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Reinforcement numbers of digraphs[☆]

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ABSTRACT

The reinforcement number of a graph G is the minimum cardinality of a set of extra edges whose addition results in a graph with domination number less than the domination number of G. In this paper we consider this parameter for digraphs, investigate the relationship between reinforcement numbers of undirected graphs and digraphs, and obtain further results for regular graphs. We also determine the exact values of the reinforcement numbers of de Bruijn digraphs and Kautz digraphs.

1. Introduction

For the terminology and notation not defined here, we refer the reader to [17,18]. In this paper a graph $G = (V, E)$ can be an undirected graph or a digraph. Let $\upsilon(G) = |V(G)|$, $\varepsilon(G) = |E(G)|$. The symbol $\Delta^+(G)$ denotes the maximum out-degree of a digraph G.

For an undirected graph $G = (V, E)$ and $v \in V(G)$, we use $N_G(v)$ to denote the set of neighbors of v, and let $N_G[v] = N_G(v) \cup \{v\}$. A vertex v *dominates* all vertices in $N_G[v]$. Analogously for a digraph $G = (V, E)$ and $v \in V(G)$, let $N_G^+(v)$ be the set of out-neighbors of v, and let $N_G^+[v] = N^+(v) \cup \{v\}$. A vertex v *dominates* all vertices in $N_G^+[v]$.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a *dominating set* if D dominates $V(G)$. The minimum cardinality over all dominating sets is called the *domination number*, and denoted by $\gamma(G)$. A dominating set D is called a *γ-set* if $|D| = \gamma(G)$.

An undirected graph G can be thought of as a *symmetric digraph* which is obtained by replacing each edge of G by two *symmetric edges*, i.e., two directed edges with the same end vertices but of opposite directions. Thus, to study the properties of digraphs is in some sense more general than for undirected graphs. A digraph, called an *orientation* of an undirected graph G, can be obtained by specifying the direction of each edge of G.

In 1990, Kok and Mynhardt [13] introduced the *reinforcement number* $r(G)$ of a graph G, which is the minimum number of extra edges whose addition to G results in a graph G' with $\gamma(G') < \gamma(G)$. They defined $r(G) = 0$ if $\gamma(G) = 1$. In this paper, we consider a graph G with $\gamma(G) > 1$. In [13], the authors established some upper bounds for undirected graphs and found a method to determine $r(G)$ in terms of $\gamma(G)$. No results are known for digraphs so far.

In this paper, we present an original investigation into the reinforcement for digraphs. In Section 2 we show that most of the results in [13] are also valid for digraphs and that for an undirected graph G, $r(G) = r(H)$ where H is the symmetric

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digraph of G. The study on reinforcement numbers for digraphs is in some sense more general than that for undirected graphs. We also prove that there exist two orientations H_1 and H_2 of an undirected graph G such that $r(H_1) \leqslant r(G) \leqslant r(H_2)$. In Section 3 we establish the upper bounds and give some characterizations for graphs that attain the bounds, which partially answer a question proposed in [5]. We obtain further results for regular graphs in Section 4 and determine the reinforcement numbers of de Bruijn digraphs and Kautz digraphs in Section 5. Finally, we conclude this paper with some remarks and problems in Section 6.

2. Reinforcement in digraphs

2.1. Fundamental

Lemma 2.1. *If G is a digraph with $\gamma(G) \geqslant 2$, then $r(G) \leqslant \upsilon(G) - \Delta^+(G) - \gamma(G) + 1$.*

Proof. It is clear that $\Delta^+(G) < \upsilon(G) - 1$ since $\gamma(G) \geqslant 2$. Let u be a vertex of maximum out-degree and let $E' = \{(u, v) : v \in V(G) \setminus N_G^+[u]\}$ a set of extra edges. Then $\gamma(G + E') = 1$ since $\{u\}$ is a dominating set in $G + E'$. Hence $r(G) \leqslant |E'| = \upsilon(G) - \Delta^+(G) - 1$. This fact means that there are $r(G) - 1$ vertices $v_1, v_2, \ldots, v_{r(G)-1}$ in $V(G) \setminus N_G^+[u]$.

Now let G' be a digraph obtained from G by adding $r(G) - 1$ extra edges (u, v_i) for each $i = 1, \ldots, r(G) - 1$. By the definition of $r(G)$, $\gamma(G) = \gamma(G') \leqslant \upsilon(G') - \Delta^+(G') = \upsilon(G) - (\Delta^+(G) + r(G) - 1)$, which yields the lemma. ■

For a digraph G, we can calculate $r(G)$ in terms of $\gamma(G)$. To the end, let us set

$$\eta(G) = \upsilon(G) - \max\{|N_G^+[X]| : X \subset V(G),\ |X| = \gamma(G) - 1\}.$$

We call a subset $X \subset V(G)$ an *η- set* if $\eta(G) = \upsilon(G) - |N_G^+[X]|$ in G.

Lemma 2.2. *$r(G) = \eta(G)$ for any digraph G with $\gamma(G) > 1$.*

Proof. Let X be an η-set in G. Then there is a subset Y of $\eta(G)$ vertices of G not dominated by X. Let G' be a digraph obtained from G by adding extra directed edges from some vertex in X to each vertex in Y such that X is a dominating set of G'. Then $\gamma(G') \leqslant |X| \leqslant \gamma(G) - 1$. It follows that $r(G) \leqslant \eta(G)$.

On the other hand, let E' be a set of $r(G)$ extra directed edges whose addition to G results in G' with $\gamma(G') = \gamma(G) - 1$. Let Y' be a γ-set of G' and $V' = \{v : (u, v) \in E', u \in Y'\}$. Then every vertex in $V(G) - V'$ is dominated by Y' in G. Hence $\eta(G) \leqslant |V(G) - N_G^+[Y']| \leqslant |V'| = r(G)$. ■

We have the following corollary from Lemma 2.2 immediately.

Corollary 2.3. *$r(G) = \upsilon(G) - \Delta^+(G) - 1$ for any digraph G with $\gamma(G) = 2$.*

This result shows that the upper bound given in Lemma 2.1 is best for $\gamma(G) = 2$.

Now we consider the relationship between reinforcement numbers of undirected graphs and digraphs.

Theorem 2.4. *For an undirected graph G, let H be its symmetric digraph. Then $\gamma(H) = \gamma(G)$ and $r(H) = r(G)$.*

Proof. Let X be a subset of vertices in G. It is clear that $N_G[X] = N_H^+[X]$. Hence X is a dominating set in G if and only if X is a dominating set in H, which implies that $\gamma(G) = \gamma(H) = \gamma$. For any set Y of $\gamma - 1$ vertices, we have $\eta(G) = \eta(H)$ since $|N_G[Y]| = |N_H^+[Y]|$. By Lemma 2.2, $r(G) = r(H)$. ■

Theorem 2.5. *For an undirected graph G, there exist two orientations H_1 and H_2 of G such that $r(H_1) \leqslant r(G) \leqslant r(H_2)$.*

Proof. For a given orientation H of G and a subset $X \subseteq V(G)$, clearly, $N_H^+[X] \subseteq N_G[X]$. This fact means $\gamma(H) \geqslant \gamma(G)$ for any orientation H since any γ-set in H is a dominating set in G.

Let D' be an η-set in G, and let H_1 be an orientation of G obtained by giving a direction from u to v for each edge $uv \in E(G)$ with $u \in D'$ and $v \notin D'$, and arbitrarily giving directions for other edges of G. Then $|N_{H_1}^+[D']| = |N_G[D']| = \upsilon(G) - \eta(G)$. Since $\gamma(H_1) \geqslant \gamma(G) > 1$, we can obtain a set X with $|X| = \gamma(H_1) - 1$ by arbitrarily adding $\gamma(H_1) - \gamma(G)$ vertices to X. Clearly $|N_{H_1}^+[X]| \geqslant |N_{H_1}^+[D']| = \upsilon(G) - \eta(G)$ and so $\eta(H_1) \leqslant \upsilon(H_1) - |N_{H_1}^+[X]| \leqslant \eta(G)$. By Lemma 2.2, we have $r(H_1) = \eta(H_1) \leqslant \eta(G) = r(G)$.

Let D be a γ-set in G, and let H_2 be an orientation of G obtained by giving a direction from u to v for each edge $uv \in E(G)$ with $u \in D$ and $v \notin D$, and arbitrarily giving directions to other edges. Clearly D is also a dominating set in H_2. Hence $\gamma(H_2) = \gamma(G) = \gamma > 1$. Let D' be an η-set in G. For any set X of $\gamma - 1$ vertices in H_2, $|N_{H_2}^+[X]| \leqslant |N_G[X]| \leqslant |N_G[D']|$, which implies that $\eta(H_2) \geqslant \upsilon(H_2) - |N[D']| = \eta(G)$. Thus $r(H_2) \geqslant r(G)$ by Lemma 2.2. ■

2.2. Selected families of digraphs

In this subsection we will determine the reinforcement numbers of special classes of digraphs. Some of them show the tightness of our results in Section 2. First we will generalize some results in [13].

Proposition 2.6. *Let $\overrightarrow{C}_n$ and $\overrightarrow{P}_n$ be a directed cycle and a directed path with $n = 2k + i$ vertices, where k is a nonnegative integer and $i \in \{1, 2\}$. Then $r(\overrightarrow{C}_n) = r(\overrightarrow{P}_n) = i$.*

Proof. Assume $\overrightarrow{C}_n = (0, 1, \ldots, n-1)$. It is clear that $\gamma(\overrightarrow{C}_n) = \lceil n/2 \rceil = k+1$ since $\{0, 2, \ldots, 2k\}$ is a dominating set. Let X be a set of k vertices in C_n. Then $|N^+_{\overrightarrow{C}_n}[X]| \leqslant 2k$ with equality if $X = \{0, 2, \ldots, 2k-2\}$. By Lemma 2.2, $r(\overrightarrow{C}_n) = \eta(\overrightarrow{C}_n) = n - 2k = i$. Similarly we have $r(\overrightarrow{P}_n) = \eta(\overrightarrow{P}_n) = i$. ■

Remark 2.7. Kok and Mynhardt [13] proved that $r(P_n) = r(C_n) = i$ where P_n and C_n are an undirected path and an undirected cycle with $n = 3k + i$ vertices. Then $r(P_n) = 1 < 2 = r(\overrightarrow{P}_n)$ and $r(C_n) = 1 < 2 = r(\overrightarrow{C}_n)$ if $n = 6k + 4$, whereas $r(\overrightarrow{P}_n) < 3 = r(P_n)$ and $r(\overrightarrow{C}_n) < 3 = r(C_n)$ if $n = 3k$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two digraphs. The *cartesian product* of G_1 and G_2 is a digraph, denoted by $G_1 \times G_2$, where $V(G_1 \times G_2) = V_1 \times V_2$. There is a directed edge from a vertex x_1x_2 to another y_1y_2, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$, in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and $(x_2, y_2) \in E(G_2)$, or $x_2 = y_2$ and $(x_1, y_1) \in E(G_1)$.

Proposition 2.8. *Let $\overrightarrow{K}_m$ and $\overrightarrow{K}_n$ be two complete digraphs, $2 \leqslant m \leqslant n$. Then $r(\overrightarrow{K}_m \times \overrightarrow{K}_n) = n - m + 1$. Furthermore, $\overrightarrow{K}_n \times \overrightarrow{K}_n$ is domination edge critical.*

Proof. By Theorem 2.4, $\gamma(\overrightarrow{K}_m \times \overrightarrow{K}_n) = \gamma(K_m \times K_n) = m$ and $r(\overrightarrow{K}_m \times \overrightarrow{K}_n) = r(K_m \times K_n) = m - n + 1$. (See [13].)

Let $G = \overrightarrow{K}_n \times \overrightarrow{K}_n$ with vertex-set $\{xy : x, y = 0, 1, \ldots, n-1\}$. Then $e = ((ij), (i'j'))$ is an edge of G if and only if $i \neq i'$ and $j \neq j'$. Let f be a permutation on $\{0, 1, \ldots, n-1\}$ such that $f(i) = j$ and $f(i') = j'$. Then $D = \{(k, f(k)) : k \neq i'\}$ is a dominating set in $G + e$. Hence $\gamma(G + e) = n - 1$. ■

The *circulant graph* $\overrightarrow{C}(n; S)$ of order n is the Cayley graph $C(Z_n, S)$, where $Z_n = \{0, 1, \ldots, n-1\}$ is the additive group of order n and S is a nonempty subset of Z_n without the identity element. It is well known that $\overrightarrow{C}(n; S)$ is a vertex-transitive digraph of degree $|S|$. If $S^{-1} = S$, then $\overrightarrow{C}(n; S)$ is symmetric and we view it undirected.

Proposition 2.9. *Let $S = \{1, 2, \ldots, k\}$ and $n = p(k+1) + q$, where $1 \leqslant q \leqslant k+1$. Then $\gamma(\overrightarrow{C}(n; S)) = p + 1$ and $r(\overrightarrow{C}(n; S)) = q$.*

Proof. Since $\overrightarrow{C}(n; S)$ is k-regular, every vertex dominates $k+1$ vertices, which implies that $\gamma(\overrightarrow{C}(n; S)) \geqslant \lceil \frac{n}{k+1} \rceil = p + 1$. On the other hand, $D = \{0, k, \ldots, pk\}$ is a dominating set in G. Hence $\gamma(\overrightarrow{C}(n; S)) = \lceil \frac{n}{k+1} \rceil = p + 1$.

For any set X of p vertices, $N^+[X] \leqslant p(k+1)$ with equality if $X = D - \{pk\}$. Hence

$$r(\overrightarrow{C}(n; S)) = \eta(\overrightarrow{C}(n; S)) = n - p(k+1) = q. \quad ■$$

2.3. Compositions of digraphs

The following proposition is straightforward by computing $\eta(G \cup H)$.

Proposition 2.10. *Let G and H be two digraphs. Then $r(G \cup H) = \min\{r(G), r(H)\}$.*

For two undirected graphs G and H, the *join* $G + H$ is defined as an undirected graph consisting of G and H with each vertex of G adjacent to every vertex of H. If G and H are digraphs then we can define two kinds of *joins* $G \to H$ and $G \leftrightarrow H$. The digraph $G \to H$ consists of G and H with extra edges from each vertex of G to every vertex of H, and $G \leftrightarrow H$ can be obtained from $G \to H$ by adding edges from each vertex of H to every vertex of G.

Proposition 2.11. *Let G and H be two digraphs. Then*

(1) $r(G \to H) = r(G)$;

(2) $r(G \leftrightarrow H) = \begin{cases} \upsilon(G) + \upsilon(H), & \text{if } \min\{\gamma(G), \gamma(H)\} = 1; \\ \min\{\upsilon(G) - \Delta^+(G) - 1, \upsilon(H) - \Delta^+(H) - 1\}, & \text{otherwise.} \end{cases}$

Proof. (1) Let D be a γ-set in G. By the definition, D is also a dominating set in $G \to H$. Hence $\gamma(G \to H) \leqslant |D| = \gamma(G)$. On the other hand, any set consisting of fewer than $\gamma(G)$ vertices cannot dominate $V(G)$, since no vertex in H dominates any vertex in G. Hence $\gamma(G \to H) = \gamma(G) = \gamma$.

Let D' be an η-set in G. Then in $G \to H$, any set X of $\gamma - 1$ vertices dominates at most $|N^+_G[D']| + \upsilon(H)$ vertices, with equality if $X = D'$. Hence

$$\begin{aligned} \eta(G \to H) &= \upsilon(G \to H) - |N^+_G[D']| - \upsilon(H) \\ &= \upsilon(G) - |N^+_G[D']| = \eta(G). \end{aligned}$$

(2) If $\min\{\gamma(G), \gamma(H)\} = 1$, then $\gamma(G \leftrightarrow H) = 1$ and $r(G \leftrightarrow H) = \upsilon(G \leftrightarrow H) = \upsilon(G) + \upsilon(H)$. Otherwise $\gamma(G \leftrightarrow H) = 2$. By Corollary 2.3,

$$\begin{aligned} r(G \leftrightarrow H) &= \upsilon(G \leftrightarrow H) - \Delta^+(G \leftrightarrow H) - 1 \\ &= \upsilon(G) + \upsilon(H) - \max\{\Delta^+(G) + \upsilon(H), \Delta^+(H) + \upsilon(G)\} - 1 \\ &= \min\{\upsilon(G) - \Delta^+(G) - 1, \upsilon(H) - \Delta^+(H) - 1\}. \end{aligned}$$

The proof is complete. ■

Next we consider an operation of graphs, called the corona. The *corona* $G \circ H$ of two undirected graphs G and H is formed from one copy of G and $\upsilon(G)$ copies of H by joining v_i to H_i, where v_i is the ith vertex of G and H_i is the ith copy of H. For digraphs G and H, if all the additional edges are from G to H, then we denote the resulting digraph by $G \overrightarrow{\circ} H$; if all the additional edges are from H to G, then we denote the resulting digraph by $G \overleftarrow{\circ} H$.

Proposition 2.12. *Let G and H be two digraphs. Then*

(1) $r(G \overrightarrow{\circ} H) = \begin{cases} \upsilon(H)+1 & \text{if } G = \overline{K_n}, n \geqslant 2; \\ \upsilon(H) & \text{otherwise.} \end{cases}$

(2) $r(G \overleftarrow{\circ} H) = r(H)$.

Proof. (1) A single vertex in $G \overrightarrow{\circ} H$ does not dominate two vertices in different copies of H; we need at least $\upsilon(G)$ vertices to dominate $\upsilon(G)$ copies of H. Hence $\gamma(G \overrightarrow{\circ} H) \geqslant \upsilon(G)$. Since $V(G)$ is a dominating set of $G \overrightarrow{\circ} H$, then $\gamma(G \overrightarrow{\circ} H) = \upsilon(G)$.

If $\upsilon(G) = 1$, then $r(G \overrightarrow{\circ} H) = \upsilon(G \overrightarrow{\circ} H) = \upsilon(H)$. Assume $\upsilon(G) \geqslant 2$ below. For any set X of $\upsilon(G) - 1$ vertices, there exists an integer i such that v_i and H_i do not belong to X (otherwise $|X| \geqslant \upsilon(G)$).

If G contains no edge, then X cannot dominate the v_i and H_i, which implies that $|N^+[X]| \leqslant \upsilon(G \overrightarrow{\circ} H) - \upsilon(H) - 1$. Let D' be a set consisting of $\upsilon(G) - 1$ vertices of G. Then $|N^+[D']| = (\upsilon(G) - 1)(\upsilon(H) + 1) = \upsilon(G \overrightarrow{\circ} H) - \upsilon(H) - 1$. Hence $r(G \overrightarrow{\circ} H) = \eta(G \overrightarrow{\circ} H) = \upsilon(H) + 1$.

If there exists an edge $(u, v) \in E(G)$, then X cannot dominate the ith copy of $V(H)$, which implies that $|N^+[X]| \leqslant \upsilon(G \overrightarrow{\circ} H) - \upsilon(H)$. Let $D' = V(G) - \{v\}$. Then D' can dominate v and so $|N^+[D']| = (\upsilon(G) - 1)(\upsilon(H) + 1) + 1 = \upsilon(G \overrightarrow{\circ} H) - \upsilon(H)$. Hence $r(G \overrightarrow{\circ} H) = \eta(G \overrightarrow{\circ} H) = \upsilon(H)$.

(2) By the definition of $G \overleftarrow{\circ} H$, we need at least $\gamma(H)$ vertices to dominate H_i and so $\gamma(G \overleftarrow{\circ} H) \geqslant \upsilon(G)\gamma(H)$. Let D_i be a γ-set in H_i. It is easy to observe that $D = \cup_{i=1}^{\upsilon(G)} D_i$ is a dominating set of $G \overleftarrow{\circ} H$. Hence $\gamma(G \overleftarrow{\circ} H) = \upsilon(G)\gamma(H)$.

For any set X of $\upsilon(G)\gamma(H) - 1$ vertices in $G \overleftarrow{\circ} H$, there exists some integer i such that $X_i = X \cap V(H_i) \leqslant \gamma(H) - 1$. By Lemma 2.2, $|N^+[X_i]| \leqslant \upsilon(H_i) - r(H_i)$ and so $|N^+[X]| \leqslant \upsilon(G \overleftarrow{\circ} H) - r(H)$. Note that any subset $D' \subset D$ with $|D'| = |D| - 1$ dominates $\upsilon(G \overleftarrow{\circ} H) - r(H)$ vertices. Hence $r(G \overleftarrow{\circ} H) = r(H)$. ■

Using Propositions 2.11 and 2.12 we can construct large graphs with required $r(G)$.

Corollary 2.13. *For a given positive integer r, there is a connected directed planar graph such that its reinforcement number is equal to r.*

Proof. Let $G = \overrightarrow{P}_2 \overrightarrow{\circ} \overline{K_r}$. Clearly G is a connected planar graph with $r(G) = r$ by Proposition 2.12. The examples for undirected graphs are similar. (Use the proposition for the undirected corona in [13].) ■

3. More upper bounds

Kok and Mynhardt [13] discussed the relationship between $r(G)$ and $\rho(G)$, the private neighborhood number of G, and obtain an upper bound for $r(G)$. We generalize that the result to digraphs. For a graph G, let $x \in X \subseteq V(G)$. The *private neighborhood of x with respect to X* is the set $PN(x, X) = \{N[x] - N[X - \{x\}]\}$ or $\{N^+[x] - N^+[X - \{x\}]\}$. Let $\rho(X) = \min\{|PN(x, X)| : x \in X\}$ and define $\rho(G) = \min\{\rho(D) : D$ is a γ-set of $G\}$ to be the *private neighborhood number* of G. It is clear that $\rho(G) \geqslant 1$ since every vertex in a minimal dominating set has at least one private neighbor.

Theorem 3.1. *For a graph G, $r(G) \leqslant \rho(G)$ with equality if $r(G) = 1$.*

Proof. The result for undirected graphs has been given in [13]. The proof for digraphs is similar. If $\gamma(G) = 1$ then $r(G) = \upsilon(G) = \rho(G)$. Assume $\gamma(G) \geqslant 2$ below. Let D be a γ-set such that there exists a vertex $v \in D$ with $|PN(v, D)| = \rho(D) = \rho(G)$. Pick a vertex $u \in D - \{v\}$ and let $H = G + \{(u, w) : w \in PN(v, D)\}$. Clearly $D - \{v\}$ is a dominating set in H. Thus $r(G) \leqslant |PN(v, D)| = \rho(G)$.

If $r(G) = 1$ then there exists a dominating set D' in $G + \{(u, v)\}$ with $|D'| = \gamma(G) - 1$. Clearly $D = D' \cup \{v\}$ is a γ-set in G and $PN(v, D) = \{v\}$. It follows that $\rho(G) \leqslant \rho(D) \leqslant |PN(v, D)| = 1$ and so $\rho(G) = 1$. ■

Note that $\sum_{v \in D} |PN(v, D)| \leqslant \upsilon(G)$. Then there exists a vertex v in any γ-set D such that $|PN(v, D)| \leqslant \upsilon(G)/|D|$. Thus the following corollary holds.

Corollary 3.2. *For a graph G, $r(G) \leqslant \rho(G) \leqslant \upsilon(G)/\gamma(G)$.*

Kok and Mynhardt [13] demonstrated that for any $\gamma(G) \geqslant 2$, the gap between $r(G)$ and $\rho(G)$ can be arbitrarily large for connected undirected graphs. By Theorem 2.4, this result is also valid for digraphs.

Theorem 3.3 (*[13]*). *For any integer r, s, t with $2 \leqslant r \leqslant s$ and $t \geqslant 2$, there exists a connected graph G such that $r(G) = r$, $\rho(G) = s$ and $\gamma(G) = t$.*

The next question is when the equalities in Theorem 3.1 hold. Dunbar et al. [5] proposed the following problems for undirected graphs, which remain open. We can ask similar questions for digraphs.

Open Problem [5] Characterize graphs G for which $r(G) = \rho(G) = \upsilon(G)/\gamma(G)$.

Now we give some results on this problem. For this purpose we refer to the *efficient dominating set*, or *E-set* for short, which is a dominating set D such that every vertex of G is dominated by a unique vertex of D. Bange et al. [2] introduced this concept as a measure of the efficiency of domination in graphs. Bange et al. [1] proved that every undirected graph has an orientation with an efficient dominating set. Clearly a dominating set D is efficient if and only if $PN(v, D) = N[v]$ or $N^+[v]$ for any $v \in D$. Furthermore, the γ-set and E-set are equivalent for regular graphs possessing an E-set.

Lemma 3.4 (*[8]*). *If G is a k-regular graph, then $\gamma(G) \geqslant \lceil \frac{\upsilon(G)}{k+1} \rceil$, with equality if and only if G has an E-set. In addition, if G has an E-set, then every E-set is a γ-set, and vice versa.*

Proof. Here we only consider digraphs. The proof for undirected graphs is similar. Since G is k-regular, $|N^+[v]| = k + 1$ for each $v \in V(G)$. Hence $\gamma(G) \geqslant \lceil \frac{\upsilon(G)}{k+1} \rceil$. It is easy to observe that equality holds if and only if there exists a dominating set D such that $PN(v, D) = N^+[v]$ for every $v \in V(G)$, equivalently, D is an E-set.

Now suppose that G has an E-set, i.e., $\gamma(G) = \frac{\upsilon(G)}{k+1}$. Then a dominating set D is a γ-set if and only if $|D| = \frac{\upsilon(G)}{k+1}$. On the other hand, D is efficient if and only if $|D| = \frac{\upsilon(G)}{k+1}$. The lemma follows. ■

Theorem 3.5. *For a graph G, $\rho(G) = \upsilon(G)/\gamma(G)$ if and only if every γ-set of G is an E-set consisting of vertices with the same (out-)degree equal to $\rho(G) - 1$.*

Proof. For any γ-set D of G,

$$\gamma(G)\rho(G) \leqslant |D| \cdot |\rho(D)| \leqslant \sum_{v \in D} |PN(v, D)| \leqslant \upsilon(G).$$

Hence $\rho(G) = \upsilon(G)/\gamma(G)$ if and only if the following equalities

$$\sum_{v \in D} |PN(v, D)| = \upsilon(G) \quad \text{and} \tag{3.1}$$

$$|PN(v, D)| = \rho(D) = \rho(G) \quad \text{for any } v \in D \tag{3.2}$$

hold for any γ-set D.

Clearly (3.1) holds if and only if D is an E-set. If G is a digraph, then (3.2) is equivalent to $\rho(G) = |PN(v, D)| = d^+(v) + 1$ for any $v \in D$, since D is efficient. Hence $d^+(v) = \rho(G) - 1$ is a constant. Similarly we obtain the result for undirected graphs. ■

Corollary 3.6. *If every vertex of a graph G belongs to a γ-set, then $r(G) = \rho(G) = \upsilon(G)/\gamma(G)$ if and only if G is regular and has an E-set.*

Proof. Assume $\rho(G) = \upsilon(G)/\gamma(G)$. Since every vertex of G belongs to a γ-set, Lemma 3.4 and Theorem 3.5 imply that G is regular and has an E-set.

Conversely, suppose that G is k-regular and has an E-set. Clearly every set of $\gamma(G) - 1$ vertices dominates at most $(\gamma(G) - 1)(k + 1) = \upsilon(G) - k - 1$ vertices. Thus $r(G) = \eta(G) \geqslant k + 1$. On the other hand, $r(G) \leqslant \rho(G) = k + 1$. Hence $r(G) = \rho(G)$. ■

4. Regular graphs

Section 3 presented some results on the characterization of $r(G) = \rho(G) = \upsilon(G)/\gamma(G)$. In view of Corollary 3.6, it is not difficult to obtain further results for regular graphs.

Theorem 4.1. *Let G be a k-regular graph. Then $r(G) = \rho(G) = \upsilon(G)/\gamma(G)$ if and only if G has an E-set.*

Proof. The necessity follows from Theorem 3.5. For the sufficiency, let D be an E-set of G. By Lemma 3.4, $\upsilon(G)/\gamma(G) = k+1$. Let X be a set of $\gamma(G) - 1$ vertices. Clearly $|N^+[X]| \leqslant (\gamma(G) - 1)(k + 1) = \upsilon(G) - k - 1$ since G is k-regular; equality holds if $X \subset D$ and $|X| = \gamma(G) - 1$. Hence $r(G) = \eta(G) = k + 1$. The theorem follows. ■

Next we consider the following problem for regular graphs.

Open Problem [5] Determine additional upper and lower bounds for $r(G)$.

Lemma 4.2. *Given two positive integers ℓ and γ, let $a_1 = \frac{\ell}{\gamma}$ and $a_n = \frac{\ell+\lceil a_{n-1}\rceil}{\gamma}$ for any integer $n \geqslant 2$. Then*

$$\lim_{n\to\infty} a_n = \begin{cases} \dfrac{\ell}{\gamma-1} & \text{if } (\gamma-1) \mid \ell\,; \\ \dfrac{\ell+1+\lfloor\frac{\ell}{\gamma-1}\rfloor}{\gamma} & \text{otherwise.} \end{cases}$$

Proof. First we show that a_n converges. Clearly a_n is monotone increasing. We prove by induction on n that a_n has an upper bound

$$b_n = \frac{\ell+1}{\gamma-1} - \frac{\ell+1}{\gamma^{\upsilon}(\gamma-1)}.$$

It is trivial for $n = 1$. Assume $a_n \leqslant b_n$. Then

$$\begin{aligned} a_{n+1} &= \frac{\ell+\lceil a_n\rceil}{\gamma} \leqslant \frac{\ell+\lceil b_n\rceil}{\gamma} \\ &\leqslant \frac{\ell+b_n+1}{\gamma} = b_{n+1}. \end{aligned}$$

Thus $a_n \leqslant b_n$ for any n. Since $\lim_{n\to\infty} b_n = \frac{\ell+1}{\gamma-1}$, a_n converges to a real number a.

Next we determine a. Let $t = \lceil a\rceil - a \in [0, 1)$. Then $1 - t$ is just the fractional part of a unless a is an integer. Note that $\lim_{n\to\infty}\lceil a_n\rceil = a$, since a_n is monotone increasing. By the definition of a_n,

$$\begin{aligned} a &= \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\ell+\lceil a_{n-1}\rceil}{\gamma}\right) \\ &= \frac{\ell+\lceil a\rceil}{\gamma} = \frac{\ell+a+t}{\gamma}, \end{aligned}$$

that is, $a = \frac{\ell+t}{\gamma-1}$. Let $l = (\gamma-1)p + q$, where $0 \leqslant q \leqslant \gamma-2$. Then $a = p + \frac{q+t}{\gamma-1}$. Since $q + t \leqslant \gamma - 2 + t < \gamma - 1$, then $\frac{q+t}{\gamma-1} < 1$ is the fractional part of a.

First consider $q = 0$, i.e., $(\gamma-1) \mid l$. If $t = 0$, then $a = \frac{\ell}{\gamma-1}$ is an integer. If $t > 0$, then a has the fractional part $1 - t = \frac{q+t}{\gamma-1} = \frac{t}{\gamma-1}$. Hence $t = 1 - \frac{1}{\gamma}$ and $a = \frac{\ell}{\gamma-1} + \frac{1}{\gamma}$. There are two possible values of a. We prove by induction that $a_n \leqslant \frac{\ell}{\gamma-1}$ for any n, which implies that $a = \frac{\ell}{\gamma-1}$. Clearly $a_1 \leqslant \frac{\ell}{\gamma-1}$. Assume $a_n \leqslant \frac{\ell}{\gamma-1}$. Then $\lceil a_n\rceil = \frac{\ell}{\gamma-1}$ since $(\gamma-1) \mid \ell$, and

$$a_{n+1} = \frac{\ell+\lceil a_n\rceil}{\gamma} \leqslant \frac{\ell+\frac{\ell}{\gamma-1}}{\gamma} = \frac{\ell}{\gamma-1}.$$

Hence $a_n \leqslant \frac{\ell}{\gamma-1}$ for any n. Then $a \neq \frac{\ell}{\gamma-1} + \frac{1}{\gamma-1}$ and so $a = \frac{\ell}{\gamma-1}$.

Now consider $q \geqslant 1$. Then a has the fractional part $1 - t = \frac{q+t}{\gamma-1}$. Hence $t = 1 - \frac{q+1}{\gamma}$ and $a = p + 1 - t = \frac{\ell+p+1}{\gamma}$. The result follows. ∎

Theorem 4.3. *For a k-regular graph G, let $\gamma = \gamma(G) \geqslant 2$ and $\ell = \gamma(k+1) - \upsilon(G)$. Then*

$$k+1-\ell \leqslant r(G) \leqslant \rho(G) \leqslant k+1-\begin{cases} \dfrac{\ell}{\gamma-1} & \text{if } (\gamma-1) \mid \ell\,; \\ \left\lceil\dfrac{\ell+1+\lfloor\frac{\ell}{\gamma-1}\rfloor}{\gamma}\right\rceil & \text{otherwise.} \end{cases}$$

Proof. Let $\gamma = \gamma(G)$. For any set D' consisting of $\gamma - 1$ vertices, $|N^+[D']| \leqslant (\gamma-1)(k+1)$. Hence $r(G) = \eta(G) \geqslant \upsilon(G) - (\gamma-1)(k+1) = k+1-\ell$.

Let $a_1 = \ell/\gamma$ and $a_n = \ell/\gamma + \lceil a_{n-1}\rceil/\gamma$ for any integer $n \geqslant 2$. We proceed by induction on n to show that $\rho(G) \leqslant k+1-a_n$ for any n.

Since $\sum_{v\in D}|PN(v, D)| \leqslant \upsilon(G)$ for any γ-set D, there exists a vertex $v \in D$ such that

$$\begin{aligned} \rho(G) &\leqslant |PN(v, D)| \leqslant \upsilon(G)/\gamma(G) \\ &= k+1-\ell/\gamma = k+1-a_1. \end{aligned}$$

Suppose that $\rho(G) \leqslant k+1-a_n$. We show $\rho(G) \leqslant k+1-a_{n+1}$. By the assumption on $\rho(G)$, there exists a γ-set D and a vertex $v \in D$ such that $|PN(v, D)| = \rho(G) \leqslant k+1-a_n$. Hence in $N^+[v]$ there are at least $\lceil a_n\rceil$ vertices not belonging to $PN(v, D)$; clearly these vertices do not belong to $PN(v', D)$ for any $v' \neq v$ and $v' \in D$. It follows that $\sum_{v \in D} |PN(v, D)| \leqslant \upsilon(G) - \lceil a_n \rceil$ and there exists some vertex $u \in D$ such that

$$\begin{aligned} \rho(G) &\leqslant |PN(u, D)| \\ &\leqslant k + 1 - (\ell + \lceil a_n \rceil)/\gamma \\ &= k + 1 - a_{n+1}. \end{aligned}$$

Thus $\rho(G) \leqslant k + 1 - a_n$ for any n. Let n tend to infinity and we obtain $\rho(G) \leqslant k + 1 - \lim_{n\to\infty} a_n$. Note that $\rho(G)$ is an integer. The theorem follows from Lemma 4.2. ∎

Remark 4.4. The lower bound in Theorem 4.3 is sharp for any positive integer k and any integer $n \geqslant k + 1$, as shown in Proposition 2.9. The upper bound is strictly smaller than $\upsilon(G)/\gamma(G)$ unless $\ell = 0$.

Note that the lower and upper bounds in Theorem 4.3 are equal for $\gamma = 2$ or $\ell = 0$. Then Corollary 2.3 and Theorem 4.1 can be immediately derived from Theorem 4.3. The following corollary is straight forward if we let $\ell = 1$.

Corollary 4.5. *Let G be a k-regular graph. If $\upsilon(G) = \gamma(G)(k + 1) - 1$, then $r(G) = \rho(G) = k$.*

5. de Bruijn digraphs and Kautz digraphs

In this section we determine the reinforcement numbers of de Bruijn Digraphs and Kautz Digraphs. Note that loops may arise. We cannot directly apply results in Section 4 to de Bruijn Digraphs and Kautz Digraphs.

First recall the definition of the de Bruijn digraph $B(d, n)$. It is a digraph with vertex-set $V = \{x_1 \cdots x_n : x_i \in \{0, 1, \ldots, d - 1\}\}$; there is a directed edge from x to y if and only if $x = x_1x_2 \ldots x_n$ and $y = x_2 \cdots x_n\alpha$, where $\alpha \in \{0, 1, \ldots, d - 1\}$. $B(d, n)$ has d^n vertices, d^{n+1} edges, and is d-regular.

Shibata and Gonda [15] defined the extended de Bruijn digraph $EB(d, n; q_1, \ldots, q_p)$. It is a digraph whose vertex-set is the set of n-dimensional vectors on d elements divided into p blocks of sizes $q_1, \ldots, q_p$, expressed as follows

$$x = (x_{11}x_{12} \cdots x_{1q_1})(x_{21}x_{22} \cdots x_{2q_2}) \cdots (x_{p1}x_{p2} \cdots x_{pq_p}),$$

where $0 \leqslant x_{ij} \leqslant d - 1$, and $q_1 + q_2 + \cdots + q_p = n$. The out-neighbors of x are those vertices having the form

$$(x_{12} \cdots x_{1q_1}\alpha_1)(x_{22} \cdots x_{2q_2}\alpha_2) \cdots (x_{p2} \cdots x_{pq_p}\alpha_p),$$

where $0 \leqslant \alpha_i \leqslant d - 1$ for each $i = 1, 2, \ldots, p$. The extended de Bruijn digraph $EB(d, n; q_1, \ldots, q_p)$ has d^n vertices, d^{n+p} edges and is d^p-regular. From the definition, if $p = 1$, i.e., the vertices are not divided, then clearly $EB(d, n; n) = B(d, n)$.

Although loops arise, we can determine $r(G)$ for $G = EB(d, n; q_1, \ldots, q_p)$ with $q_1 = \cdots = q_p = q \geqslant 1$. For a given p, a sequence $(i_1 \cdots i_p)$ on $\{0, 1, \ldots, d - 1\}$ and $j \in \{1, 2, \ldots, q\}$, let

$$D^{(j)}_{(i_1 \cdots i_p)} = \{(x_{11} \cdots x_{1q}) \cdots (x_{p1} \ldots x_{pq}) : x_{k1} = \cdots x_{kj} = i_k, k = 1, \ldots, p\}.$$

It is not difficult to verify that

$$D_{(i_1,\ldots,i_p)} = \begin{cases} D^{(1)}_{(i_1,\ldots,i_p)} - D^{(2)}_{(i_1,\ldots,i_p)} + D^{(3)}_{(i_1,\ldots,i_p)} - \cdots + D^{(q)}_{(i_1,\ldots,i_p)} & \text{if } q \text{ is odd;} \\ D^{(1)}_{(i_1,\ldots,i_p)} - D^{(2)}_{(i_1,\ldots,i_p)} + D^{(3)}_{(i_1,\ldots,i_p)} - \cdots + D^{(q-1)}_{(i_1,\ldots,i_p)} & \text{if } q \text{ is even,} \end{cases}$$

is a dominating set in G with $|D| = \lceil \upsilon(G)/(d^p + 1) \rceil$. Hence we can determine $\gamma(G)$.

Lemma 5.1 ([9]). *Let $G = EB(d, n; q_1, \ldots, q_p)$ with $q_1 = \cdots = q_p = q$ and $n = pq$. Then*

$$\gamma(G) = \begin{cases} (d^n + 1)/(d^p + 1) & \text{if } q \text{ is odd;} \\ (d^n + d^p)/(d^p + 1) & \text{if } q \text{ is even.} \end{cases}$$

Theorem 5.2. *Let $G = EB(d, n; q_1, \ldots, q_p)$ with $q_1 = \cdots = q_p = q$ and $n = pq$. Then*

$$r(G) = \begin{cases} d^p & \text{if } q \text{ is odd;} \\ 1 & \text{if } q \text{ is even.} \end{cases}$$

Proof. First assume that q is odd. Let X be a set of $\gamma(G) - 1$ vertices. Then $|N[X]| \leqslant (\gamma(G) - 1)(d^p + 1) = d^n - d^p$. Let $D' = D_{(i_1,\ldots,i_p)} - D^{(q)}_{(i_1,\ldots,i_p)}$. Note that $|N^+[D^{(q)}_{(i_1,\ldots,i_p)}]| = d^p$ as a result of loops. Hence $|N[D']| = d^n - d^p$ and $r(G) = \eta(G) = d^p$ by Lemma 2.2.

Now assume that q is even. Let X be a set of $\gamma(G) - 1$ vertices. Then $|N[X]| \leqslant (\gamma(G) - 1)(d^p + 1) = d^n - 1$. Let $D' = D_{(i_1,\ldots,i_p)} - D^{(q)}_{(i_1,\ldots,i_p)}$. Note that $N^+[D^{(q)}_{(i_1,\ldots,i_p)}] = D^{(q-1)}_{(i_1,\ldots,i_p)} \subset D'$. Hence $|N^+[D']| = |N^+[D]| - 1 = d^n - 1$. By Lemma 2.2, $r(G) = \eta(G) = 1$. ∎

Since $EB(d, n; n) = B(d, n)$, we immediately obtain the reinforcement numbers of de Bruijn digraphs if we let $p = 1$ in Theorem 5.2.

Corollary 5.3.

$$r(B(d,n)) = \begin{cases} d & \text{if } n \text{ is odd;} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Next we consider the Kautz digraph $K(d, n)$, which has vertex-set and edge-set as follows.

$$\begin{cases} V = \{x_1 \cdots x_n : x_i \in \{0, 1, \ldots, d\}, x_i \neq x_{i+1}, i = 1, \ldots, n-1\}, \text{ and} \\ E = \{(x_1 x_2 \cdots x_n, \ x_2 \cdots x_n \alpha) : \alpha \in \{0, 1, \ldots, d\}, \alpha \neq x_n\}. \end{cases}$$

$K(d, n)$ has $d^{n-1}(d + 1)$ vertices, $d^{\upsilon}(d + 1)$ edges, and is d-regular.

M. Imase and M. Itoh [10] proposed a generalization of Kautz digraphs, called the generalized Kautz digraph. It is denoted by $GK(d, n)$ and has vertex-set and edge-set as follows.

$$\begin{cases} V = \{0, 1, \ldots, n-1\}, \text{ and} \\ E = \{(x, y) : y \equiv -dx - i \ (\bmod \ n), 0 < i \leqslant d\}. \end{cases}$$

$GK(d, n)$ is a d-regular digraph with n vertices.

Lemma 5.4 ([12]). *For any positive integers d and n, $\gamma(GK(d, n)) = \lceil \frac{n}{d+1} \rceil$.*

Theorem 5.5. *For any positive integers d and n, let $n = p(d + 1) + q$ where $1 \leqslant q \leqslant d + 1$. Then $r(GK(d, n)) = q$.*

Proof. Let $G = GK(d, n)$ and $n = p(d + 1) + q$ where $1 \leqslant q \leqslant d + 1$. Then $\gamma(G) = p + 1$. Since every vertex dominates at most $d + 1$ vertices in G, then $\eta(G) \leqslant n - p(d + 1) = q$.

On the other hand, $D' = \{0, 1, \ldots, p - 1\}$ satisfies $|D'| = \gamma(G) - 1$ and $|N^+[D']| = n - q$. Hence $r(G) = \eta(G) = q$. ∎

Corollary 5.6. *For any positive integers d and n, $r(K(d, n)) = d + 1$.*

6. Conclusions

Using the results in Section 4 we can determine the reinforcement numbers for regular graphs with an E-set. The E-set has close relations to the perfect error-correcting codes and received much attention. There are many important classes of networks for which it is known exactly which graphs in each class have E-sets. These classes include hypercubes, cube-connected cycles, circulant graphs, tori, and so on [4,8,14,16]. In [8] we used these characterizations to determine the bondage number $b(G)$, which was first introduced by Fink et al. [6] as the minimum number of edges whose removal results in a graph with larger domination number. It is also easy to use those characterizations to determine $r(G)$.

Motivated by the note that the bondage number and the reinforcement number are two parallel parameters, we give a comparison of them. Fink et al. [6] conjectured that $b(G) \leqslant \Delta(G)+1$, which was disproved later, while $r(G) \leqslant \upsilon(G)/\gamma(G) \leqslant \Delta(G) + 1$ is an immediate consequence of Corollary 3.2. For a planar graph G, it was proved that $b(G) \leqslant 8$ and conjectured that $b(G) \leqslant 7$; furthermore, no planar graph with $b(G) > 6$ has been constructed yet (see, for example, [3,7,11]). However, Corollary 2.13 shows that $r(G)$ can be any positive integer. For a tree T, Fink et al. [6] show that $b(T) = 1$ or 2, while $r(T)$ can be arbitrarily large by Corollary 2.13. But for a vertex-transitive digraph G, we showed in [8] that $b(G) \geqslant \lceil \upsilon(G)/\gamma(G) \rceil$, while $r(G) \leqslant \lfloor \upsilon(G)/\gamma(G) \rfloor \leqslant b(G)$.

We conclude this paper with some problems.

Problem 1. Prove or disprove that there exists an orientation H for any undirected graph G such that $r(H) = r(G)$.

Problem 2. Determine whether or not the upper bound in Theorem 4.3 is tight for any positive integers ℓ and γ.

Problem 3. Investigate the relationship between $b(G)$ and $r(G)$ for special families of graphs.

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