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On addition of edges of graphs

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Abstract: Given positive integers t and $d(\geqslant 2)$, let $P(t,d)$ denote the minimum diameter of a graph obtained by adding t extra edges to a path with diameter d. It was proved that $P(6,4)=1$, $P(6,d)=2$ for $d=5,6,7$, and

$$\left\lceil \frac{d}{7} \right\rceil \leqslant P(6,d) \begin{cases} \leqslant \left\lceil \frac{d}{7} \right\rceil + 2 & \text{if } h = 7; \\ \leqslant \left\lceil \frac{d}{7} \right\rceil + 1 & \text{otherwise}, \end{cases}$$

for $d=7(2k-1)+h$, where $k\geqslant 1$ and $1\leqslant h\leqslant 14$. Moreover, $P(7,d)=2$ for $d=5,6,7,8$, and

$$\left\lceil \frac{d}{8} \right\rceil \leqslant P(7,d) \begin{cases} = \left\lceil \frac{d}{8} \right\rceil & \text{if } h = 1; \\ \leqslant \left\lceil \frac{d}{8} \right\rceil + 2 & \text{if } h = 2,3,4,5,6,7,8; \\ \leqslant \left\lceil \frac{d}{8} \right\rceil + 1 & \text{otherwise}, \end{cases}$$

for $d=8(2k-1)+h$, where $k\geqslant 1$ and $1\leqslant h\leqslant 16$.

Key words: diameter; altered graphs; path; edge addition; minimum diameter

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关于图的边添加

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摘要:给定任意正整数 t 和 $d(\geqslant 2)$,记 $P(t,d)$ 为在直径 d 的路上加上 t 条边后所得图的最小直径.证明了:$P(6,4)=1$;当 $d=5,6,7$ 时有 $P(6,d)=2$;当 $d=7(2k-1)+h(k\geqslant 1,\ 1\leqslant h\leqslant 14)$ 时有

$$\left\lceil \frac{d}{7} \right\rceil \leqslant P(6,d) \leqslant \begin{cases} \left\lceil \frac{d}{7} \right\rceil + 2 & \text{若 } h = 7; \\ \left\lceil \frac{d}{7} \right\rceil + 1 & \text{其他}; \end{cases}$$

当 $d=5,6,7,8$ 时有 $P(7,d)=2$;当 $d=8(2k-1)+h\ (k\geqslant 1,1\leqslant h\leqslant 16)$时有

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$$\left\lceil\frac{d}{8}\right\rceil\leqslant P(7,d)\leqslant\begin{cases}\left\lceil\frac{d}{8}\right\rceil & 若\ h=1;\\ \left\lceil\frac{d}{8}\right\rceil+2 & 若\ h=2,3,4,5,6,7,8;\\ \left\lceil\frac{d}{8}\right\rceil+1 & 其他.\end{cases}$$

关键词:直径;变更图;路;加边; 最小直径

0 Introduction

Let $G=(V,E)$ be a simple undirected graph (without loops and multiple edges) with vertex-set V and edge-set E. Let $P(t,d)$ denote the minimum diameter of a graph obtained by adding t extra edges to a path with diameter d. We follow Ref. [1] for graph-theoretical terminology and notation not defined here. It is well-known that when the underlying topology of an interconnection network of a system is modelled by a graph G, the diameter of G is an important measure for communication efficiency and message delay of the system[2]. In a real-time system, the message delay must be limited within a given period since any message obtained beyond the bound may be worthless. If the message delay exceeds a given time-bound in a network, one often needs to add some links to the network to ensure that the reach of a message can be within a required time. This situation motivates Chung and Garey[3] to propose the following well-known "edge-addition problem" in graph theory: given positive integers t and d, what is the minimum diameter $P(t,d)$ of the graph obtained by adding t edges to a path with diameter d?

It is easy to get that $P(1,d)=\left\lfloor\frac{d+1}{2}\right\rfloor$ for $d\geqslant 2$. Schoone et al[4] showed $P(2,d)=\left\lceil\frac{d+1}{3}\right\rceil$ for $d\geqslant 3$, and $P(3,d)=\left\lceil\frac{d+2}{4}\right\rceil$ for $d\geqslant 5$. Deng and Xu[5], Najim and Xu[6] proved that $\left\lceil\frac{d}{5}\right\rceil\leqslant P(4,d)\leqslant\left\lceil\frac{d}{5}\right\rceil+1$ for $d\geqslant 4$; $\left\lceil\frac{d}{6}\right\rceil\leqslant P(5,d)\leqslant\left\lceil\frac{d}{6}\right\rceil+1$ for $d\geqslant 4$, and

$$\left\lceil\frac{d}{t+1}\right\rceil\leqslant P(t,d)\leqslant\left\lceil\frac{d-2}{t+1}\right\rceil+2.$$

In this note, we prove that $P(6,4)=1$, $P(6,d)=2$ for $d=5,6,7$, and

$$\left\lceil\frac{d}{7}\right\rceil\leqslant P(6,d)\leqslant\begin{cases}\left\lceil\frac{d}{7}\right\rceil+2 & \text{if } h=7;\\ \left\lceil\frac{d}{7}\right\rceil+1 & \text{otherwise},\end{cases}$$

for $d=7(2k-1)+h$, where $k\geqslant 1$ and $1\leqslant h\leqslant 14$. And $P(7,d)=2$ for $d=5,6,7,8$, and

$$\left\lceil\frac{d}{8}\right\rceil\leqslant P(7,d)\leqslant\begin{cases}\left\lceil\frac{d}{8}\right\rceil & \text{if } h=1;\\ \left\lceil\frac{d}{8}\right\rceil+2 & \text{if } h=2,3,4,5,6,7,8;\\ \left\lceil\frac{d}{8}\right\rceil+1 & \text{otherwise},\end{cases}$$

for $d=8(2k-1)+h$, where $k\geqslant 1$ and $1\leqslant h\leqslant 16$.

1 Several lemmas

Lemma 1.1 $P(t,d)\leqslant P(t,d')$ for $d\leqslant d'$.

This trivial lemma is obtained by a direct observation from the definitions.

Lemma 1.2[5] $P(t,(2k-1)(t+1)+1)=2k$ for any positive integer k.

Lemma 1.3[3] For given positive integers t and d,

$$P(t,d)\geqslant\left\lceil\frac{d}{t+1}\right\rceil.$$

Lemma 1.4 $P(6,7(2k-1)+h)\leqslant 2k+1$ for any positive integers k and h with $2\leqslant h\leqslant 6$.

Proof Let $d=7(2k-1)+h$ and $P=x_0x_1\cdots x_d$ be a simple path. The six vertices x_{2k-1}, x_{4k-1}, x_{6k}, x_{8k}, x_{10k} and x_{12k} partition P into seven segments:

$$P_1=P(x_0,x_{2k-1}),\ P_2=P(x_{2k-1},x_{4k-1}),$$
$$P_3=P(x_{4k-1},x_{6k}),\ P_4=P(x_{6k},x_{8k}),$$
$$P_5=P(x_{8k},x_{10k}),\ P_6=P(x_{10k},x_{12k}),$$

$$P_7 = P(x_{12k}, x_d).$$

Let G be an altered graph obtained from P plus 6 extra edges $e_1 = x_0x_{4k-1}$, $e_2 = x_{4k-1}x_{8k}$, $e_3 = x_{4k-1}x_{12k}$, $e_4 = x_{2k-1}x_{6k}$, $e_5 = x_{6k}x_{10k}$, $e_6 = x_{10k}x_d$ (see Fig. 1). Define 21 cycles as follows.

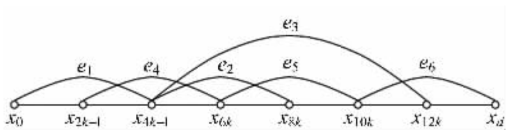

Fig. 1 Construction of Lemma 1. 4 for $k=1$ and $h=6$

$C^1 = P_1 \cup P_2 + e_1$,
$C^2 = P_1 \cup P_3 + e_1 + e_4$,
$C^3 = P_1 \cup P_4 + e_1 + e_2 + e_4$,
$C^4 = P_1 \cup P_5 + e_1 + e_2 + e_4 + e_5$,
$C^5 = P_1 \cup P_6 + e_1 + e_3 + e_4 + e_5$,
$C^6 = P_1 \cup P_7 + e_1 + e_3 + e_4 + e_5 + e_6$,
$C^7 = P_2 \cup P_3 + e_4$,
$C^8 = P_2 \cup P_4 + e_4 + e_2$,
$C^9 = P_2 \cup P_5 + e_4 + e_2 + e_5$,
$C^{10} = P_2 \cup P_6 + e_4 + e_5 + e_3$,
$C^{11} = P_2 \cup P_7 + e_4 + e_5 + e_6 + e_3$,
$C^{12} = P_3 \cup P_4 + e_2$,
$C^{13} = P_3 \cup P_5 + e_2 + e_5$,
$C^{14} = P_3 \cup P_6 + e_3 + e_5$,
$C^{15} = P_3 \cup P_7 + e_3 + e_5 + e_6$,
$C^{16} = P_4 \cup P_5 + e_5$,
$C^{17} = P_4 \cup P_6 + e_5 + e_2 + e_3$,
$C^{18} = P_4 \cup P_7 + e_5 + e_6 + e_2 + e_3$,
$C^{19} = P_5 \cup P_6 + e_2 + e_3$,
$C^{20} = P_5 \cup P_7 + e_2 + e_3 + e_6$,
$C^{21} = P_6 \cup P_7 + e_6$.

It is easy to see that,

$$\varepsilon(P_1) = 2k-1,\ \varepsilon(P_2) = 2k,\ \varepsilon(P_3) = 2k+1,$$
$$\varepsilon(P_4) = 2k,\ \varepsilon(P_5) = 2k,\ \varepsilon(P_6) = 2k,$$
$$\varepsilon(P_7) = 2k+h-7.$$

Thus, we have

$\varepsilon(C^1) = 4k$; $\varepsilon(C^{21}) \leqslant 4k$;
$\varepsilon(C^{16}) = 4k+1$; $\varepsilon(C^{20}) \leqslant 4k+2$;
$\varepsilon(C^i) = 4k+2$, for $i = 2,3,7,8,12,19$;
$\varepsilon(C^i) = 4k+3$, for $i = 4,5,9,10,13,14,17$;
$\varepsilon(C^i) \leqslant 4k+3$, for $i = 6,11,15,18$.

It is easy to see that for two vertices x and y of G, they are contained in some cycle C^i as defined above. So, we have

$$\max\{d(C^i): 1 \leqslant i \leqslant 21\} \leqslant \left\lfloor \frac{4k+3}{2} \right\rfloor = 2k+1.$$

We get $P(6, 7(2k-1)+h) \leqslant d(G) \leqslant 2k+1$ for any positive integers k and h with $2 \leqslant h \leqslant 6$. □

Lemma 1. 5 $P(7, 8(2k-1)+h) \leqslant 2k+2$ for any positive integers k and h with $2 \leqslant h \leqslant 8$.

Proof Let $d = 8(2k-1)+h$ and $P = x_0x_1\cdots x_d$ be a simple path. The seven vertices x_{2k-1}, x_{4k-1}, x_{6k}, x_{8k}, x_{10k}, x_{12k} and x_{14k} partition P into eight segments:

$$P_1 = P(x_0, x_{2k-1}),\ P_2 = P(x_{2k-1}, x_{4k-1}),$$
$$P_3 = P(x_{4k-1}, x_{6k}),\ P_4 = P(x_{6k}, x_{8k}),$$
$$P_5 = P(x_{8k}, x_{10k}),\ P_6 = P(x_{10k}, x_{12k}),$$
$$P_7 = P(x_{12k}, x_{14k}),\ P_8 = P(x_{14k}, x_d).$$

Let G be an altered graph obtained from P plus seven extra edges $e_1 = x_0x_{4k-1}$, $e_2 = x_{4k-1}x_{8k}$, $e_3 = x_{4k-1}x_{12k}$, $e_4 = x_{4k-1}x_d$, $e_5 = x_{2k-1}x_{6k}$, $e_6 = x_{6k}x_{10k}$, $e_7 = x_{10k}x_{14k}$ (see Fig. 2). Define 28 cycles as follows.

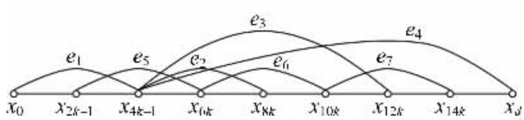

Fig. 2 Construction of Lemma 1. 5 for $k=1$ and $h=6$

$C^1 = P_1 \cup P_2 + e_1$,
$C^2 = P_1 \cup P_3 + e_1 + e_5$,
$C^3 = P_1 \cup P_4 + e_1 + e_2 + e_5$,
$C^4 = P_1 \cup P_5 + e_1 + e_2 + e_5 + e_6$,
$C^5 = P_1 \cup P_6 + e_1 + e_3 + e_5 + e_6$,
$C^6 = P_1 \cup P_7 + e_1 + e_3 + e_5 + e_6 + e_7$,
$C^7 = P_1 \cup P_8 + e_1 + e_4 + e_5 + e_6 + e_7$,
$C^8 = P_2 \cup P_3 + e_5$,
$C^9 = P_2 \cup P_4 + e_5 + e_2$,
$C^{10} = P_2 \cup P_5 + e_5 + e_6 + e_2$,
$C^{11} = P_2 \cup P_6 + e_5 + e_6 + e_3$,
$C^{12} = P_2 \cup P_7 + e_5 + e_6 + e_7 + e_3$,
$C^{13} = P_2 \cup P_8 + e_5 + e_6 + e_7 + e_4$,
$C^{14} = P_3 \cup P_4 + e_2$,
$C^{15} = P_3 \cup P_5 + e_2 + e_6$,
$C^{16} = P_3 \cup P_6 + e_3 + e_6$,
$C^{17} = P_3 \cup P_7 + e_3 + e_6 + e_7$,

$C^{18}=P_3\cup P_8+e_6+e_7+e_4$,
$C^{19}=P_4\cup P_5+e_6$,
$C^{20}=P_4\cup P_6+e_6+e_2+e_3$,
$C^{21}=P_4\cup P_7+e_6+e_7+e_2+e_3$,
$C^{22}=P_4\cup P_8+e_6+e_7+e_2+e_4$,
$C^{23}=P_5\cup P_6+e_2+e_3$,
$C^{24}=P_5\cup P_7+e_2+e_3+e_7$,
$C^{25}=P_5\cup P_8+e_2+e_4+e_7$,
$C^{26}=P_6\cup P_7+e_7$,
$C^{27}=P_6\cup P_8+e_3+e_4+e_7$,
$C^{28}=P_7\cup P_8+e_3+e_4$,

It is easy to see that,

$$\varepsilon(P_1)=2k-1,\ \varepsilon(P_2)=2k,\ \varepsilon(P_3)=2k+1,$$
$$\varepsilon(P_4)=2k,\ \varepsilon(P_5)=2k,\ \varepsilon(P_6)=2k,$$
$$\varepsilon(P_7)=2k,\ \varepsilon(P_8)=2k+h-8.$$

Thus, we have

$\varepsilon(C^1)=4k$;
$\varepsilon(C^i)=4k+1$, for $i=19,26$;
$\varepsilon(C^i)=4k+2$, for $i=2,3,8,9,14,23$;
$\varepsilon(C^i)\leqslant 4k+2$, for $i=28$;
$\varepsilon(C^i)=4k+3$, for $i=4,5,10,11,15,16,20,24$;
$\varepsilon(C^i)\leqslant 4k+3$, for $i=25,27$;
$\varepsilon(C^i)=4k+4$, for $i=6,12,17,21$;
$\varepsilon(C^i)\leqslant 4k+4$, for $i=7,13,18,22$.

It is easy to see that for two vertices x and y of G, they are contained in some cycle C^i as defined above. So, we have

$$\max\{d(C^i):1\leqslant i\leqslant 28\}\leqslant\left\lfloor\frac{4k+4}{2}\right\rfloor=2k+2.$$

Thus, $P(7,8(2k-1)+h)\leqslant d(G)\leqslant 2k+2$ for any positive integers k and h with $2\leqslant h\leqslant 8$. □

2 Proofs of main results

Theorem 2.1 $P(6,4)=1$, $P(6,d)=2$ for $d=5,6,7$, and

$$\left\lceil\frac{d}{7}\right\rceil\leqslant P(6,d)\leqslant\begin{cases}\left\lceil\frac{d}{7}\right\rceil+2 & \text{if } h=7;\\ \left\lceil\frac{d}{7}\right\rceil+1 & \text{otherwise},\end{cases}$$

for $d=7(2k-1)+h$, where $k\geqslant 1$ and $1\leqslant h\leqslant 14$.

Proof It is easy to verify that $P(6,4)=1$ and $P(6,d)=2=\left\lceil\frac{d}{7}\right\rceil+1$ if $d=5,6$ or 7. Suppose $d\geqslant 8$ below. Note that for any positive integer $d(d\geqslant 8)$, there are positive integers k and h with $k\geqslant 1$ and $1\leqslant h\leqslant 14$ such that $d=7(2k-1)+h$. By Lemma 1.3, we have $P(6,d)\geqslant\left\lceil\frac{d}{7}\right\rceil$. So we only need to prove $P(6,d)\leqslant\left\lceil\frac{d}{7}\right\rceil+1$.

If $h=1$, then

$$P(6,7(2k-1)+1)=2k\leqslant\left\lceil\frac{d}{7}\right\rceil+1$$

by Lemma 1.2.

If $h=2,3,4,5,6$, then

$$P(6,7(2k-1)+h)\leqslant 2k+1=\left\lceil\frac{d}{7}\right\rceil+1$$

by Lemma 1.4.

If $h=8,9,10,11,12,13,14$, then

$$P(6,7[2(k+1)-1]+1)=P(6,7(2k-1)+15)=2(k+1)=2k+2$$

by Lemma 1.2, and

$$P(6,7(2k-1)+h)\leqslant P(6,7(2k-1)+15)=2k+2=\left\lceil\frac{d}{7}\right\rceil+1$$

by Lemma 1.1.

Thus, we have $\left\lceil\frac{d}{7}\right\rceil\leqslant P(6,d)\leqslant\left\lceil\frac{d}{7}\right\rceil+1$ for any positive integer d ($d\geqslant 4$ and $d\neq 14k$, where $k=1,2,3,\cdots$). It is easy to get $P(6,14k)\leqslant 2k+2=\left\lceil\frac{d}{7}\right\rceil+2$. And so the theorem holds. □

Theorem 2.2 $P(7,d)=2$ for $d=5,6,7,8$, and

$$\left\lceil\frac{d}{8}\right\rceil\leqslant P(7,d)\leqslant\begin{cases}\left\lceil\frac{d}{8}\right\rceil & \text{if } h=1;\\ \left\lceil\frac{d}{8}\right\rceil+2 & \text{if } h=2,3,4,5,6,7,8;\\ \left\lceil\frac{d}{8}\right\rceil+1 & \text{otherwise};\end{cases}$$

for $d=8(2k-1)+h$, where $k\geqslant 1$ and $1\leqslant h\leqslant 16$.

Proof It is easy to verify that $P(7,d)=2=\left\lceil\frac{d}{8}\right\rceil+1$ if $d=5,6,7$ or 8. Suppose $d\geqslant 9$ below. Note that for any positive d ($d\geqslant 8$), there are

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positive integers k and h with $k\geqslant 1$ and $1\leqslant h\leqslant 16$ such that $d=8(2k-1)+h$. By Lemma 1.3, we have $P(7,d)\geqslant\left\lceil\frac{d}{8}\right\rceil$. So we only need to prove $P(7,d)\leqslant\left\lceil\frac{d}{8}\right\rceil+1$.

If $h=1$, then

$$P(7,8(2k-1)+1)=2k\leqslant\left\lceil\frac{d}{8}\right\rceil$$

by Lemma 1.2.

If $h=2,3,4,5,6,7,8$, then

$$P(7,8(2k-1)+h)\leqslant 2k+2=\left\lceil\frac{d}{8}\right\rceil+2$$

by Lemma 1.5.

If $h=9,10,11,12,13,14,15,16$, then

$$P(7,8[2(k+1)-1]+1)=P(7,8(2k-1)+17)=2(k+1)=2k+2$$

by Lemma 1.2, and

$$P(7,8(2k-1)+h)\leqslant P(7,8(2k-1)+17)=2k+2=\left\lceil\frac{d}{8}\right\rceil+1$$

by Lemma 1.1.

Thus, we have

$$\left\lceil\frac{d}{8}\right\rceil\leqslant P(7,d)\leqslant\begin{cases}\left\lceil\frac{d}{8}\right\rceil & \text{if } h=1;\\ \left\lceil\frac{d}{8}\right\rceil+2 & \text{if } h=2,3,4,5,6,7,8;\\ \left\lceil\frac{d}{8}\right\rceil+1 & \text{otherwise.}\end{cases}$$

The theorem is proved. □

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