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Note

Diameter vulnerability of graphs by edge deletion☆

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Abstract

Let $f(t,k)$ be the maximum diameter of graphs obtained by deleting t edges from a $(t+1)$-edge-connected graph with diameter k. This paper shows $4\sqrt{2t}-6 < f(t,3) \le \max\{59, 5\sqrt{2t}+7\}$ for $t \ge 4$, which corrects an improper result in [C. Peyrat, Diameter vulnerability of graphs, Discrete Appl. Math. 9 (3) (1984) 245–250] and also determines $f(2,k) = 3k-1$ and $f(3,k) = 4k-2$ for $k \ge 3$.

Keywords: Diameter; Edge-deletion; Diameter vulnerability; Edge-connected graph

1. Introduction

We follow [9] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a connected graph, where $V = V(G)$ is the vertex-set of G and $E = E(G)$ is the edge-set of G. For any two distinct vertices x and y in G, the distance $d_G(x, y)$ between x and y is the length of a shortest path between x and y in G. The diameter $D(G)$ of G is the maximum value of $d_G(x, y)$ over all pairs of vertices x and y in G.

Let $f(t,k)$ denote the maximum possible diameter of a graph obtained by deleting t edges from a $(t+1)$-edge-connected graph with diameter k, and $g(t,k)$ denote the maximum diameter of any connected graph obtained by deleting t edges from a connected graph with diameter k. By the definitions, it is clear that for given t and k if $f(t,k)$ and $g(t,k)$ are well-defined then

$$f(t,k) \le g(t,k). \qquad (1)$$

The problem determining $f(t,k)$ for given t and k, proposed by Chung and Garey [1], is of interest, for example, when studying the potential effects of link failures on the performance of a communication network, especially for networks in which the maximum time-delay or signal degradation is directly related to the diameter of the network. This problem is proved to be NP-complete by Schoone et al. [6] in general. Much work has been done on this topic, see [1–7], and also [8] for a survey of some well-known results.

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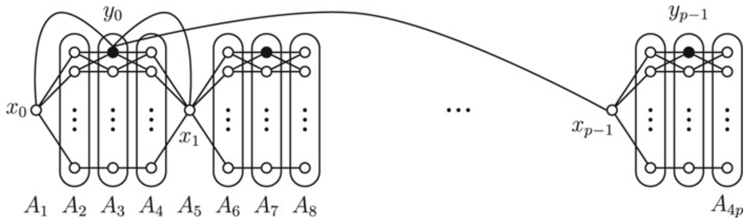

Fig. 1. The constructed graph G for the lower bound of $f(t, 3)$.

Obviously, $f(t, 1) = 2$. Chung and Garey [1] determined $f(1, k) = 2k$ and gave the bounds: $(t + 1)(k - 3) \leq f(t, k) \leq (t+1)k+t$ for $k \geq 4$. Schoone et al. [6] improved this upper bound as $(t+1)k$, determined $g(2, k) = 3k-1$ and $g(3, k) = 4k - 2$ for $k \geq 2$. In 1984, Peyrat [5] determined $f(t, 2) = 4$ and gave "bounds" of $f(t, 3)$ as follows:

$$3\sqrt{2t} - 3 \leq f(t, 3) \leq 3\sqrt{2t} + 4 \quad \text{if } t \text{ is large enough.} \tag{2}$$

In this note, we establish the bounds as follows.

$$4\sqrt{2t} - 6 < f(t, 3) \leq \max\{59, 5\sqrt{2t} + 7\} \quad \text{for } t \geq 4. \tag{3}$$

It is clear that the lower bound of $f(t, 3)$ in (3) is greater than the upper bound in (2) if $t \geq 50$, which implies that the upper bound of $f(t, 3)$ given in (2) is improper for $t \geq 50$. We also determine $f(2, k) = 3k - 1$ and $f(3, k) = 4k - 2$.

The proofs of our results are in Sections 2 and 3, respectively.

2. Bounds of $f(t, 3)$

Let $\mathbb{N}^*$ be the set of positive integers. We first state the lower bound of $f(t, 3)$.

Theorem 1. *$f(t, 3) > 4\sqrt{2t} - 6$ for any $t \geq 4$.*

Proof. For a given $t \in \mathbb{N}^*$ with $t \geq 4$, there exists some $p \in \mathbb{N}^*$ such that $p(p + 1)/2 \leq t < (p + 1)(p + 2)/2$. First assume $t = p(p + 1)/2$. We construct a graph G with $(3t + 4)p$ vertices as follows, as illustrated in Fig. 1.

The vertex-set $V(G)$ of G can be partitioned into $\{A_1, A_2, \ldots, A_{4p}\}$ such that

$$|A_i| = \begin{cases} 1 & i \equiv 1 \pmod 4; \\ t + 1 & i \not\equiv 1 \pmod 4. \end{cases}$$

Let $A_{4k+1} = \{x_k\}$ and arbitrarily choose one vertex $y_k \in A_{4k+3}$ for each $k = 0, 1, \ldots, p - 1$ (where y_k's are shown as black dots in Fig. 1) and let $Y = \{y_k : 0 \leq k \leq p - 1\}$. All the induced subgraphs $G[A_i \cup A_{i+1}]$ for $1 \leq i \leq 4p - 1$ and $G[Y]$ are complete (the readers can imagine these edges though they do not show in Fig. 1). Let $E_1 = E(G[Y])$. Link x_k to y_0 by an edge for each $k = 0, 1, \ldots, p - 1$, and let E_2 denotes the set of these p edges. It is easy to check that G is $(t + 1)$-edge-connected and of diameter 3.

Note that $|E_1 \cup E_2| = p(p - 1)/2 + p = p(p + 1)/2 = t$. If we delete all the edges in $E_1 \cup E_2$ from G, the remaining graph has diameter $4p - 1 = 2\sqrt{1 + 8t} - 3$. Hence $f(t, 3) \geq 2\sqrt{1 + 8t} - 3 > 4\sqrt{2t} - 3$.

Now, assume $p(p + 1)/2 < t < (p + 1)(p + 2)/2$. We construct a graph G' from G by adding another vertex x_p and linking x_p to y_0 and each vertex in A_{4p}. Obviously, G' is also $(t + 1)$-edge-connected and of diameter 3. Let $E' = E_1 \cup E_2 \cup \{x_p y_0\}$, then $|E'| = p(p + 1)/2 + 1 \leq t$. Noting when $t \leq (p + 1)(p + 2)/2 - 1$, we have $p \geq (\sqrt{9 + 8t} - 3)/2$. Since $G' - E'$ has diameter $4p$, we have

$$f(t, 3) \geq f(p(p + 1)/2 + 1, 3) \geq 4p \geq 2(\sqrt{9 + 8t} - 3) > 4\sqrt{2t} - 6.$$

The proof of the theorem is complete. □

It is clear that $4\sqrt{2t} - 6 > 3\sqrt{2t} + 4$ for $t > 50$. This fact shows that the upper bound of $f(t, 3)$ given in (2) is not correct for $t > 50$. However, the method proposed by Peyrat in [5] to establish the upper bound of $f(t, 3)$ is very useful. Now, by refining this method we prove the following theorem.

Theorem 2. *$f(t, 3) \leq \max\{59, 5\sqrt{2t} + 7\}$ for $t \geq 4$.*

Proof. Let G be a $(t+1)$-edge-connected graph of diameter 3, and let $E' \subset E(G)$ with $|E'| = t$. Let $G' = G - E'$ and d be diameter of G'. Then there are two vertices x and y such that $d_{G'}(x, y) = d$ and a shortest xy-path $(x = x_0, x_1, \ldots, x_d = y)$ of length d in G'. Let

$$N_i = \{z \in V(G') : d_{G'}(z, x) = i\}.$$

It is clear that $N_i \neq \emptyset$ since $x_i \in N_i$ for each $i = 0, 1, \ldots, d$, and $\{N_0, N_1, \ldots, N_d\}$ is a partition of $V(G')$. Let

$$L_i = N_i \cup \{z \in V(G') : \ d_{G'}(z, z') \leq 1 \text{ for some } z' \in N_i\}, \quad 1 \leq i \leq d.$$

For $0 \leq i \leq d$, let e_i be the number of edges in E' that has an end-vertex x_i, that is, $e_i = d_G(x_i) - d_{G'}(x_i)$. Since G is $(t+1)$-edge-connected, $d_G(x_i) \geq t + 1$. It follows that

$$|L_i| \geq d_{G'}(x_i) + 1 = d_G(x_i) + 1 - e_i \geq t + 2 - e_i,$$

that is,

$$|L_i| \geq t + 2 - e_i \quad \text{for } i = 1, 2, \ldots, d. \tag{4}$$

Let $p = \lfloor d/5 \rfloor$. Then $d \leq 5p + 4$. Let $[0, d] = \{0, 1, \ldots, d\}$ and

$$I_k = \{5k-2, 5k-1, 5k, 5k+1, 5k+2\} \cap [0, d], \quad 0 \leq k \leq p.$$

For each $k = 0, 1, \ldots, p$, choose $j_k \in \{5k-1, 5k, 5k+1\} \cap [0, d]$ such that $|N_{j_k}| = \max\{|N_{5k-1}|, |N_{5k}|, |N_{5k+1}|\}$. By (4) and the maximality of $|N_{j_k}|$, we have

$$|N_{j_k}| \geq \max\{|N_{5k-1}|, |N_{5k}|, |N_{5k+1}|\} \geq \lceil (t + 2 - e_{5k})/3 \rceil,$$

that is,

$$|N_{j_k}| \geq \lceil (t + 2 - e_{5k})/3 \rceil. \tag{5}$$

Let $J = \{j_0, j_1, \ldots, j_p\}$ and $i, j \in J$ such that $|j - i| \geq 4$. Then $L_i \cap L_j = \emptyset$ and there are no edges in G' between L_i and L_j. Let E_k be the set of edges of E' having one of their end-vertices in L_k. If there are no edges of E' between L_i and L_j then either

$$|E_i| \geq \lceil (t+1)/3 \rceil \quad \text{or} \quad |E_j| \geq \lceil (t+1)/3 \rceil. \tag{6}$$

In fact, if there exist $u \in N_i$ and $v \in N_j$ such that neither u nor v is the end-vertex of an edge of E', then all the neighbors of u (resp. v) in G are in L_i (resp. L_j). But there are no edges in G' between L_i and L_j, which implies $d_G(u, v) > 3$ contradicting the fact that the diameter of G is 3.

So, without loss of generality, we can assume that each vertex of N_i is the end-vertex of an edge of E', whose other end-vertex does not belong to L_i. There is j_k such that $i = j_k$ and, by (5),

$$|N_i| = |N_{j_k}| \geq \lceil (t + 2 - e_{5k})/3 \rceil.$$

If $e_{5k} = 0$, we have

$$|E_i| \geq |N_i| \geq \lceil (t+2)/3 \rceil.$$

If $e_{5k} \neq 0$, then

$$|E_i| \geq |N_i| - 1 + e_{5k} \geq \lceil (t + 2 - e_{5k})/3 \rceil - 1 + e_{5k} \geq \lceil (t+1)/3 \rceil.$$

This completes the proof of (6).

We now prove $f(t, 3) \leq \max\{59, 5\sqrt{2t} + 7\}$.

Let $K = \{j \in J : |E_j| \geq \lceil (t+1)/3 \rceil\}$. So if $j_k, j_{k'} \in J - K$ and $|k - k'| \geq 2$ (which implies $|j_k - j_{k'}| \geq 4$), then there is an edge of E' from L_{j_k} to $L_{j_{k'}}$ by (6). Let $s = |K|$. We have

$$t = |E'| \geq \frac{s}{2} \left\lceil \frac{t+1}{3} \right\rceil + \frac{(|J| - s - 1)(|J| - s - 2)}{2}.$$

(The flaw in the proof in [5] is here!) Since $|J| = p + 1$, we have

$$6t \geq s(t+1) + 3(p - s)(p - s - 1).$$

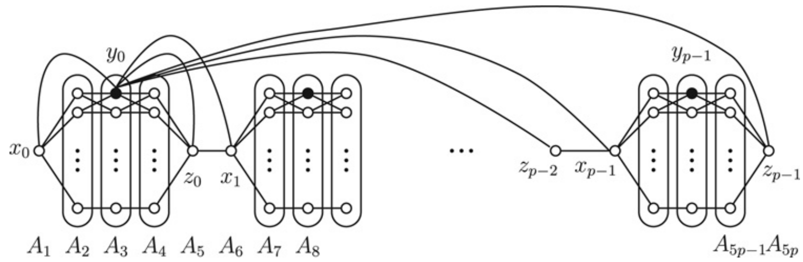

Fig. 2. Example for the lower bound for the relaxed version.

This implies $s \leq 5$. Therefore,

$$t \geq \frac{3p^2 - 3(2s+1)p + 3s^2 + 4s}{6 - s} \geq \frac{p(p-1)}{2} \quad \text{if } p \geq 12.$$

The reason that the last inequality holds is because that $6s^2 + (p^2 - 13p + 8)s \geq 0$ if $p \geq 12$. By solving the inequality $t \geq p(p-1)/2$, we have

$$12 \leq p \leq (\sqrt{1+8t} + 1)/2 < \sqrt{2t} + 3/5.$$

Finally, by the definition of p, we have

$$f(t, 3) = d \leq 5p + 4 \leq \max\{59, 5\sqrt{2t} + 7\}.$$

The proof of the theorem is complete. □

Remark 1. The proof of the above theorem is independent of $(t+1)$-edge-connectivity of G and only dependent on $\delta(G) \geq t + 1$ and $G - E'$ being connected. If we relax the condition of G, namely, if $f(t, 3)$ is the largest possible diameter of connected graphs obtained by deleting t edges from a graph G with minimum degree at least $(t+1)$ and diameter 3, then the upper bound in the above theorem is almost best possible in the point of view of preserving the main part "$5\sqrt{2t}$". This can be seen by the following example. Let G be a graph constructed as follows, and illustrated in Fig. 2. Let $\{A_1, A_2, \ldots, A_{5p}\}$ be a partition of $V(G)$ with

$$|A_i| = \begin{cases} 1 & i \equiv 1, 0 \pmod 5; \\ t+1 & \text{otherwise.} \end{cases}$$

Then G has $(3t+5)p$ vertices. For $0 \leq k \leq p-1$, let $A_{5k+1} = \{x_k\}$ and $A_{5k+5} = \{z_k\}$, and $Y = \{y_k \in A_{5k+3} : 0 \leq k \leq p-1\}$. Add edges between each pair of vertices in $A_i \cup A_{i+1}$ and each pair of vertices in Y such that all induced graphs $G[A_i \cup A_{i+1}]$ for $1 \leq i \leq 5p-1$ and $G[Y]$ are complete. Let $E_1 = E(G[Y])$. Then $|E_1| = \frac{1}{2}p(p-1)$. Link each x_k and z_k $(0 \leq k \leq p-1)$ to y_0 by an edge, and let E_2 denote the set of these $2p$ edges. Then $|E_1 \cup E_2| = \frac{1}{2}p(p-1) + 2p = \frac{1}{2}p(p+3)$. Obviously, the graph G constructed as above has diameter 3 and minimum degree at least $t+1$. Let $t = \frac{1}{2}p(p+3)$. Then, by deleting all the edges of $E_1 \cup E_2$, the resulting graph has diameter $5p - 1 = 5\lfloor \frac{\sqrt{9+8t}-3}{2} \rfloor - 1 > 5\sqrt{2t} - 8$, that is, $f(t, 3) > 5\sqrt{2t} - 8$. Note that the set E' of edges incident with vertices in $A_{5p-4} \cup A_{5p-3} \cup \cdots \cup A_{5p}$ is an edge-cut of G and $|E'| = p + 2 < t + 1$. This fact implies that G is not $(t+1)$-edge-connected. So we need to develop a new technique in order to improve the upper bound for $f(t, 3)$.

3. Values of $f(2, k)$ and $f(3, k)$

Let $P(t, d)$ be the minimum diameter of a graph obtained by adding t edges to a path of length d. The problem determining $P(t, d)$ is closely related to $g(t, k)$ since Chung and Garey [1] showed that for a connected graph G, $F \subset E(G)$ and $|F| = t$, if $D = D(G - F)$ is well defined then $D(G) \geq P(t, d)$. This fact shows that in order to establish an upper bound for $g(t, k)$, it is sufficient to consider a graph with diameter k obtained from a single path plus t extra edges, then the length of the path gives an upper bound for $g(t, k)$. Clearly, $P(1, d) = \lfloor \frac{d+1}{2} \rfloor$ for $d \geq 2$. Schoone et al. [6] determined that

$$P(2, d) = \left\lceil \frac{d+1}{3} \right\rceil \quad \text{for } d \geq 3 \quad \text{and} \quad P(3, d) = \left\lceil \frac{d+2}{4} \right\rceil \quad \text{for } d \geq 4. \tag{7}$$

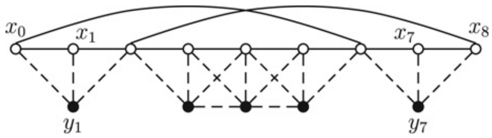

Fig. 3. Construction of $G_{2,k}$ for $k = 3$.

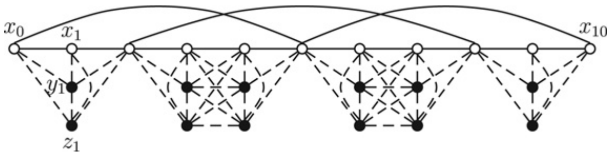

Fig. 4. Construction of $G_{3,k}$ for $k = 3$.

At the same time, using these results, they determined

$$g(1, k) = 2k, \qquad g(2, k) = 3k - 1, \qquad g(3, k) = 4k - 2 \quad \text{for } k \geq 2. \tag{8}$$

(see [8] or [1] and [6] for more details). Motivated by these facts, we propose the following conjecture.

Conjecture 1. *For a fixed t, there exists a minimum $k_0(t) \in \mathbb{N}$ such that for each $k \geq k_0(t)$*

$$f(t, k) = g(t, k) = \max_{P(t,d)=k} d.$$

Since $P(1, d) = \lfloor \frac{d+1}{2} \rfloor$ for $d \geq 2$, we have

$$f(1, k) = g(1, k) = 2k = \max_{P(1,d)=k} d,$$

for all $k \geq 1$, namely $k_0(1) = 1$. So the conjecture is true for $t = 1$. We now show that the conjecture is also true for $t = 2$ and $t = 3$ by proving $f(2, k) = 3k - 1$ and $f(3, k) = 4k - 2$ for $k \geq 3$. Combining these results with $g(2, k) = 3k - 1$, $g(3, k) = 4k - 2$ for $k \geq 2$ and $f(t, 2) = 4$ for any $t \geq 1$, we have $k_0(2) = k_0(3) = 3$.

Theorem 3. *$f(2, k) = 3k - 1$ and $f(3, k) = 4k - 2$ for $k \geq 3$.*

Proof. We first prove $f(2, k) = 3k - 1$. By (1) and (8), we only need to prove $f(2, k) \geq 3k - 1$. To this end, we only need to construct a 3-edge-connected graph with diameter k such that its diameter increases to at least $3k - 1$ when its two edges are deleted.

Let $H_{2,k}$ be a graph obtaining from a path $P_{3k-1} = (x_0, x_1, \ldots, x_{3k-1})$ plus two extra edges x_0x_{2k} and $x_{k-1}x_{3k-1}$ (see Fig. 3 for $k = 3$). Note that Schoone et al. use $H_{2,k}$ to show $g(2, k) \geq 3k - 1$ in [6] (in which, however, there is a typographical error, that is, the adding edge $x_D x_{2D}$ should be $x_0 x_{2D}$).

Since $H_{2,k}$ is not 3-edge-connected, we need to make some modification. Call a vertex of P_{3k-1} that is incident with an extra edge a fixed vertex. For each non-fixed vertex x_i, add an additional vertex y_i (the black dots in Fig. 3). For $0 \leq i \leq 3k - 1$, let $A_i = \{x_i\}$ if x_i is fixed, and $A_i = \{x_i, y_i\}$ otherwise. Add edges so that the induced graph by $A_i \cup A_{i+1}$ is complete for $0 \leq i \leq 3k - 2$ (the dashed lines in Fig. 3). The resulting graph is denoted by $G_{2,k}$. It is easy to check $G_{2,k}$ is 3-edge-connected and of diameter k. By deleting the two extra edges x_0x_{2k} and $x_{k-1}x_{3k-1}$, the remaining graph is of diameter $3k - 1$. This implies $f(2, k) \geq 3k - 1$ and completes our proof of the first equality.

Similarly, we prove $f(3, k) \geq 4k - 2$ by constructing a 4-edge-connected graph $G_{3,k}$ with diameter k. Let $H_{3,k}$ be a graph obtaining from a path P_{4k-2} plus three extra edges x_0x_{2k-1}, $x_{2k-1}x_{4k-2}$ and $x_{k-1}x_{3k-1}$ (see Fig. 4). We construct a graph $G_{3,k}$ from $H_{3,k}$ by expanding each non-fixed vertex x_i to $A_i = \{x_i, y_i, z_i\}$ and adding edges such that the induced graph by $A_i \cup A_{i+1}$ is complete for $0 \leq i \leq 4k - 3$. Then $G_{3,k}$ is a 4-edge-connected graph with diameter k. By deleting the three extra edges, the remaining graph is of diameter $4k - 2$. This implies $f(3, k) \geq 4k - 2$.

The proof of the theorem is complete. □

Remark 2. The method used in the proof of the above special case of Conjecture 1 can be applied to the general case provided the following fact is true: all fixed vertices of $H_{t,k}$ are not adjacent to each other in P, where the graph $H_{t,k}$ is constructed from a path P of length $\max_{P(t,d)=k} d$ by adding t extra edges such that $H_{t,k}$ is of diameter k. The requirement that the fixed vertices are non-adjacent in P is necessary since otherwise we cannot insure the edge-connectivity of $G_{t,k}$ which is obtained from $H_{t,k}$.

References

[1] F.R.K. Chung, M.R. Garey, Diameter bounds for altered graphs, J. Graph Theory 8 (4) (1984) 511–534.
[2] Z.-G. Deng, J.-M. Xu, On diameters of altered graphs, J. Math. Study 37 (1) (2004) 35–41.
[3] A.A. Najim, J.-M. Xu, Edge addition and edge deletion of graphs, J. Univ. Sci. Technol. China 36 (3) (2006) 254–257.
[4] A.A. Najim, J.-M. Xu, On addition and deletion of edges of graphs, J. Univ. Sci. Technol. China 36 (9) (2006) 951–955.
[5] C. Peyrat, Diameter vulnerability of graphs, Discrete Appl. Math. 9 (3) (1984) 245–250.
[6] A.A. Schoone, H.L. Bodlaeder, J. van Leeuwen, Diameter increase caused by edge deletion, J. Graph Theory 11 (3) (1987) 409–427.
[7] Y.-Z. Wu, J.-M. Xu, On diameters of altered graphs, J. Math. Res. Exposition 26 (3) (2006) 502–508.
[8] J.-M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.
[9] J.-M. Xu, Theory and Application of Graphs, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.