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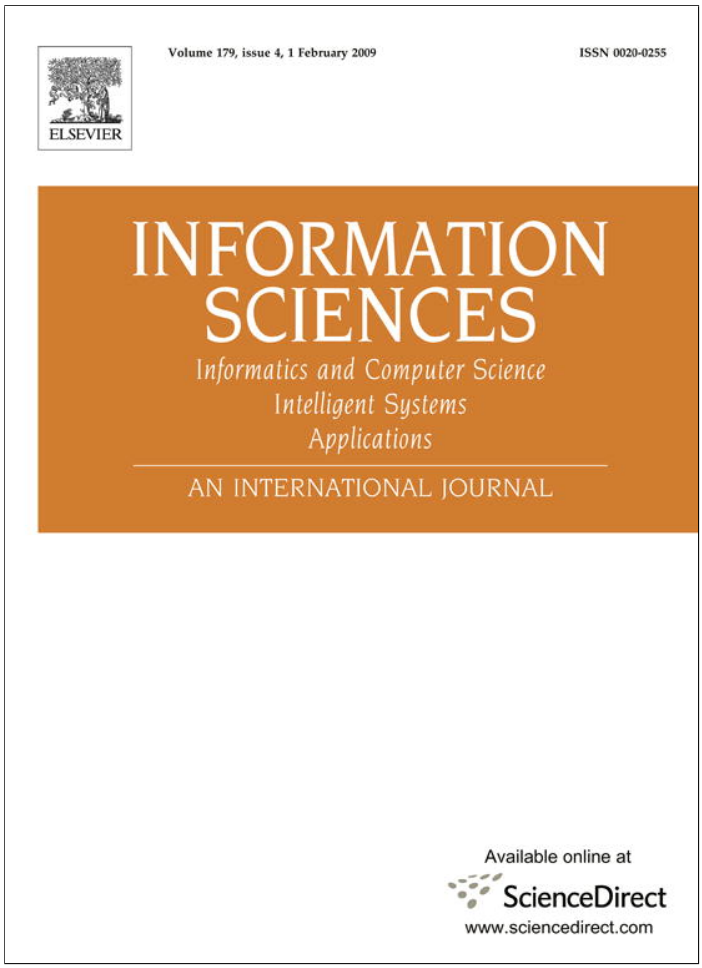

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Information Sciences 179 (2009) 404–409

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Edge-fault-tolerant bipanconnectivity of hypercubes ☆

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ARTICLE INFO

Article history:
Received 27 November 2007
Received in revised form 8 October 2008
Accepted 8 October 2008

Keywords:
Panconnectivity
Bipanconnectivity
Hypercube
Fault tolerance
Networks

ABSTRACT

This paper shows that for any two distinct vertices u and v with distance d in the hypercube Q_n ($n \geqslant 3$) with at most $2n-5$ faulty edges and each vertex incident with least two fault-free edges, there exist fault-free uv-paths of length ℓ in Q_n for every ℓ with $d+4 \leqslant \ell \leqslant 2^n-1$ and $\ell - d \equiv 0 \pmod 2$. This result improves some known results on edge-fault bipanconnectivity of hypercubes. The proof is based on the recursive structure of Q_n.

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1. Introduction

It is well known that a topological structure of an interconnection network can be modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network. Investigating structures of G is essential to design a suitable topology of interconnection network. Structures of paths and cycles, which are fundamental topologies for parallel and distributed processing, are suitable for local area networks and for the development of simple parallel algorithms with low communication cost.

A graph G is *ℓ-pancyclic* if G contain cycles of length k for every k with $\ell \leqslant k \leqslant |V|$, and G is *pancyclic* if it is g-pancyclic, where g is the girth of G. A graph is *panconnected* if for any pair of distinct vertices u and v with distance d, there exist uv-paths of length ℓ for every ℓ with $d \leqslant \ell \leqslant |V|-1$. It is well known that any bipartite graph contains no odd cycles. A bipartite graph G is *bipancyclic* if it contain cycles of length ℓ for every even ℓ with $g \leqslant \ell \leqslant |V|$, and *bipanconnected* if for any pair of distinct vertices u and v with distance d, there exist uv-paths of length ℓ for every ℓ with $d \leqslant \ell \leqslant |V|-1$ and $\ell - d \equiv 0 \pmod 2$.

In this paper, we consider the hypercube Q_n. In [2], Li et al. showed that Q_n is bipanconnected for $n \geqslant 2$. In [10], Xu et al. proved that if Q_n have at most $n-2$ faulty edges and $n \geqslant 2$, then for any two distinct vertices u and v with distance d, there exist fault-free uv-paths of length ℓ for every ℓ with $d+2 \leqslant \ell \leqslant 2^n-1$ and $\ell - d \equiv 0 \pmod 2$. In [9], Xu et al. showed that if Q_n has at most $n-1$ faulty edges and each vertex is incident with at least two fault-free edges, then each fault-free edge of Q_n lies on fault-free cycles of length ℓ for every even ℓ between 6 and 2^n for $n \geqslant 4$. For more faulty edges, if Q_n has faulty edges up to $2n-5$, each vertex is incident with at least two fault-free edges and $n \geqslant 3$, then Chan and Lee [1] showed that Q_n contains a fault-free Hamilton cycle; Tsai [6] showed that Q_n contains fault-free cycles of length ℓ for every even ℓ between 6 and 2^n; and contains a fault-free path of length 2^n-1 joining any two vertices with odd distance and a fault-free

☆ The work was supported by NNSF of China (No. 10671191).

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doi:10.1016/j.ins.2008.10.011

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path of length $2^n - 2$ joining any two vertices with even distance; Tsai and Lai [7], and Shih et al. [5] showed that each fault-free edge of Q_n lies on fault-free cycles of length ℓ for every even ℓ between 6 and 2^n. We improve the above results by the following theorem.

Theorem 1. *If Q_n has at most $2n-5$ faulty edges and each vertex is incident with at least two fault-free edges, then for any two distinct vertices u and v with distance d, there exist fault-free uv-paths of length ℓ in Q_n for every ℓ with $d+4 \leqslant \ell \leqslant 2^n - 1$ and $\ell - d \equiv 0 \pmod 2$ for $n \geqslant 3$.*

The proof is based on the recursive structure of Q_n and given in Section 3. Section 2 gives some Lemmas.

2. Some lemmas

Let $G = (V, E)$ be a graph, where V is the vertex-set and E is the edge-set. For two distinct vertices u and v in G, a uv-path P of length k is a sequence of different vertices $(x_0, x_1, \ldots, x_k)$, where $x_0 = u, x_k = v$, and $x_{i-1}x_i \in E(G)$ for each $i = 1, 2, \ldots, k$, here k is the number of edges in P, called the *length* of P, denoted by $\varepsilon(P) = k$. The distance between them, denoted by $d_G(u, v)$, is the length of a shortest uv-path in G. Let $P = (u, \ldots, t, x, y, z, \ldots, v)$ be a uv-path of length at least two. An interior vertex x in P partitions P into two sections. We use $P(u, x)$ to denote the subpath $(u, \ldots, t, x)$ of P from u to x and use $P(y, v)$ to denote the subpath $(y, z, \ldots, v)$ of P from y to v. Since xy is an edge in P, we can write the path $P = P(u, x) + xy + P(y, v)$.

The *n-dimensional hypercube* Q_n is a graph with 2^n vertices, each vertex with a distinct binary string $u_1u_2\cdots u_n$. Two vertices are linked by an edge if and only if their strings differ in exactly one position. The hypercube is a widely used and well-known interconnection model since it possesses many attractive properties [4,8].

An edge in Q_n is called the *i-dimensional edge* if its end-vertices differ in the ith position. Let E_i be the set of the i-dimensional edges. Clearly, E_i is a perfect matching of Q_n for each $i = 1, 2, \ldots, n$, and $Q_n - E_i$ is a union of two disjoint copies L_i and R_i of $(n-1)$-subcubes of Q_n, where L_i is induced by the subset of vertices whose the ith position is 0 and R_i is induced by the subset of vertices whose the ith position is 1. We write $Q_n = L_i \odot R_i$. Since Q_n is vertex-transitive and edge-transitive, without loss of generality, we may assume $i = n$ and write $Q_n = L_i \odot R_i$ as $Q_n = L \odot R$ in our discussion. We use x_Lx_R to denote an edge between L and R, where $x_L \in L$ and $x_R \in R$. Let F be a set of faulty edges in Q_n, $F_L = F \cap E(L)$, $F_R = F \cap E(R)$ and $F_i = F \cap E_i$. A subgraph H of Q_n is said to be *fault-free* if it contains no edges in F. For a vertex u in Q_n, let $f(u)$ be the number of faulty edges in Q_n incident with u.

Lemma 1 ([4]). *Let u and v be two distinct vertices with distance d in Q_n $(n \geqslant 2)$. Then there exist n internally vertex-disjoint uv-paths in Q_n such that only d of them are of length d, which lie in the same d-dimensional subcube Q_d, the other $n-d$ paths are of length $d+2$.*

Lemma 2. *Assume that $Q_n = L \odot R$ $(n \geqslant 3)$ has at most $2n-5$ faulty edges and $f(u) \leqslant n-2$ for any $u \in V(Q_n)$. Then, for any two distinct vertices $u_L \in L$ and $v_R \in R$ with distance d in Q_n, there exists a fault-free u_Lv_R-path P of length at most $d+4$ in Q_n such that $P = P(u_L, x_L) + x_Lx_R + P(x_R, v_R)$, where $x_L \in L, x_R \in R$, $P(u_L, x_L) \subset L$ and $P(x_R, v_R) \subset R$.*

Proof. Let F be a set of faulty edges in Q_n with $|F| \leqslant 2n-5$. To prove the lemma, we first consider a simple case, $d = 1$, which can help the reader to understand our proof for more general case.

In this case, $v_R = u_R$ and $u_Lu_R \in E(Q_n)$. If $u_Lu_R \notin F$, then we are done. Assume that $u_Lu_R \in F$ below. We need to prove that there is a required fault-free u_Lu_R-path of length at most 5 in Q_n.

Let $u_L = u_1u_2\cdots u_{n-1}u_n$. Then $u_R = u_1u_2\cdots u_{n-1}\bar{u}_n$. For each $i = 1, 2, \ldots, n-1$, let $x_L^i = u_1\cdots u_{i-1}\bar{u}_iu_{i+1}\cdots u_n$ and $x_R^i = u_1\cdots u_{i-1}\bar{u}_iu_{i+1}\cdots \bar{u}_n$. Then x_L^i is a neighbor of u_L in L, x_R^i is a neighbor of u_R in R, $x_L^ix_R^i \in E_n$ and $x_L^ix_R^j \notin E_n$ for each $1 \leqslant i \neq j \leqslant n-1$. Let $S = \{i : u_Lx_L^i \notin F\}, T = \{j : u_Rx_R^j \notin F\}$, and let $|S| = s$ and $|T| = t$. It is clear that $2 \leqslant s \leqslant n-1$ and $2 \leqslant t \leqslant n-1$ since $u_Lu_R \in F$ and $f(u) \leqslant n-2$ for any $u \in V(Q_n)$. Let $|S \cap T| = k$. Without loss of generality, we can assume $S = \{1, 2, \ldots, s\}$, $T = \{s-k+1, \ldots, s-k+t\}$ and $s \leqslant t$ below.

For any $i \in S$ and $j \in T$, if $j = i$ then let $P_{ii} = (x_L^i, x_R^i)$, which is an $x_L^ix_R^i$-path of length 1; if $i < j$ then let $P_{ij} = (x_L^i, z_L^{ij}, z_R^{ij}, x_R^j)$, which is an $x_L^ix_R^j$-path of length 3, where

$$\begin{aligned} z_L^{ij} &= u_1\cdots u_{i-1}\bar{u}_iu_{i+1}\cdots u_{j-1}\bar{u}_ju_{j+1}\cdots u_{n-1}u_n, \\ z_R^{ij} &= u_1\cdots u_{i-1}\bar{u}_iu_{i+1}\cdots u_{j-1}\bar{u}_ju_{j+1}\cdots u_{n-1}\bar{u}_n. \end{aligned} \tag{1}$$

Then $z_L^{ij}z_R^{ij} \in E_n$. Let

$$\mathscr{P} = \{P_{ij} : i \in S, j \in T, i \leqslant j\}.$$

Then

$$|\mathscr{P}| = st - \frac{1}{2}k(k-1) \geqslant st - \frac{1}{2}s(s-1).$$

We now show that any two paths in $\mathscr{P}$ are internally vertex-disjoint. Note that the edge u_Lu_R is not contained in $\mathscr{P}$ since $1 \leqslant i, j \leqslant n-1$. Thus, we only need to consider internally vertex-disjointness of any two paths of length 3 in $\mathscr{P}$.

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Let $P_{ij} = (x_L^i, z_L^{ij}, z_R^{ij}, x_R^j)$ be any path of length 3 in $\mathscr{P}$, where $i < j$, z_L^{ij} and z_R^{ij} are defined in (1). Suppose that there is another path P of length 3 in $\mathscr{P}$ that shares a common vertex z with P_{ij}. Then z is necessarily in $\{z_L^{ij}, z_R^{ij}\}$. If $z = z_L^{ij}$, then the vertex in P adjacent to z_L^{ij} is necessarily $x_L^j = u_1 \cdots u_{j-1}\bar{u}_j u_{j+1} \cdots u_n$. However, $x_L^j x_R^j \in E_n$ and $P_{jj} = x_L^j x_R^j$. This fact implies that $P \neq P_{jj}$. If $z = z_R^{ij}$, then the vertex in P adjacent to z_R^{ij} is necessarily $x_R^i = u_1 \cdots u_{i-1}\bar{u}_i u_{i+1} \cdots \bar{u}_n$. However, $x_L^i x_R^i \in E_n$ and $P_{ii} = x_L^i x_R^i$. This fact implies that $P \neq P_{ii}$. Thus, $P = P_{ji} = (x_L^j, z_L^{ij}, z_R^{ij}, x_R^i)$. However, $P = P_{ji} \notin \mathscr{P}$ since $j > i$, a contradiction. It follows that any two paths in $\mathscr{P}$ are internally vertex-disjoint.

Suppose that each path in $\mathscr{P}$ contains at least one edge in F. By the definition, $f(u_L)$ and $f(u_R)$ are counted in Q_n, not in L and R. Thus, the edge $u_L u_R \in F$ is counted twice, so $f(u_L) + f(u_R) = 2n - s - t - 1$. Similarly, we have $|F| \geqslant |\mathscr{P}| + f(u_L) + f(u_R) - 1$. We can induce a contradiction as follows:

$$\begin{aligned}
2n - 5 \geqslant |F| &\geqslant st - \frac{1}{2}s(s-1) + f(u_L) + f(u_R) - 1 \\
&\geqslant st - \frac{1}{2}s(s-1) + 2n - s - t - 2 \\
&= t(s-1) - \frac{1}{2}s(s+1) + 2n - 2 \\
&\geqslant s(s-1) - \frac{1}{2}s(s+1) + 2n - 2 \quad (\text{since } t \geqslant s) \\
&= \frac{1}{2}s(s-3) + 2n - 2 \\
&\geqslant 2n - 3 \quad (\text{since } s \geqslant 2).
\end{aligned}$$

It follows that there exists a fault-free path, say P_{ij}, in $\mathscr{P}$. Thus, if $i = j$ then $u_L x_L^i + P_{ii} + x_R^i u_R$ is a fault-free $u_L u_R$-path of length 3 in Q_n; if $i < j$ then $u_L x_L^i + P_{ij} + x_R^j u_R$ is a fault-free $u_L u_R$-path of length 5 in Q_n.

We now consider more general case, $2 \leqslant d \leqslant n$. Assume, without loss of generality, that $|F_L| \leqslant |F_R|$. Then L has at most $n-3$ faulty edges. We will show that there is a fault-free $u_L v_R$-path P of length at most $d+4$ in Q_n such that $P = P(u_L, x_L) + x_L x_R + P(x_R, v_R)$, where $x_L \in L, x_R \in R$, $P(u_L, x_L) \subset L$ and $P(x_R, v_R) \subset R$.

Assume that $v_L v_R \in E(Q_n)$, where $v_L \in L$. Since $L \cong Q_{n-1}$ and the distance between u_L and v_L is $d-1$ in L, by Lemma 1 there exist $n-1$ internally vertex-disjoint $u_L v_L$-paths of length $d-1$ or $d+1$ in L. Choose $n-2$ such paths $P_1, P_2, \ldots, P_{n-2}$ such that each one of them is as long as possible, which implies $\varepsilon(P_i) \geqslant 2$ for each $i = 1, 2, \ldots, n-2$. At least one of the $n-2$ paths is fault-free since L has at most $n-3$ faulty edges. If $v_L v_R \notin F$, then we are done. Assume that $v_L v_R \in F$ below.

For each $i = 1, 2, \ldots, n-2$, let $P_i = (u_L, y_L^i, \ldots, z_L^i, x_L^i, v_L)$, and let x_R^i be the common neighbor of x_L^i and v_R in R, that is, $x_R^i v_R \in E(Q_n)$. Let $H = \{i : P_i \text{ is fault-free}\}$, and $T = \{j : x_R^j v_R \notin F\}$. Let $|H| = h$ and $|T| = t$. Then $1 \leqslant h \leqslant n-2$ and $1 \leqslant t \leqslant n-2$ since at least one P_i is fault-free, $f(v_R) \leqslant n-2$ and $v_L v_R \in F$. Let $|H \cap T| = k$. Without loss of generality, we can assume $H = \{1, 2, \ldots, h\}$, $T = \{h-k+1, \ldots, h-k+t\}$ and $h \leqslant t$ below.

For any $i \in H$ and $j \in T$, if $i = j$ then let $P'_{ii} = (x_L^i, x_R^i)$, which is an $x_L^i x_R^i$-path of length 1; if $i < j$ then let $P'_{ij} = (x_L^i, z_L^{ij}, z_R^{ij}, x_R^j)$, which is an $x_L^i x_R^j$-path of length 3, where z_L^{ij} and z_R^{ij} are defined by (1). Let

$$\mathscr{P}' = \{P'_{ij} : i \in H, j \in T, i \leqslant j\}.$$

Then

$$|\mathscr{P}'| = ht - \frac{1}{2}k(k-1) \geqslant ht - \frac{1}{2}h(h-1).$$

In the same argument as the case of $d = 1$, we can show that any two paths in $\mathscr{P}'$ are internally vertex-disjoint. Suppose that each path in $\mathscr{P}$ contains at least one edge in F. Then

$$\begin{aligned}
2n - 5 \geqslant |F| &\geqslant ht - \frac{1}{2}h(h-1) + (n-2-h) + (n-1-t) = t(h-1) - \frac{1}{2}h(h+1) + 2n - 3 \\
&\geqslant h(h-1) - \frac{1}{2}h(h+1) + 2n - 3 \quad (\text{since } t \geqslant h) = \frac{1}{2}h(h-3) + 2n - 3 \geqslant 2n - 4 \quad (\text{since } h \geqslant 1).
\end{aligned}$$

This contradiction shows that there exists a fault-free path, say P'_{ij}, in $\mathscr{P}'$. Thus, if $i = j$ then $P_i(u_L, x_L^i) + P_{ii} + x_R^i v_R$ is a fault-free $u_L v_R$-path of length at most $d+2$ in Q_n; if $i < j$ then $P_i(u_L, x_L^i) + P'_{ij} + x_R^j v_R$ contains a fault-free $u_L v_R$-path of length at most $d+4$ in Q_n.

The proof of the lemma is complete. □

Lemma 3 [6]. *Assume that Q_n $(n \geqslant 3)$ has at most $2n-5$ faulty edges and $f(u) \leqslant n-2$ for any $u \in V(Q_n)$. Then Q_n contains a fault-free path of length $2^n - 1$ joining any two vertices with odd distance and a fault-free path of length $2^n - 2$ joining any two vertices with even distance.*

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Lemma 4. *Assume that Q_n $(n \geqslant 3)$ has at most $2n-5$ faulty edges and $Q_n = L \odot R$ satisfies the following properties:*

(a) *For any two distinct vertices u and v with distance d_L in L, there exist fault-free uv-paths of length ℓ in L for every ℓ with $d_L + 2 \leqslant \ell \leqslant 2^{n-1} - 1$ and $\ell - d_L \equiv 0 \pmod 2$.*
(b) *For any two distinct vertices u and v with distance d_R in R, there exist fault-free uv-paths of length ℓ in R for every ℓ with $d_R + 4 \leqslant \ell \leqslant 2^{n-1} - 1$ and $\ell - d_R \equiv 0 \pmod 2$.*

For any two distinct vertices $u_L \in L$ and $v_R \in R$ with distance d in Q_n, if there exists a fault-free $u_L v_R$-path P of length ℓ', where $P = P(u_L, x_L) + x_L x_R + P(x_R, v_R)$, $x_L \in L, x_R \in R$, $P(u_L, x_L) \subset L$ and $P(x_R, v_R) \subset R$, then there exist fault-free $u_L v_R$-paths of length ℓ for every ℓ with $\ell' \leqslant \ell \leqslant 2^n - 1$ and $\ell - \ell' \equiv 0 \pmod 2$ in Q_n.

Proof. Let $P = P(u_L, x_L) + x_L x_R + P(x_R, v_R)$ be a fault-free $u_L v_R$-path of length ℓ' in Q_n, where $x_L \in L, x_R \in R, x_L x_R \in E_n$, $P(u_L, x_L)$ in L and $P(x_R, v_R)$ in R. Let $P_L = P(u_L, x_L)$, $P_R = P(x_R, v_R)$, $\ell'_1 = \varepsilon(P_L)$ and $\ell'_2 = \varepsilon(P_R)$. Then $\ell'_1 \leqslant 2^{n-1} - 1$, $\ell'_2 \leqslant 2^{n-1} - 1$ and $\ell' = \ell'_1 + \ell'_2 + 1$. Let ℓ be any integer with $\ell' \leqslant \ell \leqslant 2^n - 1$ and $\ell - \ell' \equiv 0 \pmod 2$. We are done if $\ell = \ell'$. Assume $\ell \geqslant \ell' + 2$ below since $\ell - \ell' \equiv 0 \pmod 2$.

If $\ell' + 2 \leqslant \ell \leqslant 2^{n-1} + \ell'_2$, let $\ell_1 = \ell - \ell'_2 - 1$, then $\ell_1 \geqslant \ell' + 2 - \ell'_2 - 1 = \ell'_1 + 2 \geqslant d_L(u_L, x_L) + 2$, and $\ell_1 \leqslant 2^{n-1} - 1$, $\ell_1 - \ell'_1 \equiv 0 \pmod 2$. By the condition (a), in L there exists a fault-free $u_L x_L$-path P_1 of length ℓ_1. Then $P_1 + x_L x_R + P_R$ is a fault-free $u_L v_R$-path of length ℓ.

By the condition (a), let T_L be a longest fault-free $u_L x_L$-path in L, and let T'_L be a fault-free $u_L x_L$-path of length $\varepsilon(T_L) - 2$ in L. Since $\varepsilon(T_L) - \ell'_1 \equiv 0 \pmod 2$, we have that $\varepsilon(T_L) = 2^{n-1} - 1$ and $\varepsilon(T'_L) = 2^{n-1} - 3$ if ℓ'_1 is odd or $\varepsilon(T_L) = 2^{n-1} - 2$ and $\varepsilon(T'_L) = 2^{n-1} - 4$ if ℓ'_1 is even.

If $2^{n-1} + \ell'_2 + 1 \leqslant \ell \leqslant 2^n - 4$, let $\ell_2 = \ell - \varepsilon(T'_L) - 1$, then $\ell'_2 + 3 \leqslant \ell_2 \leqslant 2^{n-1} - 2$ if ℓ'_1 is odd and $\ell'_2 + 4 \leqslant \ell_2 \leqslant 2^{n-1} - 1$ if ℓ'_1 is even. Noting that $\ell - \ell' \equiv 0 \pmod 2$ and $\ell' = \ell'_1 + \ell'_2 + 1$, we have $\ell_2 - (\ell'_1 + \ell'_2 + 1) \equiv 0 \pmod 2$. This fact implies that if ℓ'_1 is odd then $\ell_2 - \ell'_2 \equiv 0 \pmod 2$, that is, $\ell'_2 + 4 \leqslant \ell_2$. Clearly, $\ell'_2 \geqslant d_R(x_R, v_R)$. By the condition (b), in R there exists a fault-free $x_R v_R$-path P_2 of length ℓ_2. Then $T'_L + x_L x_R + P_2$ is a fault-free $u_L v_R$-path of length ℓ.

Let T_R be a longest fault-free $x_R v_R$-path in R. Then $\varepsilon(T_R) = 2^{n-1} - 1$ if ℓ'_2 is odd or $2^{n-1} - 2$ if ℓ'_2 is even.

If $\ell = 2^n - 3$, then ℓ'_1 and ℓ'_2 have the same parity since $\ell - \ell' \equiv 0 \pmod 2$. Thus, if ℓ'_1 is odd, then the path $T'_L + x_L x_R + T_R$ is a fault-free $u_L v_R$-path of length ℓ; if ℓ'_1 is even, then the path $T_L + x_L x_R + T_R$ is a fault-free $u_L v_R$-path of length ℓ.

If $\ell = 2^n - 2$ or $2^n - 1$ then, by Lemma 3, there is a fault-free $u_L v_R$-path of length ℓ in Q_n.

By arbitrariness of ℓ, the proof is completed. □

Lemma 5. *Let F be a set of faulty edges in Q_n $(n \geqslant 4)$ with $1 \leqslant |F| \leqslant 2n-5$ and $f(u) \leqslant n-3$ for any $u \in V(Q_n)$. Then there exists some j $(1 \leqslant j \leqslant n)$ with $Q_n = L_j \odot R_j$ such that $|F \cap L_j| \leqslant 2n-7$ and $|F \cap R_j| \leqslant 2n-7$.*

Proof. If $|F| \leqslant 2n-6$, it is clear that $|F \cap L_j| \leqslant 2n-7$ and $|F \cap R_j| \leqslant 2n-7$ for any $j \in \{1, 2, 3, \ldots, n\}$ with $|F_j| \geqslant 1$ when $n \geqslant 4$, where $F_j = F \cap E_j$. Assume $|F| = 2n-5$ below. If there is some $j \in \{1, 2, 3, \ldots, n\}$ with $|F_j| \geqslant 2$, then $|F \cap L_j| \leqslant 2n-7$ and $|F \cap R_j| \leqslant 2n-7$. If $n \geqslant 6$, such a $j \in \{1, 2, 3, \ldots, n\}$ exists certainly and we are done. Thus, we only need to consider $n \in \{4, 5\}$.

If $n = 4$, then $|F| = 3$ and $f(u) \leqslant 1$ for any vertex u in Q_4 by our hypothesis. Assume that $|F_i| \leqslant 1$ for each $i = 1, 2, 3, 4$. Without loss of generality, assume $|F_1| = |F_2| = |F_3| = 1$. We need to prove that there is some $j \in \{1, 2, 3, 4\}$ such that $|F \cap L_j| \leqslant 1$ and $|F \cap R_j| \leqslant 1$.

Let X be the set of end-vertices of edges in F. Then $|X| = 6$ since $f(u) \leqslant 1$ for any vertex u in Q_4. Suppose to the contrary that one of L_i and R_i has two faulty edges for each $i = 1, 2, 3$. Then for each $i = 1, 2, 3$, there are exactly five vertices in X whose jth positions are identical. It follows that there are at least three vertices in X whose the first three positions are identical. Let x, y, z be such three vertices in X, and let $x = x_1x_2x_3x_4, y = x_1x_2x_3y_4, z = x_1x_2x_3z_4$. Since $\{x_4, y_4, z_4\} \subseteq \{0, 1\}$, at least two of x, y and x are identical. This fact means $|X| < 6$, a contradiction.

If $n = 5$, then $|F| = 5$ and $f(u) \leqslant 2$ for any vertex u in Q_5. Assume that $|F_i| = 1$ and let $e_i = u^i v^i \in F_i$ for every i with $1 \leqslant i \leqslant 5$. We need to prove that there is some $j \in \{1, 2, 3, 4, 5\}$ such that $|F \cap L_j| \leqslant 3$ and $|F \cap R_j| \leqslant 3$. Suppose to the contrary that for each $i = 1, 2, 3, 4, 5$, one of L_i and R_i has four faulty edges. Then for each $i = 1, 2, 3, 4, 5$, there are exactly 9 end-vertices of edges in F whose jth positions are identical. If let $u^1 = 0x_2x_3x_4x_5$ and $v^1 = 1x_2x_3x_4x_5$, then, for each $j = 2, 3, 4, 5$, the jth positions of u^i and v^i are different for $j = i$, and identical for $j \neq i$, that is

$$u^2 = 00x_3x_4x_5, \quad v^2 = 01x_3x_4x_5; \qquad u^3 = 0x_20x_4x_5, \quad v^3 = 0x_21x_4x_5;$$
$$u^4 = 0x_2x_30x_5, \quad v^4 = 0x_2x_31x_5; \qquad u^5 = 0x_2x_3x_40, \quad v^5 = 0x_2x_3x_41.$$

It is easy to see that $u^2 = u^1$ if $x_2 = 0$, and $v^2 = u^1$ if $x_2 = 1$. Then $u^3 = u^1$ if $x_3 = 0$, and $v^3 = u^1$ if $x_3 = 1$. It follows that $f(u^1) \geqslant 3$, a contradiction.

The lemma follows. □

Lemma 6 [3]. *Assume that Q_n $(n \geqslant 3)$ has f_v faulty vertices and f_e faulty edges. If $f_v + f_e \leqslant n-2$ then, for any two distinct fault-free vertices u and v in Q_n with distance d, there exist fault-free uv-paths of length ℓ in Q_n for every ℓ with $d + 2 \leqslant \ell \leqslant 2^n - 2f_v - 1$ and $\ell - d \equiv 0 \pmod 2$.*

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Lemma 7. *Assume that Q_n $(n \geqslant 3)$ has at most $2n-5$ faulty edges and $f(u) \leqslant n-2$ for any $u \in V(Q_n)$. If there exists a vertex x with $f(x) = n-2$ then, for any two distinct u and v with distance d, there exist fault-free uv-paths of length ℓ in Q_n for every ℓ with $d+4 \leqslant \ell \leqslant 2^n - 1$ and $\ell - d \equiv 0 \pmod 2$.*

Proof. Let u and v be two distinct vertices with distance d in Q_n, and let ℓ be any integer with $d+4 \leqslant \ell \leqslant 2^n-1$ and $\ell - d \equiv 0 \pmod 2$.

By Lemma 3, Q_n contains a fault-free uv-path of length 2^n-1 if d is odd or a fault-free uv-path of length 2^n-2 if d is even. Thus, we can assume $\ell \leqslant 2^n-3$ if d is odd or $\ell \leqslant 2^n-4$ if d is even.

Let x be a vertex in Q_n with $f(x) = n-2$. We use F' to denote the set of faulty edges not incident with x. Then $|F'| \leqslant n-3$ since $|F| \leqslant 2n-5$ and $f(x) = n-2$.

If $x \notin \{u, v\}$ then, by Lemma 6, there exists a uv-path of length ℓ for every ℓ with $d+4 \leqslant \ell \leqslant 2^n-3$ in $Q_n - F' \cup \{x\}$, so in $Q_n - F$.

If $x \in \{u, v\}$, without loss of generality, assume $u = x$, then u has two neighbors in $Q_n - F$, at least one of them, say y, is not v. Let $\ell_1 = \ell - 1$, then $d+3 \leqslant \ell_1 \leqslant 2^n-4$. Since $d_{Q_n}(y, v) = d-1$ or $d+1$, we have $d_{Q_n}(y, v) + 2 \leqslant \ell_1 \leqslant 2^n - 4$. By Lemma 6, there exists a yv-path P of length ℓ_1 in $Q_n - F' \cup \{u\}$. Then the path $uy + P$ is a fault-free uv-path of length ℓ in Q_n.

The proof is complete. □

Lemma 8 ([10]). *If Q_n $(n \geqslant 2)$ has at most $n-2$ faulty edges, then for any two different vertices u and v with distance d there exist fault-free uv-paths of length ℓ in Q_n for every ℓ with $d+2 \leqslant \ell \leqslant 2^n - 1$ and $\ell - d \equiv 0 \pmod 2$*

3. Proof of Theorem 1

We first note that if there exists a vertex x with $f(x) = n-2$ in Q_n, then the theorem follows by Lemma 7. Assume $f(x) \leqslant n-3$ for any vertex x in Q_n below. We prove the theorem by induction on $n \geqslant 3$. The base of induction for $n = 3$ holds clearly by Lemma 8. Assume that the theorem holds for $n-1$ with $n \geqslant 4$.

Let F be a set of faulty edges in Q_n with $|F| \leqslant 2n-5$. By Lemma 5, we can assume $Q_n = L \odot R$ with $|F_L| \leqslant 2n-7$ and $|F_R| \leqslant 2n-7$. Since $f(x) \leqslant n-3$ for any vertex x in Q_n, each vertex u is incident with at least two fault-free edges in L if $u \in L$ or R if $u \in R$.

Let u and v be any two distinct vertices with distance d in Q_n and ℓ be any integer with $d+4 \leqslant \ell \leqslant 2^n-1$ and $\ell - d \equiv 0 \pmod 2$. We need to prove that there exists a fault-free uv-path P of length ℓ in Q_n.

Case 1: $u \in L$ and $v \in R$.

Assume, without loss of generality, $|F_L| \leqslant |F_R|$. Then $|F_L| \leqslant n-3$.

By Lemma 2, there exists a fault-free uv-path of length at most $d+4$ in Q_n. By the induction hypothesis, $Q_n = L \odot R$ satisfies the condition (b) in Lemma 4 clearly. Since $|F_L| \leqslant n-3$, by Lemma 8, for any two vertices x and y with distance h in L there exists a fault-free xy-path of length ℓ in L for every ℓ with $h+2 \leqslant \ell \leqslant 2^{n-1}-1$ and $\ell - h \equiv 0 \pmod 2$, and so the condition (a) in Lemma 4 holds. By Lemma 4, the theorem holds.

Case 2: Both u and v are in L or R. Without loss of generality, assume u and v in L.

If $d+4 \leqslant \ell \leqslant 2^{n-1}-1$, then by the induction hypothesis, there exists a fault-free uv-path of length ℓ in L, so in Q_n.

Let P_L be a longest fault-free uv-path and P'_L be a fault-free uv-path of length $\varepsilon(P_L) - 2$ in L. Then $\varepsilon(P_L) - d \equiv 0 \pmod 2$, $\varepsilon(P'_L) - d \equiv 0 \pmod 2$. Thus, $\varepsilon(P_L) = 2^{n-1} - 1$ and $\varepsilon(P'_L) = 2^{n-1} - 3$ if d is odd, and $\varepsilon(P_L) = 2^{n-1} - 2$ and $\varepsilon(P'_L) = 2^{n-1} - 4$ if d is even. Let h denote the number of faulty edges in F_R corresponding to the edges in the path P_L (or P'_L).

If $n \geqslant 5$, since

$$\varepsilon(P_L) > \varepsilon(P'_L) \geqslant 2^{n-1} - 4 > 2(2n-5) \geqslant 2(2n-5-h) + h,$$

then there is an edge $x_L y_L$ in P_L such that the set of edges $\{x_L x_R, y_L y_R, x_R y_R\}$ is fault-free, and there is also an edge $x'_L y'_L$ in P'_L such that the set of edges $\{x'_L x'_R, y'_L y'_R, x'_R y'_R\}$ is fault-free.

If $2^{n-1} + 2 \leqslant \ell \leqslant 2^n - 3$, let $\ell_1 = \ell - \varepsilon(P'_L) - 1$, then ℓ_1 is odd and $5 \leqslant \ell_1 \leqslant 2^{n-1} - 1$. By the induction hypothesis, there is a fault-free $x_R y_R$-path P_R of length ℓ_1 in R with $5 \leqslant \ell_1 \leqslant 2^{n-1} - 1$ and ℓ_1 is odd. Then $P = P'_L(u, x_L) + x_L x_R + P_R + y_R y_L + P'_L(y_L, v)$ is a fault-free uv-path of length ℓ, where $\ell = \varepsilon(P'_L) + \varepsilon(P_R) + 1 = \varepsilon(P'_L) + \ell_1 + 1$, as required.

If $\ell = 2^{n-1} + 1$ or $\ell = 2^{n-1}$, then $P = P_L(u, x_L) + x_L x_R + x_R y_R + y_R y_L + P_L(y_L, v)$ is a fault-free uv-path of length ℓ.

If $\ell = 2^n - 1$ or $\ell = 2^n - 2$, let P'_R be a longest fault-free $x_R y_R$-path of length $2^{n-1} - 1$ in R, then $P = P_L(u, x_L) + x_L x_R + P'_R + y_R y_L + P_L(y_L, v)$ is a fault-free uv-path of length ℓ.

Now we consider the case of $n = 4$. If $d(u, v)$ is odd, let $\ell_1 = \ell - \varepsilon(P_L)$, then the above proof for $2^3 + 1 \leqslant \ell \leqslant 2^4 - 1$ is also valid. If $d(u, v)$ is even, $|F_R| = 1$ and $|F_j| = 2$, then the above proof for $2^3 \leqslant \ell \leqslant 2^4 - 2$ is also valid. If $d(u, v)$ is even and $|F_j| = 3$, it is easy to see that there exists a path P'_L between u and v such that the above proof for $2^3 \leqslant \ell \leqslant 2^4 - 2$ is also valid.

The proof is completed. □

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4. Conclusion

In this paper, we have showed that if Q_n $(n \geqslant 3)$ has at most $2n-5$ faulty edges and each vertex is incident with at least two fault-free edges, then for any two distinct vertices u and v with distance d, there exist fault-free uv-paths of length ℓ in Q_n for every ℓ with $d+4 \leqslant \ell \leqslant 2^n-1$ and $\ell - d \equiv 0 \pmod 2$. This result improves some known results on edge-fault bipanconnectivity of hypercubes.

Our result is optimal in the following sense. Chan and Lee [1] showed that there exist $2n-4$ faulty edges in Q_n $(n \geqslant 3)$ and every vertex is incident with at least two fault-free edges such that Q_n contain no Hamilton cycles. This fact shows that for two adjacent vertices u and v with uv is fault-free, there are no fault-free uv-paths of length 2^n-1 in Q_n for $n \geqslant 3$. For future research, we will apply our strategy to investigate an analogous problem for some hypercube variants and many other networks.

Acknowledgements

The authors would like to express their gratitude to the anonymous referees for their kind suggestions and useful comments on the original manuscript and two revised version, which resulted in this final version.

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