Discrete Applied Mathematics 157 (2009) 1113–1127

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Average distances and distance domination numbers[☆]

Fang Tian[a], Jun-Ming Xu[b,*]

[a] Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai, 200433, China
[b] Department of Mathematics, University of Science and Technology of China, Hefei, 230026, China

ARTICLE INFO

Article history:
Received 28 May 2005
Received in revised form 27 February 2008
Accepted 9 March 2008
Available online 25 April 2008

Keywords:
Distance
Average distance
Diameter
Domination number
Distance domination number

ABSTRACT

Let k be a positive integer and G be a simple connected graph with order n. The average distance $\mu(G)$ of G is defined to be the average value of distances over all pairs of vertices of G. A subset D of vertices in G is said to be a k-dominating set of G if every vertex of $V(G)-D$ is within distance k from some vertex of D. The minimum cardinality among all k-dominating sets of G is called the k-domination number $\gamma_k(G)$ of G. In this paper tight upper bounds are established for $\mu(G)$, as functions of n, k and $\gamma_k(G)$, which generalizes the earlier results of Dankelmann [P. Dankelmann, Average distance and domination number, Discrete Appl. Math. 80 (1997) 21–35] for $k=1$.

1. Introduction

For terminology and notation on graph theory not given here, the reader is referred to [18]. Let $G=(V,E)$ be a finite simple connected graph with vertex set $V=V(G)$ and edge set $E=E(G)$. The *distance* $d_G(x,y)$ between two vertices x and y is the length of a shortest xy-path in G. For $S\subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S and for $v\in V(G)$, $d_G(v,S)=\min\{d_G(v,u): u\in V(S)\}$. The *eccentricity* $e_G(v)$ of v is $\max\{d_G(v,x): x\in V(G)\}$. The *radius* $\mathrm{rad}(G)$ and the *diameter* $\mathrm{diam}(G)$ of G are the smallest and the largest eccentricities of the vertices in G, respectively. A vertex with $e_G(v)=\mathrm{diam}(G)$ is called a *diametral vertex*. A vertex v is a *central vertex* if $e_G(v)=\mathrm{rad}(G)$ and the *center* of G is the set of all central vertices. The *degree* of a vertex $x\in V(G)$, denoted by $\deg_G(x)$, is the number of edges incident to the vertex x. A vertex of degree one is called an *end-vertex*. Let P_n denote a path of order n and P_{xy} a path with end-vertices x and y. If the length of a path P_{xy} is equal to $\mathrm{diam}(G)$, then we call P_{xy} a *diametral path* in G.

The average (or mean) distance of G is defined to be the average over all pairs of vertices of G, i.e.,

$$\mu(G)=\frac{1}{n(n-1)}\sum_{x,y\in V} d_G(x,y).$$

Like diameter, Wiener index [13,17] or other parameters, apart from their own graph-theoretic interests, the average distance has numerous applications in analyzing problems in communication networks, geometry and physical chemistry. It is the reason why this concept has received considerable attention in the literature. There are several excellent surveys of earlier results on average distance of graphs, one of which is due to Plesnik [15]. Thus, many efforts have been made by several authors to establish the relationships between average distance and other graph parameters (see, for example, [1,2,

☆ The work was supported by NNSF of China (No. 10671191).

* Corresponding author.
E-mail address: xujm@ustc.edu.cn (J.-M. Xu).

0166-218X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2008.03.024

6–8,15,16]). For convenience, let

$$\sigma(x) = \sigma(x, G) = \sum_{y\in V} d_G(x, y), \qquad \sigma(G) = \sum_{x\in V} \sigma(x) = \sum_{(x,y)\in V\times V} d_G(x, y),$$

be the *transmission* of a vertex $x \in V$, and the *transmission* of the graph G, respectively. In order to avoid large fractions, we will often deal with $\sigma(G)$ rather than $\mu(G)$. Apart from average distance, $\sigma(G)$ also occurs in the computation of other graph-theoretical parameters, such as the forwarding index of a routing [5,12], and physical chemistry [9].

A subset I of vertices in G is said to be *k-independent* if every vertex in I is at distance at least $k + 1$ from every other vertex of I in G. The *k-independence number* of G, denoted by $\alpha_k(G)$, is defined to be the maximum cardinality among all k-independent sets of G. If $k = 1$, $\alpha_1(G)$ is $\alpha(G)$, the *independence number* of G. Dankelmann, Oellermann and Swart [7] gave the bounds on the average distance with order n and independence number $\alpha(G)$. Firby and Haviland [8] established sharp lower bounds for the average distance of G, in terms of the k-independence number $\alpha_k(G)$, and described the associated extremal graphs, thereby extending the aforementioned work of Dankelmann et al. for $k = 1$.

A subset D of vertices in G is said to be a *k-dominating set* of G if every vertex of $V(G) - D$ is within distance k from some vertex of D. The minimum cardinality among all k-dominating sets of G is called the *k-domination number* of G and is denoted by $\gamma_k(G)$. For the special case of $k = 1$, $\gamma_1(G)$ is the classic *domination number* of G. The concept of k-dominating set was introduced by Chang and Nemhauser [3,4] and finds applications in many situations and structures which give rise to graphs, see the books by Haynes, Hedetniemi and Slater [10,11].

Dankelmann [6] gave the sharp upper bounds on the average distance of a graph of given order n and domination number $\gamma(G)$, and determined the extremal graphs. In this paper, by generalizing Dankelmann's technique, we establish the sharp upper bounds on the average distance of G, in terms of k-domination number $\gamma_k(G)$, and describe the extremal graphs, extending the results of Dankelmann for $k = 1$ in [6].

The proofs of our main results are in Section 3 and some lemmas are given in Section 2.

2. Lemmas

Lemma 2.1. *Let G be a nontrivial connected graph, and k be a positive integer. Then $\gamma_k(G) = \min \gamma_k(T)$, where the minimum is taken over all spanning trees T of G.*

Proof. Let G be a nontrivial connected graph and T be a spanning tree of G. Then any k-dominating set of T is also a k-dominating set of G. Therefore, $\gamma_k(G) \le \gamma_k(T)$. Thus we have that $\gamma_k(G) \le \min \gamma_k(T)$, where the minimum is taken over all spanning trees T of G.

Now we show the reverse inequality. If G is a tree, then the theorem holds trivially. So we may assume that G is a connected graph containing cycles. Let D be a minimum k-dominating set of G and C be a cycle in G. If we can prove that D is also a k-dominating set of $G - e$ for some cycle edge $e \in E(C)$, then $\gamma_k(G - e) \le |D| = \gamma_k(G)$. By iterating the above operation finitely, we get $\gamma_k(T) \le \gamma_k(G)$ for some spanning tree T of G. Thus, we have that $\min \gamma_k(T) \le \gamma_k(G)$, where the minimum is taken over all spanning trees T of G.

If $V(C) \subseteq V(D)$, then obviously the vertices in $V(G) - D$ are also all within distance k to $G[D] - e$ for any edge $e \in E(C)$.

If $V(C) \not\subseteq V(D)$, then we select two adjacent vertices x and y in $V(C)$ such that $d_G(x, D) + d_G(y, D) = \max\{d_G(u, D) + d_G(v, D) : uv \in E(C)\}$. Now we will show that D is also a k-dominating set of $G - \{xy\}$.

First for any two adjacent vertices u and v in G, we have $|d_G(u, D) - d_G(v, D)| \le 1$. Then if w is a vertex in $V(C)$ such that $d_G(w, D) = \max\{d_G(v, D) : v \in V(C)\}$, we have that $w = x$ or $w = y$. Without loss of generality, suppose that $d_G(x, D) = \max\{d_G(v, D) : v \in V(C)\}$.

Let z be another neighbor of x different from y in $V(C)$. So we immediately have that $d_G(z, D) \le d_G(y, D)$. Thus, we get the distance between a vertex in $V(G) - D$ and D is not influenced by deleting the edge $\{xy\}$. That is to say, $d_{G-xy}(v, D) = d_G(v, D)$ for all vertices v in $V(G)$. Hence, D is also a k-dominating set of $G - e$ for some cycle edge e. ■

From Lemma 2.1, we get that every connected graph G contains a spanning tree T with the same k-domination number. That is to say, every extremal graph G with given order, k-domination number and maximum average distance is a tree. So we have to consider only trees below.

Let $S(k)$ denote a *k-generalized star* which is a tree containing one vertex whose eccentricity is at most k.

Lemma 2.2. *Let H be a graph. Then $\gamma_k(H - e) > \gamma_k(H)$ for each edge $e \in E(H)$ if and only if H is the union of several vertex disjoint k-generalized stars $S(k)$.*

Proof. Let H be a graph such that $\gamma_k(H - e) > \gamma_k(H)$ for each edge $e \in E(H)$, and D be a minimum k-dominating set of H.

If $\gamma_k(H) = 1$, by Lemma 2.1 and the property $\gamma_k(H - e) > \gamma_k(H)$ for each edge $e \in E(H)$, then H must be a tree and we can easily see that H must be a k-generalized star $S(k)$. If $\gamma_k(H) \ge 2$, then for any two vertices x and y in D, we have $d_H(x, y) \ge 2k + 1$. Otherwise, if $d_H(x, y) \le 2k$, then there must exist an edge e on the shortest path between x and y in H such that $\gamma_k(H - e) = \gamma_k(H)$.

We partition the graph H into balls of radius k, denoted $H_1, H_2, \ldots, H_{\gamma_k}$, whose centers are the vertices in D.

Then the balls are all disjoint. Furthermore, there are no edges joining any two balls. Otherwise, if there exist an edge e joining two balls, say H_i and H_j, then $\gamma_k(H-e)=\gamma_k(H)$, a contradiction to the hypothesis on H. And since H_i is an induced subgraph of H with $\gamma_k(H_i)=1$, by the aforementioned, H_i is a k-generalized star $S(k)$.

The converse is easily verified. ■

Corollary 2.3. *If G is a tree with $\gamma_k(G)\geq 2$, then there exists an edge e in a diametral path in G such that $\gamma_k(G-e)=\gamma_k(G)$.*

Proof. Let D be a minimum k-dominating set of G. We partition the graph G into balls of radius k, denoted $G_1, G_2, \ldots, G_{\gamma_k}$, whose centers are the vertices in D. Since G is connected, there must exist an edge e joining two such balls.

Then e must be in a diametral path in G, and its deletion does not change $\gamma_k(G)$. ■

Lemma 2.4 (*Meir and Moon [14]*). *$\gamma_k(G)\leq \left\lfloor \frac{n}{k+1}\right\rfloor$ for any connected graph G of order n with $n\geq k+1$.*

Definition 2.5. Suppose that $\left\lceil \frac{n}{2k+1}\right\rceil < \gamma_k \leq \left\lfloor \frac{n}{k+1}\right\rfloor$, and let s and t be, respectively, the quotient and the reminder of the division of $(2k+1)\gamma_k-n$ by k, namely $(2k+1)\gamma_k-n=sk+t$, where $s\geq 0$ and $0\leq t\leq k-1$. In particular, for $k=1$ we get $t=0$, and consequently $s=3\gamma-n$. Then we define the following numbers:

$$\begin{aligned}
A &= n-s(k+1)-t,\\
B &= s(k+1),\\
C &= 3n-s(k+1),\\
D &= 2n-s(k+1)-2t.
\end{aligned}$$

As functions of γ_k, we have

$$\begin{aligned}
A &= \frac{(2k+1)n-(k+1)(2k+1)\gamma_k+t}{k},\\
B &= \frac{(2k+1)(k+1)\gamma_k-(k+1)n-(k+1)t}{k},\\
C &= \frac{(4k+1)n-(k+1)(2k+1)\gamma_k+(k+1)t}{k},\\
D &= \frac{(3k+1)n-(2k+1)(k+1)\gamma_k-(k-1)t}{k}.
\end{aligned}$$

Note that, for $k=1$, A, B, C and D take the following values, which appear in the results by Dankelmann [6]:

$$\begin{aligned}
A &= 3n-6\gamma,\\
B &= 6\gamma-2n=2(3\gamma-n),\\
C &= 5n-6\gamma,\\
D &= 4n-6\gamma=2(2n-3\gamma).
\end{aligned}$$

Definition 2.6. For given positive integers n and γ_k, a class of graphs $\mathscr{H}_{n,\gamma_k}$ is defined as follows.

(i) If $\gamma_k \leq \frac{n}{2k+1}$, $\mathscr{H}_{n,\gamma_k}$ consists of a single path $P_{(2k+1)\gamma_k-1}=(v_1, v_2, \ldots, v_{(2k+1)\gamma_k-1})$ and independent vertices $w_1, w_2, \ldots, w_{n+1-(2k+1)\gamma_k}$ that are joined with $v_{(2k+1)\gamma_k-1}$ (see Fig. 1).

(ii) If $\gamma_k=\left\lceil \frac{n}{2k+1}\right\rceil$, $\mathscr{H}_{n,\gamma_k}$ is a single path $P_n=(v_1, v_2, \ldots, v_n)$.

(iii) If $\left\lceil \frac{n}{2k+1}\right\rceil < \gamma_k \leq \left\lfloor \frac{n}{k+1}\right\rfloor$, $\mathscr{H}_{n,\gamma_k}$ is obtained from a single path $P_{2n-(2k+1)\gamma_k+k}=(v_1, v_2, \ldots, v_{2n-(2k+1)\gamma_k+k})$ by attaching exactly one P_k to the vertex v_i for $A+k+2\leq i\leq 2n-(2k+1)\gamma_k+k$; and attaching exactly one P_t to the vertex v_{A+t+1} (see Fig. 2).

The reason for the different shapes of the extremal graphs for $\gamma_k \leq \frac{n}{2k+1}$ and $\gamma_k > \left\lceil \frac{n}{2k+1}\right\rceil$ is the fact that the path P_n is the unique graph of order n with the maximum transmission of a vertex, which has the k-domination number $\gamma_k=\left\lceil \frac{n}{2k+1}\right\rceil$.

Lemma 2.7. *Let G be a tree with order n and k-domination number $\gamma_k \leq \left\lceil \frac{n}{2k+1}\right\rceil$. Then, for each vertex $v\in V(G)$,*

$$\sigma(v,G)\leq \begin{cases} [(2k+1)\gamma_k-1]\left(n-\dfrac{2k+1}{2}\gamma_k\right), & \gamma_k\leq \dfrac{n}{2k+1};\\ \dfrac{n(n-1)}{2}, & \gamma_k=\left\lceil \dfrac{n}{2k+1}\right\rceil. \end{cases}$$

The equality holds if and only if $G=\mathscr{H}_{n,\gamma_k}$ and $v=v_1$.

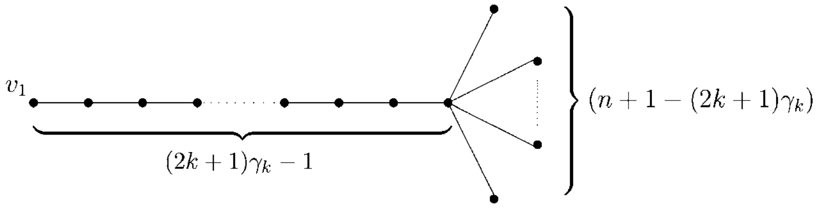

Fig. 1. The extremal graph $\mathscr{H}_{n,\gamma_k}$ with $1 \le \gamma_k \le \frac{n}{2k+1}$.

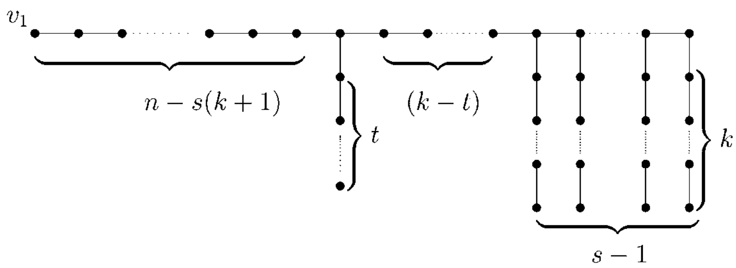

Fig. 2. The extremal graph $\mathscr{H}_{n,\gamma_k}$ with $\left\lceil \frac{n}{2k+1} \right\rceil < \gamma_k \le \left\lfloor \frac{n}{k+1} \right\rfloor$.

Proof. It is well known that in a tree, each vertex having maximum transmission is an end-vertex, i.e., a vertex with degree one (see [19] by Zelinka). Thus, we only prove this result for a diametral vertex. Let P be a diametral path, and let D be a minimum k-dominating set of G. Since every vertex of D can k-dominate at most $(2k+1)$ vertices of P, we have $\mathrm{diam}(G) \le (2k+1)\gamma_k - 1$, and thus we have

$$\begin{aligned}\sigma(v, G) &\le 1 + 2 + \cdots + (2k+1)\gamma_k - 2 + [(2k+1)\gamma_k - 1]\,(n - (2k+1)\gamma_k + 1)\\ &= [(2k+1)\gamma_k - 1]\left(n - \frac{2k+1}{2}\gamma_k\right).\end{aligned}$$

The uniqueness of the extremal graph is obvious. ■

For $\gamma_k = \left\lceil \frac{n}{2k+1} \right\rceil$, the result follows immediately.

Lemma 2.8. *Let G be a tree of order n and k-domination number $\gamma_k > \frac{n}{2k+1}$, then*

$$\mathrm{diam}(G) \le 2n - (2k+1)\gamma_k + 2k - 1.$$

Proof. The proof proceeds by induction on n. For $n \le 3k+3$, by $\frac{n}{2k+1} < \gamma_k \le \lfloor \frac{n}{k+1} \rfloor$, the value of γ_k is small. Thus, it is easy to verify that the statement holds for all graphs with maximum diameter and k-domination number at least γ_k. For $\gamma_k = \left\lceil \frac{n}{2k+1} \right\rceil$, we see that the path P_n also satisfies this statement. So, we consider this statement as $3k+4 \le n \le (2k+1)(\gamma_k - 1)$.

Suppose that the statements hold for all trees of order less than n. Let G be a tree with maximum diameter among all trees of order n and k-domination number at least γ_k. Let x and y be two vertices in G such that $d_G(x, y) = \mathrm{diam}(G)$, and let P_{xy} be a diametral path, $P_{xx_k} = (x, x_1, \ldots, x_k)$ be a subpath of P_{xy}.

First we have $\deg(x_i) = 2$ for $i = 1, \ldots, k$. Otherwise, if $\deg(x_j) \ge 3$ for some $1 \le j \le k$, then x_j must be adjacent to a vertex x'_j not on P_{xy}. Let

$$G' = G - x_{j-1}x_j + x_{j-1}x'_j.$$

Thus $\mathrm{diam}(G') = \mathrm{diam}(G) + 1$ and $\gamma_k(G') \ge \gamma_k(G)$, a contradiction to the choice of G.

Let $P = (c_1, c_2, \ldots, c_k)$ be a subpath of $P_{xy} - \{x, x_1, \ldots, x_k\}$ such that c_1 is adjacent to the vertex x_k on P_{xy}.

Case 1. $\deg(c_i) = 2$ for $i = 1, 2, \ldots, k$. Then $G - \{x, x_1, \ldots, x_k, c_1, \ldots, c_k\}$ is connected, has k-domination number $\gamma_k(G) - 1$ and has diameter at least $\mathrm{diam}(G) - (2k+1)$. Since $\gamma_k(G) - 1 > \frac{n-(2k+1)}{2k+1}$, by the induction hypothesis, we have

$$\begin{aligned}\mathrm{diam}(G) &\le \mathrm{diam}\,(G - \{x, x_1, \ldots, x_k, c_1, \ldots, c_k\}) + (2k+1)\\ &\le 2\,(n - (2k+1)) - (2k+1)(\gamma_k - 1) + 2k - 1 + 2k + 1\\ &= 2n - (2k+1)\gamma_k + 2k - 1.\end{aligned}$$

Case 2. $\deg(c_i) \ge 3$ for some $i = 1, 2, \ldots, k$. Let c_i be the nearest vertex to x_k on P_{xy} such that $\deg(c_i) \ge 3$. Let d denote one vertex farthest from c_i not on P_{xy} and c'_i be the vertex adjacent to c_i on P_{c_id}, then we have $k - i < d_G(c_i, d) \le k$.

In fact, if $d_G(c_i, d) \le k - i$, then $d_G(x_k, d) \le k$. Let

$$G' = G - c_ic_{i+1} + c_{i+1}c'_i,$$

then $\mathrm{diam}(G') = \mathrm{diam}(G) + 1$ and $\gamma_k(G') = \gamma_k(G)$, a contradiction to the choice of G.

But if $d_G(c_i, d) > k$, let

$$G' = G - c_{i-1}c_i + c_{i-1}c_i',$$

then $\mathrm{diam}(G') = \mathrm{diam}(G) + 1$, $\gamma_k(G') = \gamma_k(G)$, a contradiction to the choice of G.

Thus $\gamma_k(G - \{x, x_1, \ldots, x_k, c_1, \ldots, c_{i-1}\}) = \gamma_k(G) - 1$, and $\gamma_k(G) - 1 > \frac{n-(k+i)}{2k+1}$. The diameter of $G - \{x, x_1, \ldots, x_k, c_1, \ldots, c_{i-1}\}$ is the path P_{dy}, thus, by $k - i < d_G(c_i, d) \le k$, $\mathrm{diam}(G) - \mathrm{diam}(G - \{x, x_1, \ldots, x_k, c_1, c_2, \ldots, c_{i-1}\}) \le (2i - 1)$. By the induction hypothesis, we have

$$\begin{aligned}\mathrm{diam}(G) &\le \mathrm{diam}\left(G - \{x, x_1, \ldots, x_k, c_1, c_2, \ldots, c_{i-1}\}\right) + (2i - 1)\\ &= 2\left(n - (k + i)\right) - (2k + 1)(\gamma_k - 1) + 2k - 1 + 2i - 1\\ &= 2n - (2k + 1)\gamma_k + 2k - 1. \quad \blacksquare\end{aligned}$$

Lemma 2.9. *Let G be a tree of order n and k-domination number $\gamma_k > \left\lceil \frac{n}{2k+1} \right\rceil$. For each vertex $v \in V(G)$,*

if $k \equiv 1 \pmod 2$, then

$$\sigma(v, G) \le \sum_{i=0}^{\frac{k-1}{2}} (2n - (2k + 1)\gamma_k + k + 2i)^2 - \sum_{i=1}^{\frac{k-1}{2}} (A + k + 2i)^2 - \frac{1}{2}(A + 2k)(A + 2k + 1) + At + \frac{3}{2}t^2 + \frac{t}{2}; \tag{2.1}$$

(here, let $\sum_{i=1}^{\frac{k-1}{2}} (A + k + 2i)^2 = 0$ if $k = 1$.)

if $k \equiv 0 \pmod 2$, then

$$\begin{aligned}\sigma(v, G) \le\ &\sum_{i=0}^{\frac{k-2}{2}} (2n - (2k + 1)\gamma_k + k + 2i)^2 - \sum_{i=1}^{\frac{k}{2}} (A + k + 2i)^2\\ &+ \frac{1}{2}\left(2n - (2k + 1)\gamma_k + 2k - 1\right)\left(2n - (2k + 1)\gamma_k + 2k\right) + At + \frac{3}{2}t^2 + \frac{t}{2}.\end{aligned} \tag{2.2}$$

The equality holds if and only if $G = \mathscr{H}_{n,\gamma_k}$ and $v = v_1$.

Proof. The proof proceeds by induction on n. For $n \le 3k + 3$, by $\lceil \frac{n}{2k+1} \rceil < \gamma_k \le \lfloor \frac{n}{k+1} \rfloor$, it can verify that $v = v_1 \in \mathscr{H}_{n,\gamma_k}$ is the vertex satisfying $\sigma(v, \mathscr{H}_{n,\gamma_k})$ getting the maximum. Thus, the statement holds by some calculations. Let $n \ge 3k + 4$. Suppose that the statement holds for all trees of order less than n. Now let G be a tree and $v \in V(G)$ such that $\sigma(v, G)$ is maximum among all trees of order n and k-domination number at least γ_k. Since each vertex having maximum transmission in a tree is an end-vertex, we can assume that v is a diametral vertex. Let u be an eccentric vertex of v with $d_G(u, v) = \mathrm{diam}(G)$ and P_{uv} be a diametral path in G. Then u must be an end-vertex of P_{uv} and the neighbor u_1 is unique. By $\gamma_k > \left\lceil \frac{n}{2k+1} \right\rceil$, we get $(2k + 1)\gamma_k \ge n + (2k + 1)$. Hence, $\mathrm{diam}(G) \le 2n - (2k + 1)\gamma_k + 2k - 1 \le n - 2$. Since u must be within distance k from some vertex of G, and we aim to get an upper bound for $\sigma(v, G)$, then we can assume the existence of a subpath $P_{uu_k} = (u, u_1, u_2, \ldots, u_k)$ of P_{uv}. By the choice of u, we have $\deg_G(u_i) = 2$ for all $i = 1, 2, \ldots, k$. Otherwise, if u_i is adjacent to another vertex u_i' not on P_{uv}, then $G' = G - u_{i-1}u_i + u_{i-1}u_i'$ satisfies $\gamma_k(G') \ge \gamma_k(G)$ and $\sigma(v, G') > \sigma(v, G)$.

Hence $G - \{u, u_1, \ldots, u_k\}$ is connected and has k-domination number at least $\gamma_k(G) - 1$. By the induction hypothesis and by Lemma 2.8, we have that, for $k \equiv 1 \pmod 2$,

$$\begin{aligned}\sigma(v, G) &\le \sigma(v, G - \{u, u_1, \ldots, u_k\}) + (2n - (2k + 1)\gamma_k + 2k - 1)\\ &\quad + (2n - (2k + 1)\gamma_k + 2k - 2) + \cdots + (2n - (2k + 1)\gamma_k + k - 1)\\ &\le \sigma(v, \mathscr{H}_{n-(k+1),\gamma_k-1}) + (2n - (2k + 1)\gamma_k + 2k - 1)\\ &\quad + (2n - (2k + 1)\gamma_k + 2k - 2) + \cdots + (2n - (2k + 1)\gamma_k + k - 1)\\ &= \sum_{i=0}^{\frac{k-1}{2}} (2n - (2k + 1)\gamma_k + k + 2i)^2 - \sum_{i=1}^{\frac{k-1}{2}} (A + k + 2i)^2 - \frac{1}{2}(A + 2k)(A + 2k + 1) + At + \frac{3}{2}t^2 + \frac{t}{2},\end{aligned} \tag{2.3}$$

and for $k \equiv 0 \pmod 2$,

$$\begin{aligned}\sigma(v, G) &\le \sigma(v, G - \{u, u_1, \ldots, u_k\}) + (2n - (2k + 1)\gamma_k + 2k - 1)\\ &\quad + (2n - (2k + 1)\gamma_k + 2k - 2) + \cdots + (2n - (2k + 1)\gamma_k + k - 1)\\ &\le \sigma(v, \mathscr{H}_{n-(k+1),\gamma_k-1}) + (2n - (2k + 1)\gamma_k + 2k - 1)\\ &\quad + (2n - (2k + 1)\gamma_k + 2k - 2) + \cdots + (2n - (2k + 1)\gamma_k + k - 1)\\ &= \sum_{i=0}^{\frac{k-2}{2}} (2n - (2k + 1)\gamma_k + k + 2i)^2 - \sum_{i=1}^{\frac{k}{2}} (A + k + 2i)^2\\ &\quad + \frac{1}{2}\left(2n - (2k + 1)\gamma_k + 2k - 1\right)\left(2n - (2k + 1)\gamma_k + 2k\right) + At + \frac{3}{2}t^2 + \frac{t}{2}.\end{aligned} \tag{2.4}$$

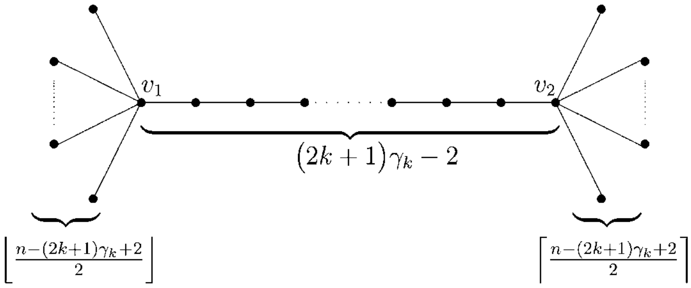

Fig. 3. The extremal graph $\mathscr{G}_{n,\gamma_k}$ with $1 \le \gamma_k \le \frac{n}{2k+1}$.

It remains to prove the uniqueness of the extremal graph. If the equality holds in (2.1) or (2.2), then it also holds in (2.3) or (2.4). By the induction hypothesis, we have that

$$G - \{u, u_1, \ldots, u_k\} = \mathscr{H}_{n-(k+1),\gamma_k-1},$$

and $v = v_1$. Notice that the vertices $u, u_1, \ldots, u_k$ are exactly at distance

$$2n - (2k+1)\gamma_k + 2k - 1,$$
$$2n - (2k+1)\gamma_k + 2k - 2,$$
$$\ldots,$$
$$2n - (2k+1)\gamma_k + k - 1$$

from v_1, which implies that $G = \mathscr{H}_{n,\gamma_k}$ and $v = v_1$. ■

From Lemmas 2.7 and 2.9, we get the following corollary.

Corollary 2.10 (*Dankelmann, Lemma 3 in [6]*). *Let G be a tree of order n and domination number γ. Then, for each vertex $v \in V(G)$,*

$$\sigma(v, G) \le \begin{cases} (3\gamma - 1)\left(n - \frac{3}{2}\gamma\right), & \text{if } \gamma \le \frac{n}{3}; \\ (2n - 3\gamma + 1)^2 - \frac{1}{2}(3n - 6\gamma + 3)(3n - 6\gamma + 2), & \text{if } \gamma > \frac{n}{3}. \end{cases}$$

The equality holds if and only if $G = \mathscr{H}_{n,\gamma}$ and $v = v_1$.

3. Main results

Now we prove the following sharp upper bounds on the average distance of a graph with given order n and k-domination number γ_k. The shape of the extremal graphs also differs depending on $\gamma_k \le \frac{n}{2k+1}$, $\gamma_k = \left\lceil \frac{n}{2k+1} \right\rceil$ or $\gamma_k > \left\lceil \frac{n}{2k+1} \right\rceil$. We will treat the three cases separately.

Definition 3.1. For positive integers n and γ_k, a class of graphs $\mathscr{G}_{n,\gamma_k}$ is defined as follows.

(i) If $\gamma_k \le \frac{n}{2k+1}$, then $\mathscr{G}_{n,\gamma_k}$ is obtained from a single path $P_{(2k+1)\gamma_k-2}$ with end-vertices v_1 and v_2, and two independent sets of vertices W_1 and W_2 of order $\left\lceil \frac{n-(2k+1)\gamma_k+2}{2} \right\rceil$ and $\left\lfloor \frac{n-(2k+1)\gamma_k+2}{2} \right\rfloor$, by joining each vertex of W_i to v_i, where $i = 1, 2$ (see Fig. 3).

(ii) If $\gamma_k = \left\lceil \frac{n}{2k+1} \right\rceil$, then $\mathscr{G}_{n,\gamma_k}$ is a single path $P_n = (v_1, v_2, \ldots, v_n)$.

Theorem 3.2. *Let G be a connected graph of order n and k-domination number $\gamma_k \le \left\lceil \frac{n}{2k+1} \right\rceil$. Then we have*

$$\mu(G) \le \begin{cases} \frac{n+1}{3} - \frac{(n-(2k+1)\gamma_k)(n-(2k+1)\gamma_k+2)(2n+(2k+1)\gamma_k-7)}{6n(n-1)}, \\ \quad \text{if } \gamma_k \le \frac{n}{2k+1} \text{ and } n-\gamma_k \text{ is even}; \\ \frac{n+1}{3} - \frac{(n-(2k+1)\gamma_k)(n-(2k+1)\gamma_k+2)(2n+(2k+1)\gamma_k-7) - 3((2k+1)\gamma_k-3)}{6n(n-1)}, \\ \quad \text{if } \gamma_k \le \frac{n}{2k+1} \text{ and } n-\gamma_k \text{ is odd}; \\ \frac{n+1}{3}, \\ \quad \text{if } \gamma_k = \left\lceil \frac{n}{2k+1} \right\rceil. \end{cases}$$

The equality holds if and only if $G = \mathscr{G}_{n,\gamma_k}$.

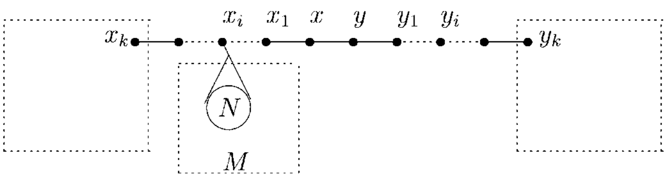

Fig. 4. The structure of G in Claim 2.

Proof. The proof proceeds by induction on n. It is easy to check that the statement holds for $n \leq 4k+2$. Assume that the statement holds for all values smaller than n.

We will prove the statement for a fixed $n\ (\geq 4k+3)$ by induction on $\gamma_k \geq 1$. Clearly, it holds for $\gamma_k = 1$, so we may assume $\gamma_k \geq 2$. If $\gamma_k = \left\lceil \frac{n}{2k+1} \right\rceil$ or $\gamma_k = \frac{n}{2k+1}$, then the result follows immediately from the fact that the graph $\mathscr{G}_{n,\left\lceil \frac{n}{2k+1} \right\rceil}$ or $\mathscr{G}_{n,\frac{n}{2k+1}}$ is isomorphic to the graph P_n. So let $\gamma_k < \frac{n}{2k+1}$.

Let G be a connected graph of order n and k-domination number γ_k that has maximum transmission. Then G is a tree by Lemma 2.1.

Since $\gamma_k \geq 2$, by Corollary 2.3, we can choose an edge xy in a diametral path P, whose deletion does not change $\gamma_k(G)$.

Let G_x and G_y denote the components of $G - xy$ that contains x and y, respectively. Since $n \geq 4k+3$, we also can choose the edge xy such that G_x and G_y contain at least $k+1$ vertices, respectively.

Claim 1. *There exists one vertex at distance k from the vertices x and y in G_x and G_y, respectively.*

Proof. Without loss of generality, we only prove the statement for G_x. Suppose that every vertex of G_x is at distance less than k to x. Then $\{x\}$ is a minimum k-dominating set for G_x. Take the farthest vertex x' of x in G_x on the path P, and let $P_{xx'}$ denote the path between x and x' on P in G_x. Since $|G_x| \geq k+1$, there exists a vertex x_1 on $P_{xx'}$ such that $\deg(x_1) \geq 3$. Suppose that x_2 is the neighbor of x_1 which is nearer to x' on $P_{xx'}$, and x_3 is a neighbor of x_1 not on $P_{xx'}$.

Let $G' = G - x_1x_2 + x_2x_3$. Thus, we have $G'_x = G_x - x_1x_2 + x_2x_3$ and $\gamma_k(G') = \gamma_k(G)$. Since $\sigma(G'_x) > \sigma(G_x)$, $\sigma(x, G'_x) > \sigma(x, G_x)$, and

$$\begin{aligned}\sigma(G) &= \left(\sum_{a,b\in V(G_x)} + \sum_{a,b\in V(G_y)} + 2 \sum_{\substack{a\in V(G_x)\\ b\in V(G_y)}} \right) d_G(a,b)\\ &= \sigma(G_x) + \sigma(G_y) + 2|V(G_x)||V(G_y)| + 2|V(G_y)|\,\sigma(x, G_x) + 2|V(G_x)|\,\sigma(y, G_y),\end{aligned}$$

we have $\sigma(G') > \sigma(G)$, a contradiction to the maximality of $\sigma(G)$. ■

By Claim 1, there exist paths of length k which belongs to P in G_x and G_y, denoted by $P_{xx_k} = (x, x_1, \ldots, x_{k-1}, x_k)$ and $P_{yy_k} = (y, y_1, \ldots, y_{k-1}, y_k)$, respectively.

Claim 2. $\deg(x) = 2$, $\deg(y) = 2$, $\deg(x_i) = 2$ *and* $\deg(y_i) = 2$ *for* $i = 1, \ldots, k-1$.

Proof. We first prove that $\deg(x) = 2$. Suppose that $\deg(x) \geq 3$ and let x' denote a neighbor of x not on P. Let $G' = G - xy + x'y$. By the same proof in Claim 1, we will get $\gamma_k(G') = \gamma_k(G)$ and $\sigma(G') > \sigma(G)$, a contradiction to the maximality of $\sigma(G)$. Similarly, we can prove $\deg(y) = 2$.

Thus, x and y are diametral vertices in G_x and G_y, respectively. By $\gamma_k(G - xy) = \gamma_k(G)$, we have x_k and y_k must be in a minimum k-dominating set of G.

Now, we prove $\deg(x_i) = 2$ for $i = 1, \ldots, k-1$. Let x_i be the nearest vertex to x on P_{xx_k} such that $\deg(x_i) \geq 3$ and y_j be the nearest vertex to y on P_{yy_k} such that $\deg(y_j) \geq 3$. Without loss of generality, we assume that $i \leq j$.

Let x'_i be a vertex farthest from x_i not on P_{xx_k}. Since x is a diametral vertex in G_x, we have $d_G(x_i, x'_i) \leq i$. Let M denote the vertices in all connected components of $G - x_i$ which contain no vertices on P_{xx_k}, and let N denote the set of vertices adjacent to x_i in M (see Fig. 4).

If $|V(G_x)| > |V(G_y)| + |M|$, then let $G' = G - x_iN + y_iN$. By $d_G(x_i, x'_i) \leq i$, $\gamma_k(G') = \gamma_k(G)$ and

$$\begin{aligned}\sigma(G') - \sigma(G) &= \sum_{\substack{a\in M\\ b\in V(G')-M}} d_{G'}(a,b) - \sum_{\substack{a\in M\\ b\in V(G)-M}} d_G(a,b)\\ &= 2|M|\left(|V(G_x)| - |M| - i\right)(2i+1) - 2|M|\left(|V(G_y)| - i\right)(2i+1)\\ &= 2|M|(2i+1)\left(|V(G_x)| - |V(G_y)| - |M|\right)\\ &> 0,\end{aligned}$$

a contradiction to the choice of G.

If $|V(G_x)| \le |V(G_y)| + |M|$, then let $G' = G - x_iN + x_{i+1}N$. By $d_G(x_i, x_i') \le i$, $\gamma_k(G') = \gamma_k(G)$ and

$$\begin{aligned}\sigma(G') - \sigma(G) &= \sum_{\substack{a\in M\\ b\in V(G')-M}} d_{G'}(a,b) - \sum_{\substack{a\in M\\ b\in V(G)-M}} d_G(a,b)\\ &= 2|M|\left((|V(G_y)| + i + 1) - (|V(G_x)| - |M| - i - 1)\right)\\ &= 2|M|\left(|V(G_y)| - |V(G_x)| + |M| + 2i + 2\right)\\ &> 0\end{aligned}$$

a contradiction to the choice of G.

Hence, no matter $|V(G_x)| > |V(G_y)| + |M|$ or $|V(G_x)| \le |V(G_y)| + |M|$, we get both the contradictions with $\sigma(G)$. Thus $\deg x_i = 2$ and $\deg y_i = 2$ for $i = 1, \ldots, k-1$. ■

By Claim 2, now suppose that G' is the graph obtained from G by identifying the $2k+2$ vertices $x_k, x_{k-1}, \ldots, x_1, x$ and $y_k, y_{k-1}, \ldots, y_1, y$ with a new vertex z and deleting loops. Then, G' has $n-(2k+1)$ vertices and $\gamma_k(G') = \gamma_k(G) - 1$ satisfying $\gamma_k(G') \le \frac{|V(G')|}{2k+1}$.

Let

$$\begin{aligned}X &= V(G_x) - \{x, x_1, \ldots, x_k\},\\ Y &= V(G_y) - \{y, y_1, \ldots, y_k\},\\ Z &= \{x_k, \ldots, x_1, x, y, y_1, \ldots, y_k\},\\ p &= |V(G_x)|,\\ q &= \gamma_k(G_x).\end{aligned}$$

By the induction hypothesis, we have,

$$\begin{aligned}\sigma(G) &= \left(\sum_{a,b\in X} + \sum_{a,b\in Y} + 2\sum_{a\in X, b\in Y}\right) d_G(a,b) + \sigma\left(G[Z]\right)\\ &\quad + 2\sum_{a\in X\cup Y}\left(\sum_{i=1}^{k} d_G(a,x_i) + d_G(a,x) + d_G(a,y) + \sum_{i=1}^{k} d_G(a,y_i)\right)\\ &= \sum_{a,b\in X} d_{G'}(a,b) + \sum_{a,b\in Y} d_{G'}(a,b) + 2\sum_{a\in X,b\in Y}\left(d_{G'}(a,b) + (2k+1)\right)\\ &\quad + \sigma\left(G[Z]\right) + 2\sum_{a\in X}\left((2k+1)d_G(a,x) + d_{G'}(a,z) + 2k+1\right)\\ &\quad + 2\sum_{a\in Y}\left((2k+1)d_G(a,y) + d_{G'}(a,z) + 2k+1\right)\\ &= \sum_{a,b\in X} d_{G'}(a,b) + \sum_{a,b\in Y} d_{G'}(a,b) + 2\sum_{a\in X,b\in Y}\left(d_{G'}(a,b) + (2k+1)\right)\\ &\quad + \sigma\left(G[Z]\right) + 2\sum_{a\in X\cup Y} d_{G'}(a,z) + 2(2k+1)\sum_{a\in X}\left(d_G(a,x)+1\right)\\ &\quad + 2(2k+1)\sum_{a\in Y}\left(d_G(a,y)+1\right)\\ &= \sigma(G') + 2(2k+1)|X||Y| + \frac{1}{3}(2k+1)(2k+2)(2k+3)\\ &\quad + 2(2k+1)\left(\sigma(x,G_x) + \sigma(y,G_y) - k(k+1) + |X| + |Y|\right)\\ &= \sigma(G') + 2(2k+1)|X||Y| + \frac{1}{3}(2k+1)(2k+2)(2k+3)\\ &\quad + 2(2k+1)\left(\sigma(x,G_x) + \sigma(y,G_y) - k(k+1) + n - (2k+2)\right)\\ &\le \sigma\left(\mathscr{G}_{n-(2k+1),\gamma_k-1}\right) + 2(2k+1)n - \frac{1}{3}(2k+6)(2k+1)(k+1)\\ &\quad + 2(2k+1)\left[(p-(k+1))(n-p-(k+1)) + \sigma(v_1,\mathscr{H}_{p,q}) + \sigma(v_1,\mathscr{H}_{n-p,\gamma_k-q})\right].\end{aligned} \tag{3.1}$$

Let

$$F(p,q) = (p-(k+1))(n-p-(k+1)) + \sigma(v_1,\mathscr{H}_{p,q}) + \sigma\left(v_1,\mathscr{H}_{n-p,\gamma_k-q}\right).$$

Case 1. $q \ge \left\lceil \frac{p}{2k+1} \right\rceil$ or $\gamma_k - q \ge \left\lceil \frac{n-p}{2k+1} \right\rceil$. Without loss of generality, we only prove $q \ge \left\lceil \frac{p}{2k+1} \right\rceil$, then $\gamma_k - q \le \gamma_k - \left\lceil \frac{p}{2k+1} \right\rceil \le \frac{n-p}{2k+1}$.

If $q = \lceil \frac{p}{2k+1} \rceil$, then

$$F(p, q) = (p - (k+1))\,(n - p - (k+1)) + \frac{p(p-1)}{2} + ((2k+1)(\gamma_k - q) - 1)\left(n - p - \frac{2k+1}{2}(\gamma_k - q)\right). \tag{3.2}$$

Since $F'_p(p, q) \ge \frac{1}{2} > 0$, then $F(p, q) \le F\left((2k+1)q, q\right)$. Since G is a graph with k-domination number γ_k and maximum transmission, then $\sigma(G) \ge \sigma(\mathscr{G}_{n,\gamma_k})$. By (3.1) and (3.2) and $\gamma_k < \frac{n}{2k+1}$, we get a contradiction as follows,

$$\begin{aligned} 0 &\le \sigma\left(\mathscr{G}_{n-(2k+1),\gamma_k-1}\right) + 2(2k+1)n - \frac{1}{3}(k+1)(2k+1)(2k+6) + 2(2k+1)F\left((2k+1)q, q\right) - \sigma\left(\mathscr{G}_{n,\gamma_k}\right) \\ &= -\frac{k+1}{2}\left(n - (2k+1)\gamma_k\right)^2 < 0. \end{aligned}$$

If $q > \left\lceil \frac{p}{2k+1} \right\rceil$, then $\sigma(v_1, \mathscr{H}_{p,q}) < \sigma\left(v_1, \mathscr{H}_{p,\lceil \frac{p}{2k+1}\rceil}\right)$. By $\gamma_k - q \le \gamma_k - \left\lceil \frac{p}{2k+1} \right\rceil \le \frac{n-p}{2k+1}$, we have $\sigma(v_1, \mathscr{H}_{n-p,\gamma_k-q}) < \sigma\left(v_1, \mathscr{H}_{n-p,\gamma_k-\lceil \frac{p}{2k+1}\rceil}\right)$. Thus, $F(p, q) < F\left(p, \lceil \frac{p}{2k+1} \rceil\right)$ and we can get the same contradiction as above.

Case 2. $q \le \frac{p}{2k+1}$ and $\gamma_k - q \le \frac{n-p}{2k+1}$.

We have,

$$\begin{aligned} F(p, q) &= -\left[p - \left(\frac{n - (2k+1)\gamma_k}{2} + (2k+1)q\right)\right]^2 + \frac{n^2}{4} + \frac{2k+1}{2}\gamma_k n \\ &\quad - \frac{(2k+1)^2}{4}\gamma_k^2 - (k+2)n + \frac{2k+1}{2}\gamma_k + (k+1)^2, \end{aligned}$$

that is,

$$\begin{aligned} F(p, q) &\le F\left((2k+1)q + \left\lfloor \frac{n - (2k+1)\gamma_k}{2} \right\rfloor, q\right) \\ &= \begin{cases} \dfrac{n^2}{4} + \dfrac{2k+1}{2}\gamma_k n - \dfrac{(2k+1)^2}{4}\gamma_k^2 - (k+2)n + \dfrac{2k+1}{2}\gamma_k + (k+1)^2; \\ \quad \text{if } n - \gamma_k \text{ is even} \\ \dfrac{n^2}{4} + \dfrac{2k+1}{2}\gamma_k n - \dfrac{(2k+1)^2}{4}\gamma_k^2 - (k+2)n + \dfrac{2k+1}{2}\gamma_k + (k+1)^2 + \dfrac{1}{4}; \\ \quad \text{if } n - \gamma_k \text{ is odd.} \end{cases} \end{aligned} \tag{3.3}$$

Since G is a graph with k-domination number γ_k and maximum transmission, then $\sigma(G) \ge \sigma(\mathscr{G}_{n,\gamma_k})$. We can calculate $\sigma\left(\mathscr{G}_{n-(2k+1),\gamma_k-1}\right)$ and $\sigma(\mathscr{G}_{n,\gamma_k})$ by the shape of $\mathscr{G}_{n,\gamma_k}$ defined in Definition 3.1. By (3.1) and (3.3), we obtain that,

$$\begin{aligned} 0 &\le \sigma\left(\mathscr{G}_{n-(2k+1),\gamma_k-1}\right) + 2(2k+1)n - \frac{1}{3}(k+1)(2k+1)(2k+6) \\ &\quad + F\left((2k+1)q + \left\lfloor \frac{n - (2k+1)\gamma_k}{2} \right\rfloor, q\right) - \sigma(G) \\ &\le \sigma\left(\mathscr{G}_{n-(2k+1),\gamma_k-1}\right) + 2(2k+1)n - \frac{1}{3}(k+1)(2k+1)(2k+6) \\ &\quad + F\left((2k+1)q + \left\lfloor \frac{n - (2k+1)\gamma_k}{2} \right\rfloor, q\right) - \sigma(\mathscr{G}_{n,\gamma_k}) \\ &= 0. \end{aligned} \tag{3.4}$$

This yields $\sigma(G) = \sigma(\mathscr{G}_{n,\gamma_k})$.

Now we only need to prove the uniqueness of the extremal graph $\mathscr{G}_{n,\gamma_k}$. Since equality $\sigma(G) = \sigma(\mathscr{G}_{n,\gamma_k})$ implies the equality in (3.2), we have $\sigma(G') = \sigma\left(\mathscr{G}_{n-(2k+1),\gamma_k-1}\right)$, $\sigma(x, G_x) = \sigma\left(v_1, \mathscr{H}_{p,q}\right)$ and $\sigma(y, G_y) = \sigma\left(v_1, \mathscr{H}_{n-p,\gamma_k-q}\right)$. By the induction hypothesis, $G' = \mathscr{G}_{n-(2k+1),\gamma_k-1}$. By the uniqueness of Lemma 2.9, we get $G_x = \mathscr{H}_{p,q}$ and $G_y = \mathscr{H}_{n-p,\gamma_k-q}$.

It is easy to see that $\mathscr{G}_{n,\gamma_k}$ is the only class of graphs satisfying all of these properties and the theorem holds. ■

Corollary 3.3 (*Dankelmann, Theorem 1 in [6]*). *Let G be a connected graph of order n and domination number $\gamma \le \frac{n}{3}$. Then we have*

$$\mu(G) \le \begin{cases} \dfrac{n+1}{3} - \dfrac{(n-3\gamma)(n-3\gamma+2)(2n+3\gamma-7)}{6n(n-1)}, & \text{if } n-\gamma \text{ is even;} \\ \dfrac{n+1}{3} - \dfrac{(n-3\gamma)(n-3\gamma+2)(2n+3\gamma-7) - 9(\gamma-1)}{6n(n-1)}, & \text{if } n-\gamma \text{ is odd.} \end{cases}$$

The equality holds if and only if $G = \mathscr{G}_{n,\gamma}$.

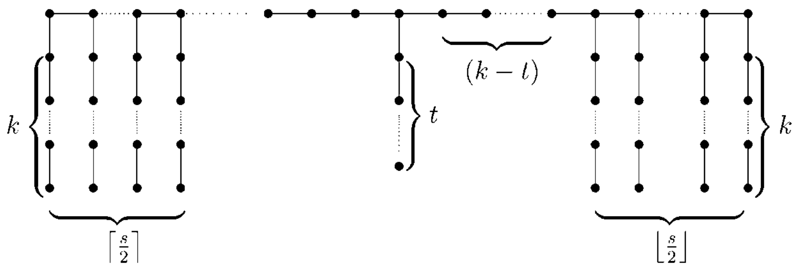

Fig. 5. The extremal graph $\mathscr{D}_{n,\gamma_k}$ with $\left\lceil \frac{n}{2k+1} \right\rceil < \gamma_k \leq \left\lfloor \frac{n}{k+1} \right\rfloor$.

Definition 3.4. For positive integers n and γ_k with $\left\lceil \frac{n}{2k+1} \right\rceil < \gamma_k \leq \left\lfloor \frac{n}{k+1} \right\rfloor$, let s and t be the quotient and the reminder of the division of $(2k+1)\gamma_k - n$ by k and let A be defined as in Definition 2.5. Let $\mathscr{D}_{n,\gamma_k}$ be the graph obtained from a single path $P_{2n-(2k+1)\gamma_k} = (v_1, v_2, \ldots, v_{2n-(2k+1)\gamma_k})$, by attaching exactly one P_k to the vertex v_i for $1 \leq i \leq \lceil \frac{s}{2} \rceil$; by attaching exactly one P_k to the vertex $v_{2n-(2k+1)\gamma_k+1-j}$ for $1 \leq j \leq \lfloor \frac{s}{2} \rfloor$; and by attaching exactly one path P_t to the vertex $v_{\lceil m \rceil + A + t - k}$ (see Fig. 5).

Theorem 3.5. *Let G be a graph of order n with k-domination number $\gamma_k > \left\lceil \frac{n}{2k+1} \right\rceil$. Let s and t be the quotient and the reminder of the division of $(2k+1)\gamma_k - n$ by k, namely $(2k+1)\gamma_k - n = sk + t$, where $s \geq 0$ and $0 \leq t \leq k-1$, and assume A, B, C, D as in Definition 2.5.*

If $\frac{\gamma_k - n - t}{k}$ is even, then

$$\mu(G) \leq \frac{n+1}{3} - \frac{B}{6n(n-1)} [((2k+1)\gamma_k - n - t - 2k)(C - 2(k+1)) + 3t(D-2)] - 2t(k-t)\left(\frac{A+t-k-1}{n(n-1)}\right). \tag{3.5}$$

If $\frac{\gamma_k - n - t}{k}$ is odd, then

$$\mu(G) \leq \frac{n+1}{3} - \frac{B-k-1}{6n(n-1)} [((2k+1)\gamma_k - n - t - 3k)(C - (k+1)) + 3t(D+2k) + 3(kD + (k-1)t - k(k+1))] - 2t(k-t)\left(\frac{A+t}{n(n-1)}\right). \tag{3.6}$$

The equality holds if and only if $G = \mathscr{D}_{n,\gamma_k}$.

Proof. The proof proceeds by induction on n. Since the bounds in (3.5) and (3.6) are strictly decreasing in γ_k, it suffices to prove the statements for all graphs with k-domination number greater than or equal to a given number γ_k. For $n \leq 3k+3$, by $\lceil \frac{n}{2k+1} \rceil < \gamma_k \leq \lfloor \frac{n}{k+1} \rfloor$, we can see, by some calculations, that the statement holds. So we can assume $n \geq 3k+4$.

Let G be a connected graph of order n and k-domination number $\gamma_k(G) \geq \gamma_k$ with maximum transmission. By Lemma 2.1, G is a tree.

Let $P_h \circ P_k$ denote the graph obtained by attaching a path P_k to each vertex of P_h. We will first show that G contains at least one induced subgraph isomorphic to $P_h \circ P_k$ for some positive integer h. Then the graph obtained by shrinking $P_h \circ P_k$ to $P_{h-1} \circ P_k$ will have k-domination number less than $\gamma_k(G)$, to which the induction hypothesis can be applied.

Let a and b be two vertices of G such that $d_G(a,b) = \mathrm{diam}(G)$, and let P_{ab} denote a diametral path in G. Let $P_{aa_k} = (a, a_1, \ldots, a_k)$ and $P_{bb_k} = (b, b_1, \ldots, b_k)$ be two subpaths of P_{ab}. Note that they do not overlap. Otherwise, we get $\gamma_k(G) = 1$, a contradiction to $\gamma_k > \lceil \frac{n}{2k+1} \rceil$ and $n \geq 3k+4$. By the choice of the vertices a and b, $\deg(a) = \deg(b) = 1$. Furthermore, $\deg(a_i) = 2$ and $\deg(b_i) = 2$, for $i = 1, 2, \ldots, k$. Otherwise, if a_i is adjacent to another vertex a_i' not on P_{ab}, then $G' = G - a_{i-1}a_i + a_{i-1}a_i'$ satisfies $\gamma_k(G') \geq \gamma_k(G)$ and $\sigma(G') > \sigma(G)$, a contradiction to the assumption of G. Let $\{c_1, \ldots, c_k\}$ and $\{d_1, \ldots, d_k\}$ be two subpaths of $P_{ab} - P_{aa_k} - P_{bb_k}$ such that c_1 is adjacent to a_k, and d_1 is adjacent to b_k. Note that they may overlap.

Case 1. $\deg(c_1) = 2$ or $\deg(d_1) = 2$. Without loss of generality, we assume that $\deg(c_1) = 2$.

If $\deg(c_i) = 2$ for $i = 1, 2, \ldots, k$, let $G' = G - \{a, a_1, \ldots, a_k, c_1, \ldots, c_k\}$. Then G' is connected and has k-domination number $\gamma_k(G) - 1 \geq \gamma_k - 1$. Let $V' = V(G')$. By Lemma 2.9 and the induction hypothesis,

$$\begin{aligned}\sigma(G) &= \left(\sum_{x,y \in V'} + 2\sum_{\substack{x \in V' \\ y \in V(G)-V'}} + \sum_{x,y \in V(G)-V'}\right) d_G(x,y) \\ &= \sigma(G') + 2\sum_{x \in V'}\left((2k+1)d_G(a,x) - \frac{(2k+1)2k}{2}\right) + \frac{(2k+1)(2k+2)2k}{3}\end{aligned}$$

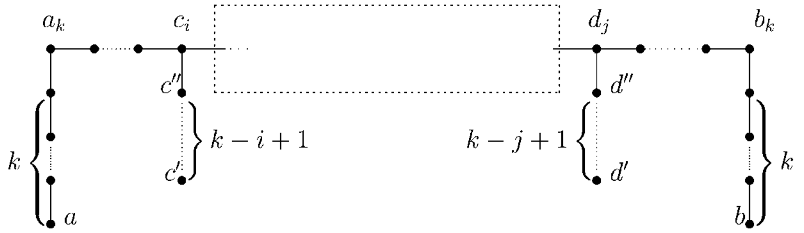

Fig. 6. The structure of G as $\deg(c_i) \geq 3$ for $i \in \{2, \ldots, k\}$ and $\deg(d_j) \geq 3$ for $j \in \{2, \ldots, k\}$.

$$= \sigma(G') + 2(2k+1)\sigma(a, G) - 2k(2k+1)n + \frac{(2k+1)(2k+2)2k}{3}$$
$$\leq \sigma(\mathscr{D}_{n-(2k+1),\gamma_k-1}) + 2(2k+1)\sigma\left(a, \mathscr{H}_{n,\gamma_k}\right) - 2k(2k+1)n + \frac{(2k+1)(2k+2)2k}{3}. \tag{3.7}$$

Since the assumption on G, we have $\sigma(G) \geq \sigma\left(\mathscr{D}_{n,\gamma_k}\right)$. By Lemma 2.9 and (3.7), we get,

$$0 \geq \sigma\left(\mathscr{D}_{n,\gamma_k}\right) - \sigma\left(\mathscr{D}_{n-(2k+1),\gamma_k-1}\right) - 2(2k+1)\sigma\left(v_1, \mathscr{H}_{n,\gamma_k}\right) + 2k(2k+1)n - \frac{(2k+1)(2k+2)2k}{3}$$
$$= \begin{cases} \dfrac{(1+k)(1+2k)}{2k}\left(n^2-t^2\right) + \left(\dfrac{(1+k)(1+2k)^3}{2k}\right)\gamma_k^2 - \left(\dfrac{(1+k)(1+2k)^2}{k}\right)\gamma_k n + 2(1+k)(1+2k)n \\ \quad -2(1+k)(1+2k)^2\gamma_k + 2k(1+k)(1+2k) \quad \text{if } \dfrac{\gamma_k-n-t}{k} \text{ is even;} \\ \dfrac{(1+k)(1+2k)}{2k}\left(n^2-t^2\right) + \left(\dfrac{(1+k)(1+2k)^3}{2k}\right)\gamma_k^2 - \left(\dfrac{(1+k)(1+2k)^2}{k}\right)\gamma_k n \\ \quad +(1+k)(1+2k)(2n+t) - 2(1+k)(1+2k)^2\gamma_k + \dfrac{3}{2}k(1+k)(1+2k) \quad \text{if } \dfrac{\gamma_k-n-t}{k} \text{ is odd.} \end{cases} \tag{3.8}$$

Let $F(n, \gamma_k)$ be the latter expression in (3.8). For constants n and $\gamma_k(G) \geq \lceil \frac{n}{2k+1} \rceil + 1 \geq \frac{n+(2k+1)}{2k+1}$, we get $\frac{dF(n,\gamma_k)}{d\gamma_k} > 0$. When $\gamma_k(G) = \frac{n+(2k+1)}{2k+1}$, then $(2k+1)\gamma_k - n = 2k+1$. Thus, we have $t = 0$ if $k = 1$, and $t = 1$ if $k \neq 1$. Hence, we have

$$0 \geq F(n, \gamma_k) \geq F\left(n, \frac{n+(2k+1)}{2k+1}\right)$$
$$= \begin{cases} -\dfrac{(1+3k+2k^2)(t^2-1)}{2k} & \text{if } \dfrac{\gamma_k-n-t}{k} \text{ is even;} \\ -\dfrac{(1+3k+2k^2)(k^2+t^2-2kt-1)}{2k} & \text{if } \dfrac{\gamma_k-n-t}{k} \text{ is odd.} \end{cases} \tag{3.9}$$

Inequality (3.9) is correct only in two cases. The first one is when $\frac{\gamma_k-n-t}{k}$ is even, $n = (2k+1)(\gamma_k-1)$, $k \geq 2$ and $t = 1$. The second one is when $\frac{\gamma_k-n-t}{k}$ is odd, $n = (2k+1)(\gamma_k-1)$, $k = 1$ and $t = 0$ or $k = 2$ and $t = 1$. Thus, equality holds in (3.7) only at the above two cases. Then $\sigma(a, G) = \sigma\left(a, \mathscr{H}_{(2k+1)(\gamma_k-1),\gamma_k}\right)$ and $\sigma(G) = \sigma(\mathscr{D}_{(2k+1)(\gamma_k-1),\gamma_k})$. By Lemma 2.9, we have $G = \mathscr{H}_{(2k+1)(\gamma_k-1),\gamma_k}$. Notice that $\mathscr{H}_{(2k+1)(\gamma_k-1),\gamma_k} = \mathscr{D}_{(2k+1)(\gamma_k-1),\gamma_k}$. Therefore, the result follows in this case.

Suppose now that $\deg(c_i) \geq 3$ for some $i \in \{2, \ldots, k\}$. Thus, $k \geq 2$. If $\deg(d_j) = 2$ for $j = 1, \ldots, k$, then the result follows with the same argument as above. So we assume that $\deg(d_j) \geq 3$ for some $j \in \{2, \ldots, k\}$ below. Let c_i be the nearest vertex to a on P_{ab} such that $\deg(c_i) \geq 3$, and d_j be the nearest vertex to b on P_{ab} such that $\deg(d_j) \geq 3$.

Let c' and d' denote the vertices, not on P_{ab}, farthest from c_i and d_j, respectively. With the same method employed in the proof of Lemma 2.7, we have $k-i < d_G(c_i, c') \leq k$ and $k-j < d_G(d_j, d') \leq k$. In fact, we can also prove that $d_G(c_i, c') = k-i+1$. If $d_G(c_i, c') \geq k-i+2$, let c'' be the neighbor of c_i on $P_{c_ic'}$ and $G' = G - c_ic_{i-1} + c''c_{i-1}$,then $\gamma_k(G') = \gamma_k(G)$ and $\sigma(G') > \sigma(G)$, a contradiction to the assumption of G. Similarly, let d'' be the neighbor of d_j on $P_{d_jd'}$, we can prove $d_G(d_j, d') = k-j+1$. Furthermore, let M (M') denote all vertices in the connected components of $G - c_i$ ($G - d_j$) which contains no vertices in P_{ab}. We can prove $G[M] = P_{c''c'}$ and $G[M'] = P_{d''d'}$. Suppose that $v \in V(M) - P_{c''c'}$ exists such that v is adjacent to some vertex in $\{c_i\} \cup V(P_{c'c''})$. If $G' = G - c_{i-1}c_i + c_{i-1}v + vc_i$, then we have $\gamma_k(G') = \gamma_k(G)$ and $\sigma(G') > \sigma(G)$, a contradiction to the assumption on G (see Fig. 6).

If $c_i = d_j$, then $\gamma_k(G) = 3$ and $\{a_k, b_k, c_i\}$ is a minimum k-dominating set for G. Since G has the maximum transmission $\sigma(G)$, we have $i = j = k$. Hence, $n = 4k+2$. By $(2k+1)\gamma_k - n = 2k+1 = sk+t$ and $k \geq 2$, we have $t = 1$. Then we have $G = \mathscr{D}_{4k+2,3}$. If $c_i \neq d_j$, we can calculate that $\sigma(G) < \sigma(\mathscr{D}_{n,\gamma_k})$ by the definition of $\sigma(G) = \sum_{(u,v)\in V\times V} d_G(u, v)$, see Figs. 5 and 6.

Case 2. Consider now the case $\deg(c_1) > 2$ and $\deg(d_1) > 2$.

We first deal with the cases when $t = 0$ and no assumption on $\frac{\gamma_k-n-t}{k}$, or $t \neq 0$ and $\frac{\gamma_k-n-t}{k}$ is odd. Let w_c be a neighbor of c_1 not on P_{ab}. Then w_c must be an end-vertex of P_k with no vertices on P_{ab}. In fact, suppose that c_1 is adjacent to an end-vertex of some $P_\ell = (w_c^1, w_c^2, \dots, w_c^\ell)$, where $\ell < k$. Let $G' = G - a_k c_1 + a_k w_c^\ell$, then $\gamma_k(G') \geq \gamma_k(G)$ and $\sigma(G') > \sigma(G)$, a contradiction to the choice of G. Denote the path P_k by $(w_c^1, w_c^2, \dots, w_c^k)$.

Hence G contains an induced subgraph $H_1 = G[\{a, a_1, \dots, a_k, c_1, w_c^1, \dots, w_c^k\}]$ with the following properties

$$\mathscr{P}_1 : \begin{cases} H_1 \text{ is isomorphic to } P_{h_1} \circ P_k \text{ for some } h_1 \geq 2; \\ \mathscr{N}_G(V(H_1)) = \{u\} \text{ for some vertex } u \in V(P_{h_1}) \\ \text{ with } \deg_{H_1}(u) = 2 \end{cases}$$

where $\mathscr{N}_G(V(H))$ denote the set of all vertices in H which are adjacent to some vertex in $G - H$.

Among all induced subgraphs H_1 of G with properties $\mathscr{P}_1$, choose one of maximum order. Then the vertex u has two neighbors in H_1, one end-vertex u_1 of $P_k = (u_1, u_2, \dots, u_k)$ and one vertex u' with degree at least 2 in H_1. Let Z denote the set of the remaining neighbors of u in $G - H_1$. We define a new graph

$$G' = G - \{u, u_1, \dots, u_k\} + u'Z,$$

i.e., we delete the vertices $\{u, u_1, \dots, u_k\}$ and join the neighbors of u in $V(G) - V(H_1)$ to u'.

With $X = V(H_1) - \{u, u_1, \dots, u_k\}$, and $Y = V(G) - V(H_1)$, we have

$$\begin{aligned}
\sigma(G) &= \left(\sum_{x,y \in X} + \sum_{x,y \in Y} + 2 \sum_{x \in X, y \in Y} \right) d_G(x,y) + 2 \sum_{x \in V(G) - \{u, u_1, \dots, u_k\}} \left(d_G(u,x) + \sum_{i=1}^{k} d_G(u_i, x) \right) \\
&\quad + \frac{1}{3} k(k+1)(k+2) \\
&= \left(\sum_{x,y \in X} + \sum_{x,y \in Y} + 2 \sum_{x \in X, y \in Y} \right) d_{G'}(x,y) + 2\left((k+1)h_1 - (k+1)\right)\left(n - (k+1)h_1\right) \\
&\quad + 2 \sum_{x \in V(G) - \{u, u_1, \dots, u_k\}} \left((k+1) d_G(u_k, x) - \frac{k(k+1)}{2} \right) + \frac{1}{3} k(k+1)(k+2) \\
&= \sigma(G') + 2(k+1)(h_1 - 1)\left(n - (k+1)h_1\right) + 2(k+1)\left(\sigma(u_k, H_1) + \sigma(u_k, G - X)\right) \\
&\quad - k(k+1)n - \frac{1}{3} k(k+1)(2k+1).
\end{aligned}$$

It is easy to check that $\gamma_k(G') = \gamma_k(G) - 1$ and $\gamma_k(G') > \left\lceil \frac{|V(G')|}{2k+1} \right\rceil$. By the induction hypothesis and

$$\gamma_k(H_1) = h_1, \qquad \gamma_k(G - X) = \gamma_k - h_1 + 1 \geq \frac{|V(G) - X|}{2k+1},$$

we have

$$\begin{aligned}
\sigma(G) &\leq \sigma\left(\mathscr{D}_{n-(k+1),\gamma_k - 1}\right) + 2(k+1)(h_1 - 1)\left(n - (k+1)h_1\right) + 2(k+1)\sigma\left(u_k, \mathscr{H}_{(k+1)h_1, h_1}\right) \\
&\quad + 2(k+1)\sigma\left(u_k, \mathscr{H}_{n-(k+1)h_1+(k+1), \gamma_k - h_1 + 1}\right) - k(k+1)n - \frac{k(k+1)(2k+1)}{3}.
\end{aligned} \tag{3.10}$$

Let $F(h_1)$ denote the latter expression in (3.10). By (2.1) and (2.2) in Lemma 2.9, we get $\sigma\left(u_k, \mathscr{H}_{(k+1)h_1, h_1}\right)$ and $\sigma\left(u_k, \mathscr{H}_{n-(k+1)h_1+(k+1), \gamma_k - h_1 + 1}\right)$. $\sigma\left(\mathscr{D}_{n-(k+1),\gamma_k-1}\right)$ is also obtained from (3.5) and (3.6) because of the induction hypothesis. By replacing them into $F(h_1)$, we get the derivative of $F(h_1)$ on h_1 as

$$\frac{\mathrm{d}(F(h_1))}{\mathrm{d}h_1} = 2(1+k)(1+2k)\left((1+k)\gamma_k - n\right) \leq 0.$$

That is, for constants n and γ_k, and $h_1 - 1 \geq \left\lceil \frac{(2k+1)\gamma_k - n - t - k}{2k} \right\rceil$, $F(h_1)$ is a decreasing function on h_1 and attains its maximum at $h_1 = \left\lceil \frac{(2k+1)\gamma_k - n - t}{2k} \right\rceil$. Thus, $h_1 = \frac{B}{2(k+1)}$ if $\frac{\gamma_k - n - t}{k}$ is even, and $h_1 = \frac{B}{2(k+1)} + \frac{1}{2}$ if $\frac{\gamma_k - n - t}{k}$ is odd. By (2.1), (2.2), (3.5), (3.6) and (3.10), we see that the right-hand side of (3.10) equals the value of $\sigma\left(\mathscr{D}_{n,\gamma_k}\right)$, that is,

$$\begin{aligned}
\sigma(G) &\leq F\left(\left\lceil \frac{(2k+1)\gamma_k - n - t}{2k} \right\rceil \right) \\
&= \sigma\left(\mathscr{D}_{n-(k+1),\gamma_k - 1}\right) - k(k+1)n - \frac{k(k+1)(2k+1)}{3}
\end{aligned}$$

$$+\begin{cases} 2(k+1)\left(\dfrac{B}{2(k+1)}-1\right)\left(n-\dfrac{B}{2}\right)+2(k+1)\sigma\left(u_k,\mathscr{H}_{\frac{B}{2},\frac{B}{2(k+1)}}\right) \\ \quad +2(k+1)\sigma\left(u_k,\mathscr{H}_{n-\frac{B}{2}+(k+1),\gamma_k-\frac{B}{2(k+1)}+1}\right) \quad \text{if } \dfrac{\gamma_k-n-t}{k} \text{ is even;} \\ 2(k+1)\left(\dfrac{B}{2(k+1)}-\dfrac{1}{2}\right)\left(n-\dfrac{B+(k+1)}{2}\right)+2(k+1)\sigma\left(u_k,\mathscr{H}_{\frac{B+(k+1)}{2},\frac{B}{2(k+1)}+\frac{1}{2}}\right) \\ \quad +2(k+1)\sigma\left(u_k,\mathscr{H}_{n-\frac{B+(k+1)}{2}+(k+1),\gamma_k-\frac{B}{2(k+1)}+\frac{1}{2}}\right) \quad \text{if } \dfrac{\gamma_k-n-t}{k} \text{ is odd.} \end{cases}$$

$$=\sigma\left(\mathscr{D}_{n,\gamma_k}\right).$$

Thus, as the cases when $t=0$ and no assumption on $\frac{\gamma_k-n-t}{k}$, or $t\neq 0$ and $\frac{\gamma_k-n-t}{k}$ is odd, the inequality of Theorem 3.5 is proved. If the equality holds in Theorem 3.5, we have $G'=\mathscr{D}_{n-(k+1),\gamma_k-1}$ and $H_1=\mathscr{H}_{(k+1)h_1,h_1}$ for $h_1=\left\lceil\frac{(2k+1)\gamma_k-n-t}{2k}\right\rceil$. By the induction hypothesis and the uniqueness of Lemma 2.9, $\mathscr{D}_{n,\gamma_k}$ is the only class of graphs satisfying all of the above properties.

Now we prove Theorem 3.5 when $\frac{\gamma_k-n-t}{k}$ is even and $t\neq 0$. By the same argument as above we can see that d_1 must be adjacent to an end-vertex w_d^1 of a path $P_k=(w_d^1,w_d^2,\ldots,w_d^k)$ which has no vertices on P_{ab}. Hence G contains another induced subgraph $H_2=G\left[\{b,b_1,\ldots,b_k,d_1,w_d^1,\ldots,w_d^k\}\right]$ with the following properties $\mathscr{P}_2$.

$$\mathscr{P}_2:\begin{cases} H_2 \text{ is isomorphic to } P_{h_2}\circ P_k \text{ for some } h_2\geq 2\,; \\ \mathscr{N}_G\left(V(H_2)\right)=\{v\} \text{ for some vertex } v\in V(P_{h_2}) \\ \text{with } \deg\limits_{H_2}(v)=2. \end{cases}$$

In addition to the graph H_1, among all subgraphs H_2 of G with the property $\mathscr{P}_2$, choose one of maximum order. Then the vertex v has two neighbors in H_2, one end-vertex v_1 of $P_k=(v_1,\ldots,v_k)$ and one vertex v' with degree at least 2 in H_2. Let $\mathcal{Z}$ denote the remaining neighbors of v in $G-H_2$. Let

$$G'=G-\{u,u_1,\ldots,u_k\}-\{v,v_1,\ldots,v_k\}+u'Z+v'\mathcal{Z},$$

that is, we delete the vertices $u,u_1,\ldots,u_k$ and $v,v_1,\ldots,v_k$, joining the neighbors of u and v in $V(G)-V(H_1)-V(H_2)$ to u' and v', respectively. Let

$$G''=G-H_1-H_2+\{u,u_1,\ldots,u_k\}+\{v,v_1,\ldots,v_k\}.$$

Let

$$\begin{aligned}\mathscr{X}&=V(H_1)-V(\{u,u_1,\ldots,u_k\}),\\ \mathscr{Y}&=V(H_2)-V(\{v,v_1,\ldots,v_k\}),\\ W&=V(G)-V(H_1)-V(H_2).\end{aligned}$$

By $\sigma(G)=\sum_{(x,y)\in V\times V}d_G(x,y)$, we obtain,

$$\begin{aligned}\sigma(G)={}&\left(\sum_{x,y\in\mathscr{X}}+\sum_{x,y\in\mathscr{Y}}+\sum_{x,y\in W}+2\sum_{x\in\mathscr{X},y\in\mathscr{Y}}+2\sum_{x\in\mathscr{X},y\in W}+2\sum_{x\in\mathscr{Y},y\in W}\right)d_G(x,y)\\ &+2\sum_{x\in\mathscr{X}\cup\mathscr{Y}\cup W}\left(d_G(u,x)+\sum_{i=1}^{k}d_G(u_i,x)+d_G(v,x)+\sum_{i=1}^{k}d_G(v_i,x)\right)\\ &+\frac{2}{3}(k+1)(k+2)k+2\sum_{\substack{x\in\{u,u_1,\ldots,u_k\},\\ y\in\{v,v_1,\ldots,v_k\}}}d_G(x,y)\\ ={}&\Bigg[\left(\sum_{x,y\in\mathscr{X}}+\sum_{x,y\in\mathscr{Y}}+\sum_{x,y\in W}\right)d_{G'}(x,y)+2\sum_{x\in\mathscr{X},y\in\mathscr{Y}}(d_{G'}(x,y)+2)\\ &+2\sum_{x\in\mathscr{X},y\in W}(d_{G'}(x,y)+1)+2\sum_{x\in\mathscr{Y},y\in W}(d_{G'}(x,y)+1)\Bigg]\\ &+\Bigg[2\sum_{x\in W}\left(d_G(u,x)+\sum_{i=1}^{k}d_G(u_i,x)+d_G(v,x)+\sum_{i=1}^{k}d_G(v_i,x)\right)\\ &+\frac{2(k+1)(k+2)k}{3}+2\sum_{\substack{x\in\{u,u_1,\ldots,u_k\}\\ y\in\{v,v_1,\ldots,v_k\}}}d_G(x,y)+\sigma(G[W])\Bigg]-\sigma(G[W])\\ &+2\sum_{x\in\mathscr{X}}((k+1)d_G(u_k,x)+(k+1)d_G(v_k,x)-k(k+1))\end{aligned}$$

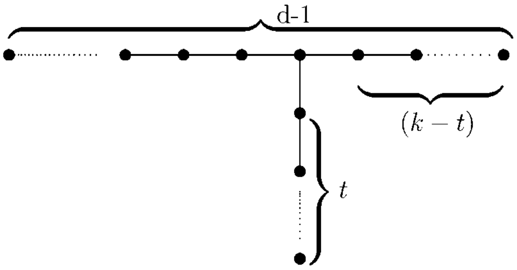

Fig. 7. The structure of $G[W]$ as the equality holds in (3.12).

$$\begin{aligned}
&+2\sum_{x\in\mathscr{Y}}((k+1)d_G(u_k,x)+(k+1)d_G(v_k,x)-k(k+1))\\
&=\sigma(G')+4(k+1)^2(h_1-1)(h_2-1)+2(k+1)\left(n-(h_1+h_2)(k+1)\right)(h_1+h_2-2)\\
&\quad+\sigma(G'')+2\sum_{x\in\mathscr{X}}((k+1)d_G(u_k,x)+(k+1)\left(d_G(u_k,x)+d\right)-k(k+1))\\
&\quad+2\sum_{x\in\mathscr{Y}}((k+1)\left(d_G(v_k,x)+d\right)+(k+1)d_G(v_k,x)-k(k+1))-\sigma(G[W])\\
&=\sigma(G')+\sigma(G'')+4(k+1)\sigma\left(u_k,\mathscr{H}_{h_1(k+1),h_1}\right)\\
&\quad+4(k+1)\sigma\left(v_k,\mathscr{H}_{h_2(k+1),h_2}\right)+4(k+1)^2(h_1-1)(h_2-1)\\
&\quad+2(k+1)^2(h_1+h_2-2)(d-k)-4k(k+1)^2\\
&\quad+2(k+1)\left(n-(h_1+h_2)(k+1)\right)(h_1+h_2-2)-\sigma(G[W])
\end{aligned}\tag{3.11}$$

where $d=d_G(u,v)$.

It is easy to check that $\gamma_k(G')=\gamma_k(G)-2$ and $\gamma_k(G')>\left\lceil\frac{|V(G')|}{2k+1}\right\rceil$; $\gamma_k(G'')=\gamma_k(G)-h_1-h_2+2$ and $\gamma_k(G'')>\left\lceil\frac{|V(G'')|}{2k+1}\right\rceil$. By the induction hypothesis and (3.11), we have

$$\begin{aligned}
\sigma(G)\le\;&\sigma\left(\mathscr{D}_{n-2(k+1),\gamma_k-2}\right)+\sigma\left(\mathscr{D}_{n-(h_1+h_2-2)(k+1),\gamma_k-(h_1+h_2-2)}\right)\\
&+4(k+1)\sigma\left(u_k,\mathscr{H}_{h_1(k+1),h_1}\right)+4(k+1)\sigma\left(v_k,\mathscr{H}_{h_2(k+1),h_2}\right)\\
&+4(k+1)^2(h_1-1)(h_2-1)+2(k+1)^2(h_1+h_2-2)(d-k)-4k(k+1)^2\\
&+2(k+1)\left(n-(h_1+h_2)(k+1)\right)(h_1+h_2-2)-\sigma(G[W]).
\end{aligned}\tag{3.12}$$

Let $F(h_1,h_2)$ denote the latter expression in (3.12). The equality holds in (3.12) if and only if $G'=\mathscr{D}_{n-2(k+1),\gamma_k-2}$ and $G''=\mathscr{D}_{n-(h_1+h_2-2)(k+1),\gamma_k-(h_1+h_2-2)}$. Thus, by the structure of $\mathscr{D}_{n-2(k+1),\gamma_k-2}$ or $\mathscr{D}_{n-(h_1+h_2-2)(k+1),\gamma_k-(h_1+h_2-2)}$, we get that the shape of $G[W]$ (see Fig. 7), and $d=n+1-(k+1)(h_1+h_2)-t$. By $\sigma(G[W])=\sum_{(x,y)\in V(G[W])\times V(G[W])}d_G(x,y)$ and Fig. 7, we have

$$\begin{aligned}
\sigma(G[W])=\;&\frac{(d-1)d(d-2)}{3}+\frac{t(t+1)(t-1)}{3}+t(k-t)(k-t+3)\\
&+t(d+t-k-1)(d+t-k)+t(t-1)(d-1).
\end{aligned}\tag{3.13}$$

By (3.12), (3.13) and $d=n+1-(k+1)(h_1+h_2)-t$, we get that

$$\frac{\partial(F(h_1,h_2))}{\partial h_1}<0\quad\text{and}\quad\frac{\partial(F(h_1,h_2))}{\partial h_2}<0.$$

That is, for constants n and γ_k,

$$h_1-1\ge\frac{(2k+1)\gamma_k-n-t-2k}{2k}\quad\text{and}\quad h_2-1\ge\frac{(2k+1)\gamma_k-n-t-2k}{2k},$$

$F(h_1,h_2)$ attains its maximum at $h_1=\frac{(2k+1)\gamma_k-n-t}{2k}$ and $h_2=\frac{(2k+1)\gamma_k-n-t}{2k}$. Thus,

$$h_1=h_2=\frac{B}{2(k+1)}\quad\text{and}\quad h_1+h_2=\frac{B}{k+1}.$$

Now we can get

$$\sigma\left(u_k,\mathscr{H}_{h_1(k+1),h_1}\right)=\sigma\left(v_k,\mathscr{H}_{h_2(k+1),h_2}\right)=\sigma\left(v,\mathscr{H}_{\frac{B}{2},\frac{B}{2(k+1)}}\right)$$

by (2.1) and (2.2) in Lemma 2.9; $\sigma\left(\mathscr{D}_{n-2(k+1),\gamma_k-2}\right)$ and

$$\sigma\left(\mathscr{D}_{n-(h_1+h_2-2)(k+1),\gamma_k-(h_1+h_2-2)}\right)=\sigma\left(\mathscr{D}_{n-(B-2(k+1)),\gamma_k-\frac{B}{k+1}+2}\right)$$

by the induction hypothesis; $\sigma(G[W])$ by (3.13), and

$$d = n + 1 - (k + 1)(h_1 + h_2) - t.$$

Hence, we can calculate that, for $\frac{\gamma_k - n - t}{k}$ even and $t \neq 0$,

$$\begin{aligned}
\sigma(G) &\leq F\left(\frac{(2k+1)\gamma_k - n - t}{2k}, \frac{(2k+1)\gamma_k - n - t}{2k}\right) \\
&= \sigma\left(\mathscr{D}_{n-2(k+1),\gamma_k-2}\right) + \sigma\left(\mathscr{D}_{n-(B-2(k+1)),\gamma_k-\frac{B}{k+1}+2}\right) - \sigma(G[W]) \\
&\quad + 2(k+1)\left(B - 2(k+1)\right)\left(n + 1 - B - t - k\right) + 2(k+1)\left(n - B\right)\left(\frac{B}{k+1} - 2\right) \\
&\quad + 4(k+1)\sigma\left(v_1, \mathscr{H}_{\frac{B}{2}, \frac{B}{2(k+1)}}\right) + 4(k+1)\sigma\left(v_1, \mathscr{H}_{\frac{B}{2}, \frac{B}{2(k+1)}}\right) + 4(k+1)^2\left(\frac{B}{2(k+1)} - 1\right)^2 \\
&= \sigma(\mathscr{D}_{n,\gamma_k}).
\end{aligned}$$

The uniqueness of the graph can be easily verified by the the induction hypothesis. ■

Corollary 3.6 (*Dankelmann, Theorem 2 in [6]*). *Let G be a graph of order n with domination number $\gamma \geq \frac{n}{3}$. Then*

$$\mu(G) \leq \begin{cases} \dfrac{n+1}{3} - \dfrac{(3\gamma - n)(3\gamma - n - 2)(5n - 6\gamma - 4)}{3n(n-1)}, & \text{if } n - \gamma \text{ is even}; \\ \dfrac{n+1}{3} - \dfrac{(3\gamma - n - 1)(3\gamma - n - 3)(5n - 6\gamma - 2) + 6(2n - 3\gamma - 1)}{3n(n-1)}, & \text{if } n - \gamma \text{ is odd}. \end{cases}$$

The equality holds if and only if $G = \mathscr{D}_{n,\gamma}$.

Acknowledgements

The authors would like to express their gratitude to the anonymous referees for their kind suggestions and useful comments on the original manuscript, which resulted in this revised version.

References

[1] R.A. Beezer, J.E. Riegsecker, B.A. Smith, Using minimum degree to bound average distance, Discrete Math. 226 (2001) 365–371.
[2] D. Bienstock, E. Gyori, Average distance in graphs with removed elements, J. Graph Theory 12 (1988) 375–390.
[3] G.J. Chang, k-domination and graph covering problems, Ph.D. Thesis, School of OR and IE, Cornell University, Ithaca, NY, 1982.
[4] G.J. Chang, G.L. Nemhauser, The k-domination and k-stability problems on sun-free chordal graphs, SIAM J. Algebr. Discrete Methods 5 (1984) 332–345.
[5] F.R.K. Chung, E.G. Coffman, M.I. Reiman, B. Simon, The forwarding index of communication networks, IEEE Trans. Inform. Theory 33 (2) (1987) 224–232.
[6] P. Dankelmann, Average distance and domination number, Discrete Appl. Math. 80 (1997) 21–35.
[7] P. Dankelmann, O.R. Oellermann, H.C. Swart, Average distance and independence number, Discrete Appl. Math. 51 (1994) 75–83.
[8] P. Firby, J. Haviland, Independence and average distance in graphs, Discrete Appl. Math. 75 (1997) 27–37.
[9] I. Gutman, O.E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, Berlin, 1986.
[10] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[11] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Domination in Graphs (Advanced Topics), Marcel Dekker, New York, 1998.
[12] M.C. Heydemann, J.C. Meyer, D. Sotteau, On forwarding indices of networks, Discrete Appl. Math. 23 (1989) 103–123.
[13] H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bull. Chem. Soc. Jpn. 4 (1971) 2332–2339.
[14] A. Meir, J.W. Moon, Relations between packing and covering numbers of a tree, Pacific J. Math. 61 (1) (1975) 225–233.
[15] J. Plesnik, On the sum of all distances in a graph or digraph, J. Graph Theory 8 (1984) 1–21.
[16] I. Tomescu, On the sum of all distances in chromatic blocks, J. Graph Theory 18 (1994) 83–102.
[17] H. Wiener, Structural determination of paraffin boiling points, J. Am. Chem. Soc. 69 (1949) 17–20.
[18] J.-M. Xu, Theory and Application of Graphs, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.
[19] B. Zelinka, Medians and peripherians of trees, Arch. Math.(Brno) 4 (1968) 87–95.