The Forwarding Indices of Wrapped Butterfly Networks

Xinmin Hou and Jun-Ming Xu
Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China

Min Xu
School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China

Let G be a connected graph. A routing in G is a set of fixed paths for all ordered pairs of vertices in G. The forwarding index of G is the minimum of the largest number of paths specified by a routing passing through any vertex of G taken over all routings in G. This article investigates the forwarding index of a wrapped butterfly graph, determines the exact value for the directed case, and gives an upper bound for undirected case. © **2008 Wiley Periodicals, Inc. NETWORKS, Vol. 53(4), 329–333 2009**

Keywords: forwarding index; wrapped butterfly digraph; wrapped butterfly graph

1. INTRODUCTION

It is well known that a topological structure of a network can be modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network. The study of structural properties of a graph is helpful for designing a suitable topology of a network.

In a communication network, in order to ensure that data are rapidly and safely transmitted from the source to the destination, a convenient way is to predefine a routing, which specifies a path for each ordered pair of vertices, carrying the data transmitted from the source to the destination. It is possible that the fixed paths specified by a given routing pass too frequently through certain vertices, which means that the routing overloads the vertex. The load of any vertex is limited by the capacity of the vertex, otherwise it would affect the efficiency of transmission, even resulting in malfunction of the network. To measure the load of a vertex, Chung et al. [5] proposed the concept of the forwarding index. The forwarding index of a graph G is the minimum of the largest number of paths specified by a routing passing through any vertex of G taken over all routings in G. Clearly, minimizing the forwarding index of a graph will result in maximizing the network capacity. Thus, it becomes very significant to determine the forwarding index of a given graph. However, Saad [22] found that for an arbitrary graph determining its vertex-forwarding index is NP-complete even if the diameter of the graph is two.

It is still of interest to determine the forwarding indices of some special class of graphs that have been used in designing of networks as suitable topologies. Several authors have studied different kinds of graphs, such as Heydemann et al. [12] for hyper-cubes and undirected toroidal meshes, Hou [14] for folded hypercubes, Xu and Xu [26] for augmented cubes, Xu et al. [25] for directed toroidal meshes and generalized hypercubes, Yan et al. [27] for cube-connected cycles, and so on.

The butterfly graph has been extensively used in parallel computer architectures as a suitable topology [4, 10, 19–21]. Some desirable properties of the butterfly graph has been considered in the literature, such as the cycle structure and hamiltonicity [1–3, 15, 24], the development of communication and routing algorithms [6, 7, 16–18], and the spectra [8, 9]. In this article, we investigate the forwarding index of a wrapped butterfly graph, determine the exact value for the directed case and give an upper bound for the undirected case.

The rest of the article is organized as follows. In Section 2, we give some definitions and notations used here. The forwarding index of a wrapped butterfly digraph is determined in Section 3 and an upper bound of the forwarding index of a wrapped butterfly undirected graph is given in Section 4.

2. PRELIMINARIES

Let G be a connected graph (or a strongly connected digraph) of order n. A routing R of G is a set of $n(n-1)$ paths $R(u, v)$ specified for all ordered pairs (u, v) of vertices in G. We define the load of a vertex v in a given routing R of G, denoted by $\xi(G, R, v)$, the number of paths in R going through v, where v is not an end vertex. The forwarding index of a graph G with a routing R is the maximum number of paths in

Received October 2006; accepted July 2008
Correspondence to: X. Hou; e-mail: xmhou@ustc.edu.cn
Contract grant sponsor: NNSF of China; Contract grant numbers: 10701068, 10671191
DOI 10.1002/net.20271
Published online 10 October 2008 in Wiley InterScience (www.interscience.wiley.com).

R going through any vertex v and is denoted by $\xi(G,R)$, that is,

$$\xi(G,R) = \max_{v\in V(G)} \xi(G,R,v).$$

The forwarding index of G, denoted by $\xi(G)$, is defined as the minimum forwarding index over all possible routings of G, that is,

$$\xi(G) = \min_R \xi(G,R).$$

For given positive integers Δ and n, let $\mathbf{Z}_n = \{0,1,\ldots,n-1\}$ be an additive group of integers modular n and $\mathbb{Z}_\Delta^n = \{x_0x_1\cdots x_{n-1} : x_i \in \mathbf{Z}_\Delta\}$. The wrapped butterfly digraph $B_\Delta(n)$ has vertices labeled by ordered pairs $(l;\mathbf{x})$, where $l \in \mathbf{Z}_n$ (l is called a level) and $x \in \mathbb{Z}_\Delta^n$; a vertex $(l;x_0x_1\cdots x_{n-1})$ is adjacent to a vertex $(l+1;x_0\cdots x_{l-1}\alpha x_{l+1}\cdots x_{n-1})$ for every $\alpha \in \mathbf{Z}_\Delta$. Clearly, $B_\Delta(n)$ is a strongly connected Δ-regular digraph with order $N = n\Delta^n$.

The wrapped butterfly graph $UB_\Delta(n)$ can be defined from $B_\Delta(n)$ by replacing each arc $((l;x),(l+1;y))$ by an edge $\{(l;x),(l+1;y)\}$. Clearly, $UB_\Delta(n)$ is a 2Δ-regular graph with order $N = n\Delta^n$ for $n \geq 3$.

It is known that both $B_\Delta(n)$ and $UB_\Delta(n)$ are Cayley graphs. The diameter of $B_\Delta(n)$ is $2n-1$ and the diameter of $UB_\Delta(n)$ is $\lfloor \frac{3n}{2} \rfloor$ (see, for example, [8,11,23,24]).

The following result is powerful for computing the forwarding index of a Cayley digraph (or a Cayley graph).

Lemma 1 ([11,12]). *If $G = (V,E)$ is a Cayley digraph (or a Cayley graph) of order n, then, for any vertex u in G,*

$$\xi(G) = \sum_{v\in V\setminus\{u\}} d(u,v) - (n-1),$$

where $d(u,v)$ denotes the distance from u to v in G.

3. THE FORWARDING INDEX OF $B_\Delta(n)$

The following two properties of $B_\Delta(n)$ can be found in [8].

Lemma 2 ([8]). *For any two vertices $(l;\mathbf{x})$ and $(l';\mathbf{y})$ in $B_\Delta(n)$, we have*

$$\text{(a) if } l \leq l',\ \text{then } d((l;\mathbf{x}),(l',\mathbf{y})) = \begin{cases} l'-l & \text{if } x_i = y_i,\ 0 \leq i < l \text{ or } l' \leq i < n, \\ n+l'-l & \text{otherwise;} \end{cases}$$

$$\text{(b) if } l \leq l',\ \text{then} d((l;\mathbf{x}),(l';\mathbf{y})) = \begin{cases} n-(l-l') & \text{if } x_i = y_i,\ l' \leq i < l \\ 2n-(l-l') & \text{otherwise.} \end{cases}$$

Lemma 3 ([8]). *The wrapped butterfly digraph $B_\Delta(n)$ has girth n.*

In fact, any vertex $(l;\mathbf{x})$ in $B_\Delta(n)$ lies on a unique cycle of length n.

Lemma 4. *Let $(l;\mathbf{x})$ be a given vertex in $B_\Delta(n)$ and $1 \leq k \leq 2n-1$. Then, in $B_\Delta(n)$,*

(1) there are Δ^k vertices at distance k from $(l;\mathbf{x})$ if $1 \leq k \leq n-1$;
(2) there are $\Delta^n - 1$ vertices at distance k from $(l;\mathbf{x})$ if $k = n$;
(3) there are $\Delta^n - \Delta^{k-n}$ vertices at distance k from $(l;\mathbf{x})$ if $n < k \leq 2n-1$.

Proof. Because $B_\Delta(n)$ is vertex-transitive, we need only to compute the number of vertices at distance k, $1 \leq k \leq 2n-1$, from the vertex $(0;\mathbf{0})$. Note that in each level of $B_\Delta(n)$, there are Δ^n vertices.

Consider all vertices $(l;x_0x_1\cdots x_{n-1})$ in the level l, $1 \leq l \leq n-1$. By Lemma 2, the distance $d((0;\mathbf{0}),(l;\mathbf{x})) = l$ if $x_i = 0$ for $l \leq i < n$, or $n+l$ otherwise. By the definition of $B_\Delta(n)$ and Lemma 2, we can easily check that all vertices at distance l and $n+l$ from $(0;\mathbf{0})$, $1 \leq l \leq n-1$, must be contained in the level l. So, the number of vertices at distance l from $(0;\mathbf{0})$ is Δ^l, and the number of vertices at distance $n+l$ from $(0;\mathbf{0})$ is $\Delta^n - \Delta^l$. The results (1) and (3) follow.

Consider the vertices in the level 0. All vertices at distance n from $(0;\mathbf{0})$ are at the level 0 and the vertex $(0;\mathbf{0})$ lies on a unique cycle of length n, thus, there are $\Delta^n - 1$ vertices at distance n from $(0;\mathbf{0})$. The result (2) follows. ∎

Theorem 5. *Let $B_\Delta(n)$ be a wrapped butterfly digraph. Then*

$$\xi(B_\Delta(n)) = \frac{3n(n-1)}{2}\Delta^n - \frac{n(\Delta^n-1)}{\Delta-1} + 1$$

Proof. Because $B_\Delta(n)$ is a Cayley digraph, by Lemma 1 we need only to compute $\sum_{v\neq u} d(u,v)$ for any given vertex $u \in V(B_\Delta(n))$.

By Lemma 4, we have

$$\begin{aligned}\sum_{v\neq u} d(u,v) &= \sum_{i=1}^{n-1} i\Delta^i + n(\Delta^n-1) + \sum_{i=1}^{n-1}(n+i)(\Delta^n-\Delta^i) \\ &= \Delta^n \sum_{i=1}^{n-1}(n+i) - n\sum_{i=1}^{n-1}\Delta^i + n(\Delta^n-1) \\ &= \frac{3n(n-1)}{2}\Delta^n - n\frac{\Delta^n-\Delta}{\Delta-1} + n\Delta^n - n \\ &= \frac{3n(n-1)}{2}\Delta^n - \frac{n(\Delta^n-1)}{\Delta-1} + n\Delta^n.\end{aligned}$$

It follows that,

$$\begin{aligned}\xi(B_\Delta(n)) &= \frac{3n(n-1)}{2}\Delta^n - \frac{n(\Delta^n-1)}{\Delta-1} + n\Delta^n - (n\Delta^n-1) \\ &= \frac{3n(n-1)}{2}\Delta^n - \frac{n(\Delta^n-1)}{\Delta-1} + 1\end{aligned}$$

as required. ∎

4. THE FORWARDING INDEX OF $UB_\Delta(n)$

Throughout this section, we assume that $n \geq 3$. By Lemma 3, the girth of $B_\Delta(n)$ is n. Then, by the definition of $B_\Delta(n)$, there is a unique shortest path between two given vertices in $B_\Delta(n)$ with distance k, $1 \leq k \leq n-1$. Note that the girth of

$UB_\Delta(n)$ is 4 (also see [16]), which implies that the shortest path between two given vertices in $UB_\Delta(n)$ with distance k, $2 \le k \le \lfloor \frac{3n}{2} \rfloor$ is not unique. This fact implies that the structure of shortest paths in $UB_\Delta(n)$ is more complicated than the one in $B_\Delta(n)$. Hence, the computation of $\xi(UB_\Delta(n))$ is more difficult than the one of $\xi(B_\Delta(n))$. In this section, we give an upper bound for the forwarding index of $UB_\Delta(n)$.

Lemma 6. *For any two distinct vertices $(l;\mathbf{x})$ and $(l';\mathbf{y})$ in $UB_\Delta(n)$ with $l \le l'$, we have*

(a) if $l' - l \le \lfloor \frac{n}{2} \rfloor$, then

$$d((l;\mathbf{x}),(l';\mathbf{y})) \le \begin{cases} l'-l & \text{if } x_i = y_i, 0 \le i < l \text{ or } l' \le i < n, \\ n-(l'-l) & \text{if } x_i = y_i, l \le i < l', \\ n+(l'-l) & \text{otherwise;} \end{cases}$$

(b) if $l' - l > \lfloor \frac{n}{2} \rfloor$, then

$$d((l;\mathbf{x}),(l';\mathbf{y})) \le \begin{cases} n-(l'-l) & \text{if } x_i = y_i, l \le i < l', \\ l'-l & \text{if } x_i = y_i, 0 \le i < l \text{ or } l' \le i < n, \\ 2n-(l'-l) & \text{otherwise.} \end{cases}$$

Proof. If $x_i = y_i$ for each $i, 0 \le i < l$ or $l' \le i \le n-1$, then

$$\begin{aligned} P_1 : &(l; x_0 \cdots x_{l-1} x_l x_{l+1} \cdots x_{n-1}) \to \\ &(l+1; x_0 \cdots x_{l-1} y_l x_{l+1} \cdots x_{n-1}) \to \cdots \\ &\to (l'-1; x_0 \cdots x_{l-1} y_l y_{l+1} \cdots y_{l'-2} x_{l'-1} \cdots x_{n-1}) \\ &\to (l'; x_0 \cdots x_{l-1} y_l \cdots y_{l'-1} x_{l'} \cdots x_{n-1}) \end{aligned}$$

is a path of length $l'-l$ between $(l;\mathbf{x})$ and $(l';\mathbf{y})$; if $x_i = y_i$ for each i, $l \le i < l'$, then

$$\begin{aligned} P_2 : (l; x_0 \cdots x_{l-1} x_l x_{l+1} \cdots x_{n-1}) & \\ &\to (l-1; x_0 \cdots y_{l-1} x_l x_{l+1} \cdots x_{n-1}) \to \cdots \\ &\to (l; x_0 y_1 \cdots y_{l-1} x_l \cdots x_{n-1}) \\ &\to (0; y_0 \cdots y_{l-1} x_l \cdots x_{n-1}) \\ &\to (n-1; y_0 \cdots y_{l-1} x_l \cdots x_{n-2} y_{n-1}) \to \cdots \\ &\to (l'; y_0 \cdots y_{l-1} x_l \cdots x_{l'-1} y_{l'} \cdots y_{n-1}) \end{aligned}$$

is a path of length $n-(l'-l)$ between the two given vertices; otherwise, there is a path with n more steps than P_1 or P_2 to reach $(l';\mathbf{y})$, for example one can take the path from $(l;\mathbf{x})$ to $(l;\mathbf{y})$ with n steps then to $(l';\mathbf{y})$ with $l'-l$ or $n-(l'-l)$ steps. Note that the paths constructed above may not be shortest paths between $(l;\mathbf{x})$ and $(l';\mathbf{y})$.

Therefore, the distance

$$d((l;\mathbf{x}),(l';\mathbf{y})) \le \begin{cases} l'-l & \text{if } x_i = y_i, 0 \le i < l \text{ or } l' \le i < n, \\ n-(l'-l) & \text{if } x_i = y_i, l \le i < l', \\ \min\{n+(l'-l), 2n-(l'-l)\} & \text{otherwise.} \end{cases}$$

The result follows. ∎

Lemma 7. *Let $UB_\Delta(n)$ be a wrapped butterfly graph. Then, in the level l with $0 \le l \le n-1$,*

(1) there are Δ^l vertices at distance at most l from $(0;\mathbf{0})$, $\Delta^{n-1}-1$ vertices at distance at most $n-l$ from $(0;\mathbf{0})$, and $\Delta^n - \Delta^l - \Delta^{n-1} + 1$ vertices at distance at most $n+l$ from $(0;\mathbf{0})$ for $1 \le l \le \lfloor \frac{n}{2} \rfloor$;

(2) there are Δ^{n-l} vertices at distance at most $n-l$ from $(0;\mathbf{0})$, $\Delta^l - 1$ vertices at distance at most l from $(0;\mathbf{0})$, and $\Delta^n - \Delta^l - \Delta^{n-l} + 1$ vertices at distance at most $2n-l$ from $(0;\mathbf{0})$ for $l > \lfloor \frac{n}{2} \rfloor$;

(3) for $l = 0$, let $A_i = \{(0; x_0 x_1 \cdots x_{i-1} 0 \cdots 0) | 0 < x_{i-1} < \Delta\}$, $1 \le i \le \lfloor \frac{n}{2} \rfloor$. Then each vertex in A_i is at distance at most $2i$ from $(0;\mathbf{0})$. Let $B_i = \{(0; 0 \cdots 0 x_i \cdots x_{n-1}) | 0 < x_i < \Delta\}$, $\lfloor \frac{n}{2} \rfloor + 1 \le i \le n-1$. Then each vertex in B_i is at distance at most $2(n-i)$ from $(0;\mathbf{0})$. Let $C = \{(0;\mathbf{X}) | \mathbf{X} \ne \mathbf{0}\} - \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} A_i - \bigcup_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1} B_i$. Then each vertex in C is at distance at most n from $(0;\mathbf{0})$.

Proof. The results (1) and (2) can be verified directly from Lemma 6.

To prove (3), we note that

$$\begin{aligned} P_1 : (0;\mathbf{0}) &\to (1; x_0 0 \cdots 0) \to (2; x_0 x_1 0 \cdots 0) \to \cdots \\ &\to (i; x_0 \cdots x_{i-1} 0 \cdots 0) \to (i-1; x_0 \cdots x_{i-1} 0 \cdots 0) \to \cdots \\ &\to (0; x_0 \cdots x_{i-1} 0 \cdots 0) \end{aligned}$$

is a path between $(0;\mathbf{0})$ and $(0; x_0 \cdots x_{i-1} 0 \cdots 0) \in A_i$ of length $2i$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, and

$$\begin{aligned} P_2 : (0;\mathbf{0}) &\to (n-1; 0 \cdots 0 x_{n-1}) \\ &\to (n-2; 0 \cdots 0 x_{n-2} x_{n-1}) \to \cdots \\ &\to (i; 0 \cdots 0 x_i \cdots x_{n-1}) \to (i+1; 0 \cdots 0 x_i \cdots x_{n-1}) \to \cdots \\ &\to (n-1; 0 \cdots 0 x_i \cdots x_{n-1}) \to (0; 0 \cdots 0 x_i \cdots x_{n-1}) \end{aligned}$$

is a path between $(0;\mathbf{0})$ and $(0; 0 \cdots 0 x_i \cdots x_{n-1}) \in B_i$ of length $2(n-i)$ for $\lfloor \frac{n}{2} \rfloor + 1 \le i \le n-1$. Hence the first and the second statements of (3) follow. The third statement of (3) follows from (a) of Lemma 6 directly. ∎

It is easy to compute that $|A_i| = \Delta^i - \Delta^{i-1}$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, $|B_i| = \Delta^{n-i} - \Delta^{n-i-1}$ for $\lfloor \frac{n}{2} \rfloor + 1 \le i \le n-1$, and

$$|C| = \Delta^n - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} |A_i| - \sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1} |B_i| - 1 = \Delta^n - \Delta^{\lfloor \frac{n}{2} \rfloor} - \Delta^{\lceil \frac{n}{2} \rceil - 1} + 1.$$

Theorem 8. *Let $UB_\Delta(n)$ be a wrapped butterfly graph. Then*

$$\xi(UB_\Delta(n)) < \frac{5n^2-4n}{4}\Delta^n - 2n\left(\Delta^{\lfloor \frac{n}{2} \rfloor} + \Delta^{\lceil \frac{n}{2} \rceil - 1}\right) + (3n+1).$$

Proof. By Lemma 7, we have

$$\sum_{l;\mathbf{x}\neq(0;\mathbf{0})} d((0;\mathbf{0}),(l;\mathbf{x}))$$
$$\leq \sum_{l=1}^{\lfloor n/2\rfloor}\Big[l\Delta^l+(n-l)(\Delta^{n-1}-1)$$
$$+(n+l)(\Delta^n-\Delta^l-\Delta^{n-l}+1)\Big]$$
$$\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}\Big[(n-l)\Delta^{n-l}+l(\Delta^l-1)$$
$$+(2n-l)(\Delta^n-\Delta^{n-l}-\Delta^l+1)\Big]$$
$$+\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}2i(\Delta^i-\Delta^{i-1})+\sum_{i=\lfloor\frac{n}{2}\rfloor+1}^{n-1}2(n-i)$$
$$\times(\Delta^{n-i}-\Delta^{n-i-1})+n\big(\Delta^n-\Delta^{\lfloor\frac{n}{2}\rfloor}-\Delta^{\lceil\frac{n}{2}\rceil-1}+1\big)$$
$$=\sum_{l=1}^{\lfloor n/2\rfloor}\Big[(n+l)\Delta^n-n\Delta^l-2l(\Delta^{n-1}-1)\Big]$$
$$+\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}\Big[(2n-l)\Delta^n-n\Delta^{n-l}-2(n-l)(\Delta^l-1)\Big]$$
$$+\sum_{l=1}^{\lfloor\frac{n}{2}\rfloor}2l(\Delta^l-\Delta^{l-1})+\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}2(n-l)$$
$$\times(\Delta^{n-l}-\Delta^{n-l-1})+n\big(\Delta^n-\Delta^{\lfloor\frac{n}{2}\rfloor}-\Delta^{\lceil\frac{n}{2}\rceil-1}+1\big)$$
$$=\left(\sum_{l=1}^{\lfloor n/2\rfloor}(n+l)+\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}(2n-l)\right)\Delta^n$$
$$-n\sum_{l=1}^{\lfloor n/2\rfloor}\Delta^l-n\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}\Delta^{n-l}$$
$$-2\sum_{l=1}^{\lfloor n/2\rfloor}l(\Delta^{n-l}-\Delta^l+\Delta^{l-1}-1)$$
$$-2\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}(n-l)(\Delta^l-\Delta^{n-l}+\Delta^{n-l-1}-1)$$
$$+n\big(\Delta^n-\Delta^{\lfloor\frac{n}{2}\rfloor}-\Delta^{\lceil\frac{n}{2}\rceil-1}+1\big)$$
$$<\left(\sum_{l=1}^{\lfloor n/2\rfloor}(n+l)+\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}(2n-l)\right)\Delta^n$$
$$-n\big(\Delta^{\lfloor\frac{n}{2}\rfloor}+\Delta^{\lceil\frac{n}{2}\rceil-1}-2\big)$$
$$+n\big(\Delta^n-\Delta^{\lfloor\frac{n}{2}\rfloor}-\Delta^{\lceil\frac{n}{2}\rceil-1}+1\big)$$
$$\leq\frac{5n^2-4n}{4}\Delta^n+n\big(\Delta^n-2\Delta^{\lfloor\frac{n}{2}\rfloor}-2\Delta^{\lceil\frac{n}{2}\rceil-1}+3\big),$$

the fourth inequality follows from

$$n\sum_{l=1}^{\lfloor n/2\rfloor}\Delta^l=\frac{n\Delta\big(\Delta^{\lfloor\frac{n}{2}\rfloor}-1\big)}{\Delta-1}>n\big(\Delta^{\lfloor\frac{n}{2}\rfloor}-1\big),$$
$$n\sum_{l=\lfloor\frac{n}{2}\rfloor+1}^{n-1}\Delta^{n-l}=\frac{n\Delta\big(\Delta^{n-\lfloor\frac{n}{2}\rfloor-1}-1\big)}{\Delta-1}>n\big(\Delta^{\lceil\frac{n}{2}\rceil-1}-1\big)$$

and $\Delta^{n-l}-\Delta^l+\Delta^{l-1}-1>0$, $\Delta^l-\Delta^{n-l}+\Delta^{n-l-1}-1>0$.

It follows from Lemma 1 that

$$\xi(UB_\Delta(n))=\sum_{l;\mathbf{x}\neq(0;\mathbf{0})}d((0;\mathbf{0}),(l;\mathbf{x}))-(n\Delta^n-1)$$
$$<\frac{5n^2-4n}{4}\Delta^n+n\big(\Delta^n-2\Delta^{\lfloor\frac{n}{2}\rfloor}$$
$$-2\Delta^{\lceil\frac{n}{2}\rceil-1}+3\big)-(n\Delta^n-1)$$
$$\leq\frac{5n^2-4n}{4}\Delta^n-2n\big(\Delta^{\lfloor\frac{n}{2}\rfloor}+\Delta^{\lceil\frac{n}{2}\rceil-1}\big)+(3n+1)$$

as required. The result follows. ∎

Remark. *In* [12], *Heydemann et al. proved that for any graph G of order n, maximum degree* Δ,

(i) $2\xi(G)+2(n-1)\leq\Delta\pi(G)$,
(ii) $\pi(G)\leq\in(G)+2(n-l)$,

where $\pi(G)$ *denotes the edge-forwarding index of G.*

In [23], *Shahrokhi and Székely proved that the edge-forwarding index of* $UB_2(n)$ *is equal to* $\frac{5n^2}{4}2^{n-1}(l+o(1))$. *Then by (i) and (ii), it is not difficult to compute that*

$$\left[\frac{5n^2}{4}(1+o(1))-4n\right]2^{n-1}+2\leq\xi(UB_2(n))$$
$$\leq\left[\frac{5n^2}{4}(1+o(1))-n\right]2^n+1.$$

Clearly, the upper bound given in Theorem 8 is less than the above upper bound.

REFERENCES

[1] D. Barth and A. Raspaud, Two edge-disjoint Hamiltonian cycles in the butterfly graph, Inform Process Lett 51 (1994), 175–179.

[2] J.-C. Bermond, E. Darrot, O. Delmas, and S. Perennes, Hamilton circuits in the directed wrapped butterfly network, Discrete Appl Math 84 (1998), 21–42.

[3] J.-C. Bermond, E. Darrot, O. Delmas, and S. Perennes, Hamilton cycle decomposition of the butterfly network, Parallel Process Lett 8 (1998), 371–385.

[4] Butterfly$^{\text{TM}}$ Parallel Processor Overview, BBN Report No. 6148, Version 1, BBN Advanced Computers, Inc., Cambridge, MA, March 1986.

[5] F.K. Chung, E.G. Coffman, M.I. Reiman, and B. Simon, The forwarding index of communication networks, IEEE Trans Inform Theory 33 (1987), 224–232.

[6] R. Cole, B. Maggs, and R. Sitaraman, Routing on butterfly networks with random faults, Proc. IEEE Annu Symp Foundations Comput Sci, IEEE Computer Society Press, Los Alamitos, CA, 1995, pp. 558–570.

[7] F. Comellas and G. Giménez, Genetic programming to design communication algorithms for paralllel architectures, Parallel Process Lett 8 (1998), 549–560.

[8] F. Comellas, M.A. Fiol, J. Gimbert, and M. Mitjana, The spectra of wrapped butterfly digraphs, Networks 42 (2003), 15–19.

[9] R. Elsässer, R. Královič, and B. Monien, Scalable sparse topologies with small spectrum, Proc. 18th Annual Symp. on Theoretical Aspects of Computer Science (STACS), A. Ferreira, and H. Reichel (Editors), LNCS 2010, Springer, Berlin, 2001, pp. 218–229.

[10] A. Gottlieb, "An overview of the NYU Ultracomputer Project", Experimental parallel computing architectures, J.J. Dongarra, Editor, Elsevier Science Publishers, B. V., Armsterdam, The Netherlands, 1987, pp. 25–29.

[11] M.C. Heydemann, "Cayley graphs and inteconnection networks", Graph symmetry: Algebraic methods and applications, G. Hahn, and G. Saludussi (Editors), Kluwer Academic Publishers, 1997, pp. 167–226.

[12] M.C. Heydemann, J.C. Meyer, and D. Sotteau, On forwarding indices of networks, Discrete Appl Math 23 (1989), 103–123.

[13] M.C. Heydemann, J.C. Meyer, D. Sotteau, and J. Opatrny, Forwarding indices of consistent routings and their complexity, Networks 24 (1994), 75–82.

[14] X. Hou, M. Xu, and J.-M. Xu, Forwarding indices of folded n-cubes, Discrete Appl Math 145 (2005), 490–492.

[15] S.-C. Hwang and G.-H. Chen, Cycles in butterfly graphs, Networks 35 (2000), 161–171.

[16] R. Klasing, B. Monien, R. Peine, and E.A. Stöhr, Broadcasting in butterfly and de Bruijn networks, Discrete Appl Math 53 (1994), 183–197.

[17] B.M. Maggs and R.K. Sitaraman, Simple algorithms for routing on butterfly networks with bounded queues, SIAM J Comput 28 (1999), 984–1003.

[18] B.M. Maggs and B. Vöcking, Improved routing and sorting on multibutterflies, Algorithmica 28 (2000), 430–437.

[19] T. Nakata, S. Matsushita, N. Tanabe, N. Kajihara, H. Onozuka, Y. Asano, and N. Koike, Parallel programming on Cenju: A multiprocessor system for mudular circuit simulation, NEC Res Dev 32 (1991), 421–429.

[20] W. Oed, Cray Y-MP C90: System features and early benchmark results, Parallel Comput 18 (1992), 947–954.

[21] G.F. Pfister, W.C. Brantley, D.A. George, S.L. Harvey, W.J. Kleinfelder, K.P. McAuliffe, E.A. Melton, V.A. Norton, and J. Weiss, "An introduction to the IBM Research Parallel Processor Prototype (RP3)", Experimental parallel computing architectures, J.J. Dongarra (Editor), Elsevier Science Publishers, B. V., Armsterdam, The Netherlands, 1987, pp. 123–140.

[22] R. Saad, Complexity of the forwrding index problem, SIAM J Discrete Math 6 (1993), 418–427.

[23] F. Shahrokhi and L.A. Székely, Constructing integral uniform flows in symmetric networks with application to the edge-forwarding index problem, Discrete Appl Math 108 (2001), 175–191.

[24] S.A. Wong, Hamilton cycles and paths in butterfly graphs, Networks 26 (1995), 145–150.

[25] J.-M. Xu, M. Xu, and X.-M. Hou, Forwarding indices of Cartesian product graphs, Taiwanese J Math 10 (2006), 1305–1315.

[26] M. Xu and J.-M. Xu, The forwarding index of augmented cubes, Inform Process Lett 101 (2007), 185–189.

[27] J. Yan, J.-M. Xu, and C. Yang, Forwarding index of cube-connected cycles, Discrete Appl Math, (2008), doi:10.1016/j.dam.2008.04.011.