For more properties of de Bruijn digraphs, the reader is referred to Section 3.2 in [9].

Assume $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$ are two distinct vertices of $B(d,n)$. If the distance from x to y is equal to l, then the unique shortest (x,y)-path

$$\begin{aligned} P: \quad x = x_1x_2\cdots x_n &\to x_2x_3\cdots x_ny_{n-l+1} \to x_3\cdots x_ny_{n-l+1}y_{n-l+2} \to \\ &\cdots \to x_l\cdots x_ny_{n-l+1}\cdots y_{n-1} \to x_{l+1}\cdots x_ny_{n-l+1}\cdots y_n = y. \end{aligned}$$

can be expressed as the following sequence:

$$P = x_1x_2 \cdots x_{l+1} \cdots x_n y_{n-l+1} \cdots y_n,$$

in which any subsequence of length n is a vertex in P.

A pair of directed edges are said to be *symmetric* if they have the same end-vertices but different orientations. The de Bruijn digraph contains pairs of symmetric edges. If there are a pair of symmetric edges between two vertices x and y, then it is not difficult to see that the coordinates of x and y are alternately in two different digits a and b, that is, $x = abab \cdots ab$ and $y = baba \cdots ba$ if n is even, while $x = abab \cdots aba$ and $y = baba \cdots bab$ if n is odd, where $a \neq b$.

We follow [9] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a strongly connected digraph (loops and parallel edges are here allowed). An edge-set F of G is called a *restricted edge-cut* (R-edge-cut, in short) if $G - F$ is not strongly connected and every strongly connected component has at least two vertices. The *restricted edge-connectivity* $\lambda'(G)$ is the minimum cardinality over all R-edge-cuts in G. We observe that there are no R-edge-cuts in $B(2,1)$, $B(2,2)$ and $B(3,1)$, and call these digraphs *trivial*, and otherwise *nontrivial*.

Lemma 1 If $B(d,n)$ is nontrivial, then $\lambda'(B(d,n)) \leq 2d - 2$ for any $d \geq 2$ and $n \geq 2$.

Proof Let G be a nontrivial $B(d,n)$, and suppose that x and y are two different vertices in G with a pair of symmetric edges between them. Then the set of edges $E_G^+(\{x,y\})$ is an edge-cut in G and $|E_G^+(\{x,y\})| = 2d - 2$. Thus, we only need to show that $E_G^+(\{x,y\})$ is an R-edge-cut. To the end, it is sufficient to show that $G - \{x,y\}$ is strongly connected. Let $u = u_1u_2 \cdots u_n$ and $v = v_1v_2 \cdots v_n$ be an arbitrary pair of vertices in $G - \{x,y\}$. We can obtain the result by showing that u and v are strongly connected in $G - \{x,y\}$.

Without loss of generality, we assume n is even, $x = abab \cdots ab$ and $y = baba \cdots ba$, where $a \neq b$ and $a, b \in \{0, 1, \cdots, d-1\}$.

We first consider the case of $n > 2$. Let $z = aab \cdots aba$ and $w = ab \cdots abaa$. Then z is an in-neighbor of x, and w is an out-neighbor of y. Moreover, $(z, w) \in E(B(d,n))$. Suppose that the distance from u to z is equal to l and the distance from w to v is equal to l'. Denote the shortest (u,z)-path by $Q = u_1u_2 \cdots u_l aab \cdots aba$ and the shortest (w,v)-path by $Q' = ab \cdots abaav_{n-l'+1} \cdots v_n$. When $l \leq n-2$, any subsequence of length n in Q contains aa, so Q contains neither x nor y. When $l = n-1$ any subsequence of length n in Q contains aa except the first subsequence of length n, which is u. So Q contains neither x nor y for $l \leq n-1$. For $l = n$, Q contains y only when $u = u_1bab \cdots ab$ with $u_1 \neq a$, which is an in-neighbor of y. Similarly, Q' contains neither x nor y for $l' \leq n-1$, and contains x only when $v = bab\cdots bv_n$ with $v_n \neq a$, which is an out-neighbor of x. We show that u can reach v in $B(d,n) - \{x,y\}$ by constructing a (u,v)-walk according to the following three cases, respectively.

Case 1 If both Q and Q' contain neither x nor y, then u can reach v in $B(d,n) - \{x,y\}$ via a (u,v)-walk $Q + (z,w) + Q'$.

Case 2 If $u = u_1 bab\cdots ab$ with $u_1 \neq a$ and Q' contains neither x nor y, then y is an out-neighbor of u. Let $z_1 = baba\cdots abb$. Then z_1 is another out-neighbor of u. Let $z_2 = abab\cdots bba$, which is an out-neighbor of z_1. Then $Q_1 = abab\cdots bbaabab\cdots abaa$ is a (z_2,w)-walk of length n, and contains neither x nor y since any subsequence of length n in Q_1 contains bb or aa. Thus, u can reach v in $B(d,n) - \{x,y\}$ via a (u,v)-walk $(u,z_1) + (z_1,z_2) + Q_1 + Q'$.

Case 3 If $u = u_1 bab\cdots ab$ with $u_1 \neq a$ and $v = bab\cdots abv_n$ with $v_n \neq a$, then $(u,v) \in E(B(d,n))$, and u can reach v in $B(d,n) - \{x,y\}$ via the edge.

When $n = 2$, we have $d \geq 3$ since $\lambda'(B(2,2))$ doesn't exist. Then $x = ab, y = ba$. Without loss of generality, we can assume $u = u_1u_2, v = v_1v_2$. Then $P = u_1u_2v_1v_2$ is the shortest path from u to v. If the vertex $z = u_2v_1 \notin \{x,y\}$, then we are done. If $z = ab$, then $u = u_1a, v = bv_2$. Since $d \geq 3$, we can construct another (u,v)-walk: u_1acbv_2 where $c \in \{0,1,\cdots,d-1\} \setminus \{a,b\}$. The walk is in $B(d,2) - \{x,y\}$. If $z = ba$, we can also construct a (u,v)-walk in $B(d,2) - \{x,y\}$ in the same way. So u can reach v via a (u,v)-walk in $B(d,2) - \{x,y\}$.

Similarly, v can reach u via a (v,u)-walk in $B(d,n) - \{x,y\}$. Thus, u and v are strongly connected in $B(d,n) - \{x,y\}$. The lemma follows. ∎

Lemma 2 Let H be a subgraph of $B(d,n)$. For $n \geq 2$, if $|V(H)| = t$, then $|E(H)| \leq \frac{1}{2}(t^2+1)$.

Proof From the definition, it is clear that $B(d,n)$ has the following properties for $n \geq 2$:

(i) any two pairs of symmetric edges are not adjacent;

(ii) any two vertices with a self-loop, if any, are not adjacent;

(iii) the end-vertices of any pair of symmetric edges have no self-loops.

Let V_1 be the set of the vertices with a self-loop in H. Suppose H_1 is the subgraph of H induced by V_1 and that H_2 is the subgraph of H induced by $V_2 = V(H) \setminus V_1$. Use E_3 to denote the set of the edges between V_1 and V_2 in H. Then

$$E(H) = E(H_1) \cup E(H_2) \cup E_3.$$

Assume $|V_1| = p$. By the property (ii), $|E(H_1)| = |V_1| = p$. Let $E_{21} = \{(x,y) : (x,y) \in E(H) \,\text{and}\, (y,x) \in E(H)\}$. By the properties (i) and (iii), E_{21} is a matching of H_2 and, hence, $|E_{21}| \leq \lfloor \frac{1}{2}(t-p) \rfloor$. Let $E_{22} =$

$E(H_2) \setminus E_{21}$. Since E_{22} contains no symmetric edges, $|E_{22}| \leq \binom{t-p}{2} = \frac{1}{2}(t-p)(t-p-1)$. It follows that

$$\begin{aligned} E(H_2) &= |E_{21}| + |E_{22}| \leq \left\lfloor \frac{1}{2}(t-p) \right\rfloor + \frac{1}{2}(t-p)(t-p-1) \\ &\leq \frac{1}{2}(t-p) + \frac{1}{2}(t-p)(t-p-1) \\ &= \frac{1}{2}(t-p)^2. \end{aligned}$$

By the property (iii), for any vertex $x \in V_1$ and any vertex $y \in V_2$ there is at most one edge between them. Therefore, $|E_3| \leq p(t-p)$. It follows that

$$\begin{aligned} |E(H)| &= |E(H_1)| + |E(H_2)| + |E_3| \\ &\leq p + \frac{1}{2}(t-p)^2 + p(t-p) \\ &= \frac{1}{2}(t^2 - p^2 + 2p) \\ &\leq \frac{1}{2}(t^2 + 1), \end{aligned}$$

where the last inequality is true because $-p^2 + 2p \leq 1$ for any p. The lemma follows. ∎

Let G be a nontrivial $B(d,n)$ and F be a minimum R-edge-cut of G. Then, $V(G)$ can be partitioned into two disjoint nonempty sets X and Y such that $F = E(X,Y)$, where $E(X,Y)$ denotes the set of the edges from X to Y in G. Let X_0 and Y_0 be the sets of the initial and terminal vertices of the edges of F, respectively. Let

$$\begin{array}{ll} d_G(x, X_0) = \min\{d_G(x,u) : \ u \in X_0\}, & m = \max\{d_G(x, X_0) : \ x \in X\}; \\ d_G(Y_0, y) = \min\{d_G(v,y) : \ v \in Y_0\}, & m' = \max\{d_G(Y_0, y) : \ y \in Y\}. \end{array}$$

For any $x_0 \in X_0$ and $y_0 \in Y_0$, let

$$\begin{array}{l} X^-_m(x_0) = \{x \in X : \ d_G(x, x_0) \leq m\}, \\ Y^+_{m'}(y_0) = \{y \in Y : \ d_G(y_0, y) \leq m'\}. \end{array}$$

Since G is d-regular, we have

$$\begin{array}{l} |X^-_m(x_0)| \leq 1 + d + d^2 + \cdots + d^m; \\ |Y^+_{m'}(y_0)| \leq 1 + d + d^2 + \cdots + d^{m'}. \end{array}$$

Noting that $|X_0| \leq |F|$ and $|Y_0| \leq |F|$, we have that

$$\begin{array}{rcl} |X| \leq \sum\limits_{x_0 \in X_0} |X^-_m(x_0)| & \leq & |F|(1 + d + d^2 + \cdots + d^m); \\ |Y| \leq \sum\limits_{y_0 \in Y_0} |Y^+_{m'}(y_0)| & \leq & |F|(1 + d + d^2 + \cdots + d^{m'}). \end{array} \tag{1}$$

We now consider the relationship between m and m'. Choose $x \in X$ and $y \in Y$ such that $d_G(x, X_0) = m$ and $d_G(Y_0, y) = m'$. Since any (x, y)-path in G must go through F, there exists an edge $e = (x_0, y_0) \in F$ such that

$$d_G(x, x_0) + 1 + d_G(y_0, y) = d_G(x, y) \leq n.$$

Because of the choices of x and y, we have $d_G(x, x_0) \geq m$ and $d_G(y_0, y) \geq m'$. Thus,

$$m' \leq d_G(y_0, y) \leq n - d_G(x, x_0) - 1 \leq n - m - 1.$$

It follows from (1) that

$$|V(G)| \leq |F| \, \frac{d^{m+1} + d^{n-m} - 2}{d - 1}. \tag{2}$$

Since G is d-regular, $|E(X, Y)| = |E(Y, X)|$. Without loss of generality, we can suppose $m \leq m'$ in the following discussion.

Lemma 3 If F is a minimum R-edge-cut of $B(d, n)$, then $|F| \geq 2d-2$ for any $d \geq 2$ and $n \geq 2$.

Proof Let F be a minimum R-edge-cut of $B(d, n)$. Suppose to the contrary that $|F| \leq 2d - 3$. We will deduce a contradiction by considering two cases.

Case 1 $m = 0$. In this case, we have $X = X_0$. Let $t = |X|$. Then $t \geq 2$ since F is an R-edge-cut. So $2 \leq t \leq |F| \leq 2d - 3$ and $d \geq 3$. Let H be the subgraph of $B(d, n)$ induced by X. We consider the number of the edges of H. On the one hand, $|E(H)| = dt - |F| \geq dt - (2d - 3)$. On the other hand, by Lemma 2, $|E(H)| \leq \frac{1}{2}(t^2 + 1)$. It follows that

$$dt - (2d - 3) \leq \frac{1}{2}(t^2 + 1),$$

which implies that

$$t^2 - 2dt + 4d - 5 \geq 0.$$

It, however, is impossible since the convex function $f(t) = t^2 - 2dt + 4d - 5 < 0$ for $2 \leq t \leq 2d - 3$ and $d \geq 3$.

Case 2 $m \geq 1$. In this case, we have $m \leq n - 2$ and $n \geq 3$ since $1 \leq m \leq m'$ and $m + m' \leq n - 1$. Note that the function $f(m) = d^{m+1} + d^{n-m}$ is convex on the interval $[1, n-2]$ and $f(1) = f(n-2) = d^{n-1} + d^2$. It follows from (2) that, if $|F| \le 2d-3$ and $d \ge 2$, then

$$
\begin{array}{rl}
d^n = & |V(B(d,n))| \le |F|\,\dfrac{d^{m+1}+d^{n-m}-2}{d-1} \\
\le & (2d-3)\,\dfrac{d^{n-1}+d^2-2}{d-1} \\
= & \left\{\begin{array}{ll} 4d^2-2d-6, & \text{for } n=3; \\ 2d^3+d^2-2d-6, & \text{for } n=4; \\ 2d^{n-1}-d^{n-2}-\cdots-d^3+d^2-2d-6, & \text{for } n\ge 5. \end{array}\right.
\end{array}
\tag{3}
$$

Note that for $d \ge 2$,

$$
\begin{array}{l}
d^3-(4d^2-2d-6)=(d-2)(d^2-2d-2)+2>0, \\
d^4-(2d^3+d^2-2d-6)=d(d-2)(d^2-1)+6>0,
\end{array}
\tag{4}
$$

and, for $n \ge 5$,

$$
\begin{array}{rl}
 & d^n-(2d^{n-1}-d^{n-2}-\cdots-d^3+d^2-2d-6) \\
> & d^n-2d^{n-1}+d^3-d^2+2d-6 \\
= & (d-2)(d^{n-1}+d^2+d+4)+2 \\
> & 0.
\end{array}
\tag{5}
$$

By (3), (4) and (5), we obtain a contradiction $d^n < d^n$.

Thus, we have $|F| \ge 2d-2$ if F is a minimum R-edge-cut of $B(d,n)$. The lemma follows. ∎

3 Proof of Theorem

By the definition, it is clear that $\lambda'(B(2,1))$, $\lambda'(B(2,2))$ and $\lambda'(B(3,1))$ do not exist. By Lemma 1 and Lemma 3, we only need to show $\lambda'(B(d,1))$ $= 2d-4$ for $d \ge 4$.

Note that $B(d,1)$ is a complete digraph of order d plus a self-loop at every vertex. Let $F = E(X,Y)$ be an R-edge-cut with $|F| = \lambda'(B(d,1))$, and $|X| = t$. Then $t \ge 2$ and $|Y| = d-t \ge 2$. So, $2 \le t \le d-2$. For any pair of vertices x, y, there are a pair of symmetric edges between them. Thus, $\lambda'(B(d,1)) = |F| = t(d-t) \ge 2d-4$ for $2 \le t \le d-2$. On the other hand, choose $F = E_B^+(\{0,1\})$. Since every vertex of $B(d,1)$ has a self-loop and every pair of vertices have a pair of symmetric edges between them, F is an R-edge-cut for $d \ge 4$. Thus, $|F| = 2(d-1)-2 = 2d-4$, which implies $\lambda'(B(d,1)) \le 2d-4$. so $\lambda'(B(d,1)) = 2d-4$. ∎

Corollary 1 (Soneoka [8]) The de Bruijn digraph $B(d,n)$ is super edge-connected for any $d \ge 2$ and $n \ge 1$.

Proof Since $B(d,1)$ is a complete digraph of order d with a loop at every vertex, it is clear that $B(d,1)$ is super edge-connected for any $d \ge 2$.

It is easy to see that $B(2,2)$ is super edge-connected. By Theorem 1, for $d \geq 2$ and $n \geq 2$, except $B(2,2)$, $\lambda'(B(d,n)) = 2d-2 > d-1 = \lambda(B(d,n))$, which means that $B(d,n)$ is super edge-connected. ∎

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