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Cycle embedding in hypercubes with faulty vertices and edges

DU Zheng-zhong[1], JING Jin[1], MA Mei-jie[2], XU Jun-ming[1]

(1. *Department of Mathematics, University of Science and Technology of China, Hefei* 230026, *China*;
2. *Department of Mathematics, Zhejiang Normal University, Jinhua* 321004, *China*)

Abstract: It was shown that for a faulty Q_n with f_v faulty vertices and f_e faulty edges, there exists a fault-free cycle of length at least 2^n-2f_v provided $f_v+f_e\leqslant 2n-4$, $f_e\leqslant 2n-5$, $n\geqslant 3$ and each vertex of the faulty Q_n is incident with at least two non-faulty edges, which improves some known results.

Key words: cycle; graph; hypercube; fault tolerance
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容错超立方体网络的圈嵌入

杜正中[1],经 耹[1],马美杰[2],徐俊明[1]

(1.中国科学技术大学数学系,安徽合肥 230026;2.浙江师范大学数学系,浙江金华 321004)

摘要:证明了对于有 f_v 个故障点和 f_e 条故障边的容错超立方体网络 Q_n,如果 $f_v+f_e\leqslant 2n-4$, $f_e\leqslant 2n-5$, $n\geqslant 3$ 且每个节点至少保留两条非故障边,那么 Q_n 中存在长至少为 2^n-2f_v 的非故障圈. 这个结果改进了许多已知结果.

关键词:圈;图;超立方体网络;容错性

0 Introduction

To find a cycle of given length in a graph is a cycle embedding problem. Linear arrays and cycles, which are two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. Many efficient algorithms designed on linear arrays and cycles for solving a variety of algebraic problems and graph problems can be found in Refs. [1,2].

In this paper we consider the problem of embedding a cycle in a hypercube network with vertex and/or edge faults. This problem has received many researchers' attention in recent years[3~13]. Let f_v and f_e be the number of faulty vertices and edges, respectively. Fu[4] showed that a fault-free cycle of length at least 2^n-2f_v can be

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Biography: DU Zheng-zhong, male, born in 1979, PhD. Research field: graphs and combinatorics.
Corresponding author: XU Jun-ming, Prof. E-mail: xujm@ustc.edu.cn

embedded in Q_n when $f_v \leqslant 2n-4$. Latifi et al[8] showed that a fault-free Hamiltonian cycle can be embedded in Q_n when $f_e \leqslant n-2$. In case of considering both faulty vertices and edges, Hsieh[5] showed that a fault-free cycle of length at least 2^n-2f_v can be embedded in Q_n for $n \geqslant 3$ when $f_e+f_v \leqslant 2n-4$ and $f_e \leqslant n-2$.

Every component in a network may have different reliability, so it can be safely assumed that in some subsets of components, all the components will not fail simultaneously. These reasons have motivated research on Hamiltonian properties of conditional faulty hypercubes. If each vertex is incident with at least two non-faulty edges and $f_e \leqslant 2n-5$, Chan and Lee[3] showed that Q_n still contains a fault-free Hamiltonian cycle. Based on this requirement, in this paper, we improve the above-mentioned results of Refs. [3,4, 5,8] by proving the following theorem.

Theorem 0.1 For a faulty $Q_n (n \geqslant 3)$ with f_v faulty vertices and f_e faulty edges, there exists a fault-free cycle of length at least 2^n-2f_v provided that $f_v+f_e \leqslant 2n-4$, $f_e \leqslant 2n-5$ and each vertex of Q_n is incident with at least two non-faulty edges.

1 Some notations and lemmas

We follow Ref. [14] for the graph-theoretical terminologies and notations not defined here. A graph $G=(V, E)$ always means a simple and connected graph, where $V=V(G)$ is the vertex-set and $E=E(G)$ is the edge-set of G. A uv-path is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \cdots, v_m \rangle$, in which $u=v_0$, $v=v_m$ and all the vertices $v_0, v_1, v_2, \cdots, v_m$ are different from each other. The length of a path P is the number of edges in P. Let $d_G(u, v)$ be the length of a shortest uv-path in graph G. A cycle is a path with at least three vertices such that the first vertex is the same as the last one. A cycle is called a Hamiltonian cycle if it contains all vertices of G and a uv-path is called a Hamiltonian path if it contains all vertices of G.

An n-dimensional binary hypercube Q_n is a graph with 2^n vertices, each vertex denoted by an n-bit binary string $u=u_n u_{n-1} \cdots u_2 u_1$. Two vertices are adjacent if and only if their strings differ in exactly one bit position. It has been proven that Q_n is a vertex and edge transitive bipartite graph (see, for example, Ref. [15]).

By definition, for any $k \in \{1,2,\cdots,n\}$, Q_n can be expressed as $Q_n = L_k \odot R_k$, where L_k and R_k are the two $(n-1)$-subcubes of Q_n induced by the vertices with the k bit position is 0 and 1, respectively. We call edges between L_k and R_k to be k-dimensional, which form a perfect matching of Q_n. Clearly, for any edge e of Q_n, there is some $k \in \{1,2,\cdots,n\}$ such that e is k-dimensional. Use u_L and u_R to denote two vertices in L_k and R_k, respectively, linked by the k-dimensional edge $u_L u_R$ in Q_n.

For a faulty set $F=F_v \cup F_e$, let $f_v=|F_v|$ and $f_e=|F_e|$, where $F_v \subset V(Q_n)$ and $F_e \subset E(Q_n)$. For any $k \in \{1,2,\cdots,n\}$, we always express Q_n as $Q_n = L_k \odot R_k$, and let $F_L = F \cap L_k$ and $F_R = F \cap R_k$. Let $f_v^L = |F_L \cap F_v|$ and we denote f_v^R, f_e^L, f_e^R similarly. Use F_k to denote the set of k-dimensional faulty edges and $f_e^k = |F_k|$.

Lemma 1.1[5] Let u and v be two arbitrary distinct fault-free vertices in Q_n with $f_v+f_e \leqslant n-2$ and $n \geqslant 3$. Then there is a fault-free uv-path whose length is at least 2^n-2f_v-1 if $d_{Q_n}(u,v)$ is odd.

Lemma 1.2[4] There exists a fault-free cycle of length at least 2^n-2f_v in Q_n if $f_v \leqslant 2n-4$ and $n \geqslant 3$.

Lemma 1.3[5] There exists a fault-free cycle of length at least 2^n-2f_v in Q_n if $f_e \leqslant n-2$, $f_v+f_e \leqslant 2n-4$ and $n \geqslant 3$.

Lemma 1.4[3] There exists a Hamiltonian cycle in Q_n with at most $2n-5$ faulty edges if each vertex of Q_n is incident with at least two non-faulty edges and $n \geqslant 3$.

2 Proof of Theorem 0.1

In this section, we give the proof of Theorem 0.1 stated in Introduction.

If $f_e = 0$, then the theorem follows from

Lemma 1.2, and so assume $f_e \geqslant 1$ below. We proceed by induction on $n \geqslant 3$.

It is not difficult to verify that Q_3 with one faulty vertex and one faulty edge contains a fault-free cycle of length 6. Thus, the theorem holds for $n=3$. Assume the induction hypothesis for $n-1$ with $n \geqslant 4$.

Since $f_e \leqslant 2n-5$ and each vertex of Q_n is incident with at least two fault-free edges, there are at most two vertices incident with $n-2$ faulty edges. It is easy to see that if there are two vertices in Q_n incident with $n-2$ faulty edges, then these two vertices are linked by a faulty edge. We choose a faulty edge $e \in F_e$ according to the following rules (mentioned in Ref. [10]):

(Ⅰ) If there are two vertices incident with $n-2$ faulty edges, then we choose a faulty edge e that links these two vertices.

(Ⅱ) If there exists only one vertex u incident with $n-2$ faulty edges, then we choose a faulty edge e that is incident with u.

(Ⅲ) If every vertex is incident with at most $n-3$ faulty edges, then we choose any faulty edge e from F_e.

Let the chosen faulty edge e be k-dimensional edge. We express Q_n as $Q_n = L_k \odot R_k = L \odot R$. Based on the choice of $e \in F_k$, each vertex in L (or R) is incident with at least two fault-free edges of L (or R). Without loss of generality, we may assume that $f_v^L + f_e^L \geqslant f_v^R + f_e^R$.

We first assume $f_e^k \geqslant 2$. Then

$$f_v^L + f_e^L + f_v^R + f_e^R \leqslant 2n-6,\ f_e^L + f_e^R \leqslant 2n-7.$$

By the induction hypothesis, there exists a fault-free cycle C_L in $L_k - F$ of length at least $2^{n-1} - 2f_v^L$.

If $2^{n-1} - 2f_v^L > 2f_e^k + 2f_v^R$, there is an edge $u_L v_L$ on C_L such that $\{u_L u_R, v_L v_R, u_R, v_R\} \cap F = \varnothing$. Since $f_v^L + f_e^L \geqslant f_v^R + f_e^R$, we get

$$f_v^R + f_e^R \leqslant \frac{2n-6}{2} = n-3.$$

By Lemma 1.1, there is a fault-free $u_R v_R$-path P_R in $R_k - F$ of length at least $2^{n-1} - 2f_v^R - 1$. Then $C_L - u_L v_L + u_L u_R + v_L v_R + P_R$ is a cycle of length at least $2^n - 2f_v$ in $Q_n - F$ (see Fig. 1(a)).

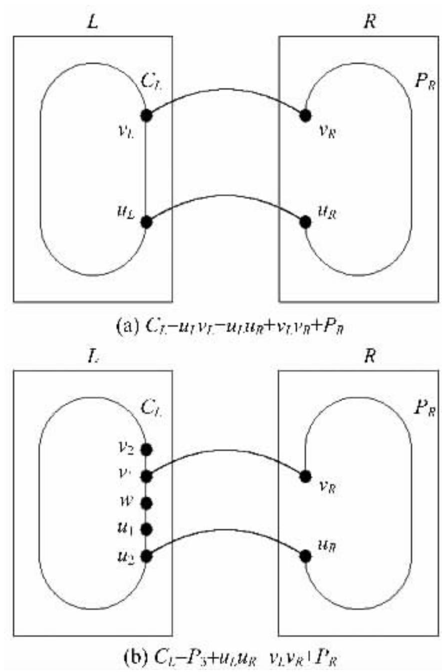

Fig. 1 Cycles of length at least $2^n - 2f_v$ in $Q_n - F$

Note that $2^{n-1} - 2f_v^L > 2f_e^k + 2f_v^R$ holds if and only if $n \geqslant 5$ or $f_e^L + f_e^R > 0$. And so, in order to prove the theorem, we only consider the case of $n=4$ and $f_e^L + f_e^R = 0$.

For $n=4$, the result holds if $f_e \leqslant 2$ by Lemma 1.3 or if $f_v = 0$ by Lemma 1.4. Assume $f_e = 3$ and $f_v = 1$ below. We need to find a fault-free cycle of length at least 14 in Q_4. Since $f_e^L + f_e^R = 0$, we have $f_e^k = 3$. Without loss of generality, assume $f_v^L = 1$. For $L - F$ shown in Fig. 2, if three faulty edges adjacent to black vertices, we choose $\{u_L, v_L\}$ satisfied that $\{u_L u_R, v_L v_R, u_R, v_R\} \cap F = \varnothing$. There is a fault-free $u_L v_L$-path P_L of length 6 in $L - F$ and a $u_R v_R$-path P_R of length 6 in R. Then $P_L + u_L u_R + v_L v_R + P_R$ is a cycle of length 14 in $Q_4 - F$. If some faulty edges are not adjacent to black vertices, we can find an edge $u_L v_L$ in $L - F$ such that $\{u_L u_R, v_L v_R\} \cap F = \varnothing$. There is a fault-free cycle C_L of length 6 in $L - F$ containing $u_L v_L$ and a $u_R v_R$-path P_R of length 7 in R. Then $C_L - u_L v_L + u_L u_R + v_L v_R + P_R$ is a cycle of length 14 in $Q_4 - F$.

We now suppose $f_e^k = 1$. Then

$$f_v^L + f_e^L + f_v^R + f_e^R \leqslant 2n-5,\ f_e^L + f_e^R \leqslant 2n-6.$$

If $f_e \leqslant n-2$, the result holds by Lemma 1.3. Assume $f_e \geqslant n-1$ below. Since $f_v^L + f_e^L \geqslant f_v^R + f_e^R$,

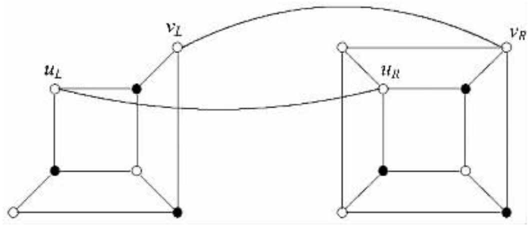

Fig. 2　Q_4 with three faulty edges and one faulty vertex

we get $f_v^R + f_e^R \leqslant \left\lfloor \frac{2n-5}{2} \right\rfloor = n-3$. We consider three cases.

Case 1　$f_e^L = 2n-6$. For the chosen edge $e \in F_k = \{e\}$, let $E^e = \{uv \in F_e^L \mid$ none of u and v is incident with $e\}$. Since each vertex of L is incident with at least two fault-free edges in L by the choice of the edge $e \in F_k = \{e\}$, there are at most $n-3$ faulty edges adjacent to the faulty edge e in L. Hence $|E^e| \geqslant 2n-6-(n-3) = n-3 \geqslant 1$. Let $e_1 \in E^e$. We may remark the edge e_1 as a temporarily fault-free edge. Then $f_v^L + f_e^L - 1 \leqslant 2n-6$, $f_e^L \leqslant 2n-7$. By the induction hypothesis, there is a fault-free cycle C_L of length at least $2^{n-1} - 2f_v^L$ in L. If $e_1 \in C_L$, let $u_L v_L = e_1$. Otherwise, we choose $u_L v_L \in C_L$ such that $\{u_L u_R, v_L v_R, u_R, v_R\} \cap F = \varnothing$. There is a $u_R v_R$-path P_R of length at least $2^{n-1} - 2f_v^R - 1$ in R. Then $C_L - u_L v_L + u_L u_R + v_L v_R + P_R$ is a cycle of length at least $2^n - 2f_v$ in $Q_n - F$ (see Fig. 1(a)).

Case 2　$f_e^L \leqslant 2n-7$ and $f_v^L + f_e^L = 2n-5$. Let $w \in F_v \cap L$. We may remark the vertex w as temporarily fault-free. Then $f_v^L + f_e^L - 1 \leqslant 2n-6$, $f_e^L \leqslant 2n-7$. By the induction hypothesis, there is a fault-free cycle C_L of length at least $2^{n-1} - 2(f_v^L - 1)$ in L.

If $w \notin C_L$, we choose an edge $u_L v_L \in C_L$ such that $\{u_L u_R, v_L v_R, u_R, v_R\} \cap F = \varnothing$. There is a $u_R v_R$-path P_R of length at least $2^{n-1} - 2f_v^R - 1$ in $R - F$ by Lemma 1.1. Then $C_L - u_L v_L + u_L u_R + v_L v_R + P_R$ is a cycle of length at least $2^n - 2(f_v - 1)$ in $Q_n - F$ (see Fig. 1(a)).

If $w \in C_L$, let $u_1, v_1 \in C_L$ be adjacent to w and $u_2, v_2 \in C_L$ be adjacent to u_1, v_1, respectively, where $w \notin \{u_1, v_1\}$. Since $f_e^k = 1$, we may choose $\{u_L, v_L\} = \{u_1, v_2\}$ (or $\{u_2, v_1\}$) such that $\{u_L u_R, v_L v_R, u_R, v_R\} \cap F = \varnothing$. Since $f_v^R + f_e^R \leqslant n-3$ and the distance between u_R and v_R is 1 or 3, by Lemma 1.1, there is a $u_R v_R$-path P_R of length at least $2^n - 2f_v^R - 1$ in R. Let P_3 be the $u_L v_L$-path in C_L of length 3. Then $C_L - P_3 + u_L u_R + v_L v_R + P_R$ is a cycle of length at least $2^n - 2f_v$ in $Q_n - F$ (see Fig. 1(b)).

Case 3　$f_e^L \leqslant 2n-7$, $f_v^L + f_e^L \leqslant 2n-6$. By the induction hypothesis, there exists a fault-free cycle C_L in $L - F$ of length at least $2^{n-1} - 2f_v^L$. Since $f_e^L + f_e^R = f_e - f_e^k \geqslant n-2 \geqslant 2$, $n \geqslant 4$, then $2^{n-1} - 2f_v^L > 2f_e^k + 2f_v^R$. There is an edge $u_L v_L$ on C_L such that $\{u_L u_R, v_L v_R, u_R, v_R\} \cap F = \varnothing$. Since $f_v^R + f_e^R \leqslant n-3$, by Lemma 1.1, there is a fault-free $u_R v_R$-path P_R of length at least $2^{n-1} - 2f_v^R - 1$ in R. Then $C_L - u_L v_L + u_L u_R + v_L v_R + P_R$ is a cycle of length at least $2^n - 2f_v$ in $Q_n - F$ (see Fig. 1(a)).

Theorem 0.1 is proved. □

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as $l \geqslant 3$, for $j=1,2$,

$$m \geqslant 1 \geqslant \frac{3}{2l-3} \Rightarrow$$

$$s(u)-s(v_j) > m(2l-3) \cdot 2^{2l-1} - 3 \cdot 2^{2l-1} \geqslant 0.$$

□

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