

Reliability of Interconnection Networks Modeled by Cartesian Product Digraphs

Chao Yang and Jun-Ming Xu

Department of Mathematics, University of Science and Technology of China, Hefei 230026, China

We determine that the connectivity and the edge-connectivity of the Cartesian product $G_1 \times G_2$ of two strongly connected and finite digraphs G_1 and G_2 are equal to $\min\{n_1\kappa_2, n_2\kappa_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}$ and $\min\{n_1\lambda_2, n_2\lambda_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}$, respectively, where $n_i, \kappa_i, \lambda_i, \delta_i^+, \delta_i^-$ are the order, the connectivity, the edge-connectivity, the minimum out-degree and the minimum in-degree of G_i , respectively, for $i = 1, 2$. © 2008 Wiley Periodicals, Inc. NETWORKS, Vol. 52(4), 202–205 2008

Keywords: connectivity; edge-connectivity; Cartesian product digraphs

1. INTRODUCTION

Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of an interconnection network. It is well known that when the underlying topology of an interconnection network is modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are two important features determining reliability and fault tolerance of the network. It is also well known that, for designing large-scale interconnection networks, the Cartesian product is an important method to obtain large graphs from smaller ones, with a number of parameters that can be easily calculated from the corresponding parameters for those small initial graphs. The Cartesian product preserves many nice properties such as regularity, existence of Hamilton cycles and Euler circuits, and transitivity of the initial graphs (see, e.g., [10]). In this note, we deal with the connectivity and the edge-connectivity of the Cartesian product of graphs.

We use the symbols n_i, δ_i, κ_i , and λ_i to denote the order, the minimum degree, the connectivity, and the edge-connectivity

of a graph G_i , respectively, for $i = 1, 2$. In 1957, Sabidussi [8] first discussed the connectivity of the Cartesian product $G_1 \times G_2$ of two undirected graphs G_1 and G_2 and proved that $\kappa(G_1 \times G_2) \geq \kappa_1 + \kappa_2$ if G_1 and G_2 are connected. In 1998, Xu [9] generalized this result to strongly connected digraphs. In 1999, Chiue and Shieh [4] proved that for connected undirected graphs G_1 and G_2 , $\lambda(G_1 \times G_2) \geq \lambda_1 + \lambda_2$. The authors [12, 13] completely determined the connectivity and the edge-connectivity of the Cartesian product of two connected undirected graphs G_1 and G_2 : namely,

$$\kappa(G_1 \times G_2) = \min\{n_1\kappa_2, n_2\kappa_1, \delta_1 + \delta_2\},$$

$$\lambda(G_1 \times G_2) = \min\{n_1\lambda_2, n_2\lambda_1, \delta_1 + \delta_2\}.$$

Very recently, Lu et al. [6] have further considered the super edge-connectivity of Cartesian product graphs.

Balbuena et al. [1] considered a generalization of the Cartesian product of graphs, the product graph $G_1 \times G_2$ of two undirected graphs G_1 and G_2 , and obtained

$$\kappa(G_1 \times G_2) \geq \min\{\kappa_1 n_2, (\delta_1 + 1)\kappa_2, \delta_1 + \delta_2\}.$$

Moreover, they stated some sufficient conditions that guarantee these product graphs to be maximally connected or superconnected.

Motivated by the technique in [1], in this note, we completely determine the connectivity and the edge-connectivity of the Cartesian product of two digraphs G_1 and G_2 as follows:

$$\kappa(G_1 \times G_2) = \min\{n_1\kappa_2, n_2\kappa_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\},$$

$$\lambda(G_1 \times G_2) = \min\{n_1\lambda_2, n_2\lambda_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\},$$

where G_i is a strongly connected digraph with minimum out-degree δ_i^+ and minimum in-degree δ_i^- , for $i = 1, 2$.

The proofs of these results are in Section 3. In Section 2, we present some preliminaries and the definition of the Cartesian product of two digraphs.

2. PRELIMINARIES

We follow [11] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a strongly connected digraph. A subset $S \subset V(G)$ (resp. $B \subseteq E(G)$) is said

Received December 2006; accepted September 2007

Correspondence to: J.-M. Xu; e-mail: xujm@ustc.edu.cn

Contract grant sponsor: NNSF of China; Contract grant number: 10671191

DOI 10.1002/net.20231

Published online 11 January 2008 in Wiley InterScience (www.interscience.wiley.com).

© 2008 Wiley Periodicals, Inc.

to be a separating set (an edge-separating set) of G if $G - S$ (resp. $G - B$) is not strongly connected. The *connectivity* of G is defined as $\kappa(G) = \min\{|S| : S \text{ is a separating set of } G\}$ if G is not a complete digraph; and $\kappa(G) = n - 1$ if G is a complete digraph of order n . The *edge-connectivity* of G is defined as $\lambda(G) = \min\{|B| : B \text{ is an edge-separating set of } G\}$. For any digraph G , we have $\kappa \leq \lambda \leq \delta = \min\{\delta^+, \delta^-\}$.

Let D and D' be two disjoint nonempty proper subsets of $V(G)$ or subgraphs of G . The symbols $N_G^+(D)$ and $N_G^-(D)$ denote the set of out-neighbors of D and the set of in-neighbors of D , $E_G^+(D)$, and $E_G^-(D)$ denote the set of out-going edges of D and the set of in-coming edges of D , respectively, and $E_G(D, D')$ denotes the set of edges with tails in D and heads in D' . The following two facts are simple and useful for the proofs of our main results.

Remark 1. *Let G be a digraph. If it is not strongly connected and has a finite number of strongly connected components, then G has at least two strongly connected components C and C' such that $N_G^+(C) = \emptyset$ and $N_G^-(C') = \emptyset$.*

Remark 2. *Let G be a digraph and let D be a nonempty proper subset of $V(G)$. Then $|D| + |N_G^+(D)| \geq \delta^+(G) + 1$ and $|D| + |N_G^-(D)| \geq \delta^-(G) + 1$. Also $|D| + |E_G^+(D)| \geq \delta^+(G) + 1$ and $|D| + |E_G^-(D)| \geq \delta^-(G) + 1$.*

The Cartesian product digraph $G_1 \times G_2 = (V, E)$ of digraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has the vertex set $V = V_1 \times V_2 = \{xy : x \in V_1, y \in V_2\}$, and $(x_1x_2, y_1y_2) \in E$ if and only if either $x_1 = y_1$ and $(x_2, y_2) \in E_2$ or $x_2 = y_2$ and $(x_1, y_1) \in E_1$. Obviously, $\delta^+(G_1 \times G_2) = \delta_1^+ + \delta_2^+$ and $\delta^-(G_1 \times G_2) = \delta_1^- + \delta_2^-$.

We now introduce some notations for convenience. For $x \in V(G_1)$ and $y \in V(G_2)$, the symbols G_2^x and G_1^y denote the subgraph of $G_1 \times G_2$ induced by $\{x\} \times V(G_2)$ and $V(G_1) \times \{y\}$, respectively. For a separating set S and an edge-separating set B of $G_1 \times G_2$, denote

$$\begin{aligned} S_x &= S \cap V(G_2^x), & S^y &= S \cap V(G_1^y), \\ B_x &= B \cap E(G_2^x), & B^y &= B \cap E(G_1^y), \\ B_{xz} &= B \cap E_G(G_2^x, G_2^z) \text{ if } (x, z) \in E(G_1). \end{aligned}$$

For a strongly connected component C of $G - S$ (or $G - B$), denote

$$\begin{aligned} C_1 &= \{x \in V_1 : xy \in V(C) \text{ for some } y \in V_2\}, \\ C_2 &= \{y \in V_2 : xy \in V(C) \text{ for some } x \in V_1\}. \end{aligned}$$

Lemma 1. *Let G_1 and G_2 be two strongly connected digraphs, S and B be a separating set and an edge-separating set of $G_1 \times G_2$, respectively, and C be a strongly connected component of $G_1 \times G_2 - S$ (resp. $G_1 \times G_2 - B$) without out-neighbors. If $C_1 \neq V_1$, then there exists an edge $(x, x') \in E_{G_1}^+(C_1)$ such that*

$$|S_x| + |S_{x'}| \geq \delta_2^+ + 1 \quad (\text{resp. } |B_x| + |B_{x'}| \geq \delta_2^+ + 1). \quad (1)$$

Proof. Let $G = G_1 \times G_2$. Since $C_1 \neq V_1$ and G_1 is strongly connected, $E_{G_1}^+(C_1) \neq \emptyset$ and there exist $x \in C_1$ and $x' \in V_1 \setminus C_1$ such that $(x, x') \in E_{G_1}^+(C_1)$. Let $D = V(C) \cap V(G_2^x)$ and $F = N_G^+(D) \cap V(G_2^{x'})$. Then

$$N_{G_2^x}^+(D) \subseteq S_x, \quad F \subseteq S_{x'} \quad (\text{resp. } E_{G_2^x}^+(D) \subseteq B_x, E_G(D, F) \subseteq B_{x'})$$

since C has no out-neighbors in $G - S$ (resp. $G - B$), which means that

$$\begin{aligned} |S_x| + |S_{x'}| &\geq |N_{G_2^x}^+(D)| + |F| \\ (\text{resp. } |B_x| + |B_{x'}| &\geq |E_{G_2^x}^+(D)| + |E_G(D, F)|). \end{aligned} \quad (2)$$

Note that $|F| = |D| = |E_G(D, F)|$ since $(x, x') \in E(G_1)$. It follows from (2) that

$$\begin{aligned} |S_x| + |S_{x'}| &\geq |N_{G_2^x}^+(D)| + |D| \\ (\text{resp. } |B_x| + |B_{x'}| &\geq |E_{G_2^x}^+(D)| + |D|). \end{aligned} \quad (3)$$

It follows from (3) and Lemma 2 that

$$|S_x| + |S_{x'}| \geq \delta_2^+ + 1 \quad (\text{resp. } |B_x| + |B_{x'}| \geq \delta_2^+ + 1)$$

as required. \blacksquare

3. MAIN RESULT

Theorem 1. *For every two nontrivial strongly connected digraphs G_1 and G_2 ,*

$$\kappa(G_1 \times G_2) = \min\{n_1\kappa_2, n_2\kappa_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}.$$

Proof. Let $G = G_1 \times G_2$. Clearly,

$$\kappa(G) \leq \min\{\delta^+(G), \delta^-(G)\} = \min\{\delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}.$$

If G_2 is not a complete digraph, let S_0 be a minimum separating set of G_2 . Then $V_1 \times S_0$ is a separating set of G , which implies $\kappa(G) \leq n_1\kappa_2$. If G_2 is a complete digraph, then $\kappa_2 = \delta_2^+$, therefore

$$\kappa(G) \leq \delta_1^+ + \delta_2^+ \leq (\delta_1^+ + 1)\delta_2^+ \leq n_1\kappa_2.$$

By symmetry, we have $\kappa(G) \leq n_2\kappa_1$.

So it remains to show

$$\kappa(G_1 \times G_2) \geq \min\{\kappa_1n_2, \kappa_2n_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}.$$

Evidently, G is not a complete digraph since neither G_1 nor G_2 is trivial. Let S be a minimum separating set in G .

Assume $C_1 \neq V_1$ and $C_2 \neq V_2$ for each strongly connected component C of $G - S$. Then

$$|S_x| \geq 1 \quad \text{for each } x \in V_1. \quad (4)$$

Otherwise, there is a strongly connected component C' containing G_2^x for some $x \in V_1$, which implies $C_2' = V_2$, a contradiction. Let C be a strongly connected component of $G - S$ without out-neighbors. Since $C_1 \neq V_1$, by Lemma 1

there exist $x \in C_1$ and $x' \in V_1 \setminus C_1$ such that $|S_x| + |S_{x'}| \geq \delta_2^+ + 1$. It follows from (4) that

$$\begin{aligned} |S| &= |S_x| + |S_{x'}| + \sum_{z \in V_1 \setminus \{x, x'\}} |S_z| \\ &\geq \delta_2^+ + 1 + (n_1 - 2) \\ &\geq \delta_2^+ + 1 + (\delta_1^+ + 1) - 2 \\ &= \delta_1^+ + \delta_2^+. \end{aligned}$$

Now, suppose there exists such a strongly connected component C of $G - S$ where either $C_1 = V_1$ or $C_2 = V_2$. By symmetry, we can without loss of generality assume $C_1 = V_1$. By Lemma 1, let $C' (\neq C)$ be a strongly connected component without out-neighbors in $G - S$ (if C' has no in-neighbors, the proof is similar). Then

$$|S_x| \geq \kappa_2 \quad \text{if } x \in C'. \quad (5)$$

If $C'_1 = V_1$, then from (5) we have

$$|S| \geq \sum_{x \in C'_1 = V_1} |S_x| \geq n_1 \kappa_2.$$

If $C'_1 \neq V_1$, then by Lemma 1 there exist $x \in C'_1$ and $x' \in V_1 \setminus C'_1$ such that $|S_x| + |S_{x'}| \geq \delta_2^+ + 1$. And we can also see that

$$|S_z| \geq 1 \quad \text{for each } z \in N_{G_1}^+(C'). \quad (6)$$

It follows from (1), (6), and Lemma 2 that

$$\begin{aligned} |S| &\geq |S_x| + |S_{x'}| + \sum_{z \in C'_1 \setminus \{x\}} |S_z| + \sum_{z \in N_{G_1}^+(C') \setminus \{x'\}} |S_z| \\ &\geq \delta_2^+ + 1 + |C'_1| - 1 + |N_{G_1}^+(C')| - 1 \\ &= \delta_2^+ + (|C'_1| + |N_{G_1}^+(C')|) - 1 \\ &\geq \delta_2^+ + (\delta_1^+ + 1) - 1 \\ &= \delta_1^+ + \delta_2^+. \end{aligned}$$

This completes the proof. \blacksquare

Corollary 1. ([13]) *For every two nontrivial connected undirected graphs G_1 and G_2 ,*

$$\kappa(G_1 \times G_2) = \min\{n_1 \kappa_2, n_2 \kappa_1, \delta_1 + \delta_2\}.$$

Theorem 2. *For every two nontrivial strongly connected digraphs G_1 and G_2 ,*

$$\lambda(G_1 \times G_2) = \min\{n_1 \lambda_2, n_2 \lambda_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}.$$

Proof. To prove the equality, we only need to prove that

$$\lambda(G_1 \times G_2) \geq \min\{\lambda_1 n_2, \lambda_2 n_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}$$

since the reverse inequality clearly holds. Let $G = G_1 \times G_2$ and B be a minimum edge-separating set of G .

Suppose that $C_1 \neq V_1$ and $C_2 \neq V_2$ for each strongly connected component C of $G - B$. Then

$$\begin{aligned} |B_x| &\geq 1 \quad \text{for each } x \in V_1 \text{ and} \\ |B^y| &\geq 1 \quad \text{for each } y \in V_2. \end{aligned}$$

Thus,

$$|B| = \sum_{x \in V_1} |B_x| + \sum_{y \in V_2} |B^y| \geq n_1 + n_2 \geq \delta_1^+ + \delta_2^+ + 2.$$

Now, suppose there exists such a strongly connected component C of $G - B$ where either $C_1 = V_1$ or $C_2 = V_2$. By symmetry, we can without loss of generality assume $C_1 = V_1$. Let $C' (\neq C)$ be a strongly connected component without out-neighbors in $G - B$. Note that

$$|B_x| \geq \lambda_2 \quad \text{for } x \in C'. \quad (7)$$

If $C'_1 = V_1$, then from (7) we have

$$|B| \geq \sum_{x \in C'_1 = V_1} |B_x| \geq n_1 \lambda_2.$$

If $C'_1 \neq V_1$, then by Lemma 1 there exists an edge $(x, x') \in E_{G_1}^+(C'_1)$ such that $|B_x| + |B_{x'}| \geq \delta_2^+ + 1$. And we can also see that

$$|B_e| \geq 1 \quad \text{for each } e \in E_{G_1}^+(C'). \quad (8)$$

It follows from (1), (8), and Lemma 2 that

$$\begin{aligned} |B| &\geq |B_x| + |B_{x'}| + \sum_{z \in C'_1 \setminus \{x\}} |B_z| + \sum_{e \in E_{G_1}^+(C') \setminus \{xx'\}} |B_e| \\ &\geq \delta_2^+ + 1 + |C'_1| - 1 + |E_{G_1}^+(C')| - 1 \\ &= \delta_2^+ + (|C'_1| + |E_{G_1}^+(C')|) - 1 \\ &\geq \delta_2^+ + (\delta_1^+ + 1) - 1 \\ &= \delta_1^+ + \delta_2^+. \end{aligned}$$

This completes the proof. \blacksquare

Corollary 2. ([12]) *For every two nontrivial connected undirected graphs G_1 and G_2 ,*

$$\lambda(G_1 \times G_2) = \min\{n_1 \lambda_2, n_2 \lambda_1, \delta_1 + \delta_2\}.$$

4. REMARKS

It is well known that the Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 gives an important method for designing large-scale interconnection networks. For example, the very popular, versatile, and efficient hypercube Q_n can be expressed as $Q_n = \underbrace{K_2 \times K_2 \times \cdots \times K_2}_n$, where K_2

is a complete graph of order two. In this note, we completely determine the connectivity and the edge-connectivity of $G_1 \times G_2$ for two strongly connected digraphs G_1 and

G_2 , which are equal to $\min\{n_1\kappa_2, n_2\kappa_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}$ and $\min\{n_1\lambda_2, n_2\lambda_1, \delta_1^+ + \delta_2^+, \delta_1^- + \delta_2^-\}$, respectively; these generalize two corresponding results for undirected graphs, namely Corollary 1 and Corollary 2. As another application, we immediately obtain $\kappa(Q_n) = \lambda(Q_n) = n$.

We note that Chartrand and Harary [3] introduced permutation graphs. For a graph G and a permutation π of $V(G)$, the permutation graph G^π is defined by taking two disjoint copies of G and adding a matching joining each vertex v in the first copy to $\pi(v)$ in the second copy. The connectivity and edge-connectivity of G^π were studied in [2, 5, 7]. It is clear that permutation graphs cannot contain the Cartesian product graphs. However, if we take π in G^π as the identity permutation, then the permutation graph $G^\pi = G \times K_2$.

We also note that Balbuena et al. [1] defined the product graph $G_1 \times G_2$ of two undirected graphs G_1 and G_2 , which is a generalization of both the Cartesian product graphs and the permutation graphs. Thus, it is interesting to determine the connectivity and edge-connectivity of $G_1 \times G_2$.

REFERENCES

- [1] C. Balbuena, P. García-Vázquez, and X. Marcote, Reliability of interconnection networks modeled by a product of graphs, *Networks* 48 (2006), 114–120.
- [2] C. Balbuena, X. Marcote, and P. García-Vázquez, On restricted connectivities of permutation graphs, *Networks* 45 (2005), 113–118.
- [3] G. Chartrand and F. Harary, Planar permutation graphs, *Ann Inst H Poincaré Sec B* 3 (1967), 433–438.
- [4] W.-S. Chiue and B.-S. Shieh, On connectivity of the Cartesian product of two graphs, *Appl Math Comput* 102 (1999), 129–137.
- [5] W. Goddard, M.E. Raines, and P.J. Slater, Distance and connectivity measures in permutation graphs, *Discrete Math* 271 (2003), 61–70.
- [6] M. Lu, G.-L. Chen, and J.-M. Xu, On super edge-connectivity of Cartesian product graphs, *Networks* 49 (2007), 135–157.
- [7] B. Piazza, Edge-connectivity of permutation graphs, *Congr Numer* 65 (1988), 7–16.
- [8] G. Sabidussi, Graphs with given group and given graph theoretical properties, *Can J Math* 9 (1957), 515–525.
- [9] J.-M. Xu, Connectivity of Cartesian product digraphs and fault-tolerant routings of generalized hypercubes, *Appl Math J Chin Univ B* 13 (1998), 179–187.
- [10] J.-M. Xu, Topological structure and analysis of interconnection networks, Kluwer Academic Publishers, Dordrecht, 2001.
- [11] J.-M. Xu, Theory and application of graphs, Kluwer Academic Publishers, Dordrecht, 2003.
- [12] J.-M. Xu and C. Yang, Connectivity of Cartesian product graphs, *Discrete Math* 306 (2006), 159–165.
- [13] J.-M. Xu and C. Yang, Connectivity and super-connectivity of Cartesian product graphs, *Ars Combinatoria*, accepted.