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Bounded edge-connectivity and edge-persistence of Cartesian product of graphs☆

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ABSTRACT

The bounded edge-connectivity $\lambda_k(G)$ of a connected graph G with respect to $k\,(\geq d(G))$ is the minimum number of edges in G whose deletion from G results in a subgraph with diameter larger than k and the edge-persistence $D^+(G)$ is defined as $\lambda_{d(G)}(G)$, where $d(G)$ is the diameter of G. This paper considers the Cartesian product $G_1 \times G_2$, shows $\lambda_{k_1+k_2}(G_1 \times G_2) \geq \lambda_{k_1}(G_1) + \lambda_{k_2}(G_2)$ for $k_1 \geq 2$ and $k_2 \geq 2$, and determines the exact values of $D^+(G)$ for $G = C_n \times P_m$, $C_n \times C_m$, $Q_n \times P_m$ and $Q_n \times C_m$.

1. Introduction

We follow [24] for graph-theoretical terminology and notation not defined here. Throughout this paper, a graph $G = (V, E)$ always means a connected and simple graph (without loops and multiple edges), where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. It is well known that the underlying topology of an interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

Let x and y be two distinct vertices in a graph $G = (V, E)$. The distance $d_G(x, y)$ between x and y is the number of edges in the shortest xy-path, and the diameter of G is $d(G) = \max\{d_G(x, y) : x, y \in V(G)\}$. It is quite natural that, when an interconnection network is modeled by a graph G, the diameter $d(G)$ directly depicts transmission delay of the network if the store-forward time of messages is the same at every vertex. Thus, the diameter is often taken as a measure of efficiency, which is an important parameter to measure the performance of an interconnection network. In order to improve or increase the efficiency of message transmission we need to minimize the diameter of the graph. This is the reason why this concept has received considerable attention in the literature. Many famous graph-theoreticians were interested in this topics, such as Erdős, Rényi, and Sós in [8–10], Alon, Gyárfás, and Ruszinkó [1], Harary [12], Chung [6,7], and so on. The interested reader is referred to the survey paper [3] for early results.

Since some link faults may happen when a network is put into use, it is practically meaningful and important to consider faulty networks. In other words, the removal of some edges in a graph may result in increasing of diameter of the remaining graph, which motivated Chung and Garey [7] to propose the following concept. The edge-fault-tolerant diameter $D_t(G)$ of a t-edge-connected graph G is defined as

$$D_t(G) = \max\{d(G - F) : F \subset E(G), |F| < t\}.$$

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On the other hand, in a real-time system, the message delay must be limited within a given period since any message obtained beyond the bound may be worthless. A natural question is how many faulty links at most can synchronously happen in the network to ensure message delay within the effective bounds. In the language of graph theory, this problem can be stated as follows. At most how many edges can be removed from a graph to ensure no increase of diameter of the remaining graph. In the literature, this question is called the "edge-deletion problem". However, this problem is quite difficult in general, since it has been proved to be NP-complete by Schoone, Bodlaender and van Leeuwen [20].

To investigate further this problem mentioned above, Exoo [11], motivated from Boesch et al. [4], proposed a measure of network vulnerability, called the edge-persistence. The *edge-persistence* $D^+(G)$ of a graph G is the minimum number of edges whose deletion from G increases the diameter of G. For example, $D^+(P_m) = D^+(C_n) = 1$, where P_m is a path of order m and C_n is a cycle of length n. Motivated by Lovász, Neumann-Lara and Plummer [14], Xu [23] generalized this concept to more general case, called the bounded edge-connectivity.

For any positive integer k and $x, y \in V(G)$, the *xy-bounded edge-connectivity* $\lambda_k(G; x, y)$ with respect to k is the minimum number of edges in G whose deletion destroys all xy-paths of length at most k. The *bounded edge-connectivity* of G with respect to k is defined as

$$\lambda_k(G) = \min\{\lambda_k(G; x, y) : x, y \in V(G)\}.$$

Clearly, $\lambda_k(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G. If $k \leq d(G) - 1$, then $\lambda_k(G) = 0$. Thus, we assume that $k \geq d(G)$ in this paper. Specially, $\lambda_1(G) = 1$ if and only if $G = K_m$ is a complete graph of order $m \geq 2$. It is also clear that $\lambda_k(G) = D^+(G)$ if $k = d(G)$, and $\lambda_{n-1}(G) = \lambda(G)$, the classical edge-connectivity of G, if $n = |V(G)|$. Thus, the bounded edge-connectivity is a generalization of both the edge-persistence and the classical edge-connectivity.

In [23], Xu established the relationships between $\lambda_k(G)$ and $D_t(G)$ as follows. For any connected graph G,

(a) $\lambda_k(G) = t \Leftrightarrow D_t(G) \leq k < D_{t+1}(G)$ if G is $(t + 1)$-edge-connected, or

(b) $D_t(G) = k \Leftrightarrow \lambda_{k-1}(G) < t \leq \lambda_k(G)$ if G is t-edge-connected.

The three parameters $\lambda_k(G)$, $D_t(G)$ and $D^+(G)$ can be viewed as important measures of the vulnerability of communication networks modeled as graphs and, thus, have received much research attention in the past years, see, for example, [4,5,7,11,12,15–18,20–23,25].

We consider the Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2. For graphs G_1 and G_2, the Cartesian product $G_1 \times G_2$ is the graph with vertex-set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge-set $E(G_1 \times G_2) = \{(x_1, x_2)(y_1, y_2) | x_1 = y_1$ and $x_2y_2 \in E(G_2)$ or $x_2 = y_2$ and $x_1y_1 \in E(G_1)\}$.

It is well known that the Cartesian product is an important research topic in graph theory (see, e.g., [13]). It is also well known that, for designing large-scale interconnection networks, the Cartesian product is an important method to obtain large graphs from smaller ones, with a number of parameters that can be easily calculated from the corresponding parameters for those small initial graphs. The Cartesian product preserves many nice properties such as regularity, existence of Hamilton cycles and Euler circuits, and transitivity of the initial graphs (see, e.g., [23]). In fact, many well-known networks can be constructed by the Cartesian products of some simple graphs. For example, the n-dimensional hypercube Q_n is the Cartesian product of n complete graphs of order 2, a torus is the Cartesian product of two cycles, and a mesh is the Cartesian product of two paths.

What we are interested in is the bounded edge-connectivity and edge-persistence of the Cartesian product of graphs. Graham and Harary [12] showed $D^+(Q_n) = n - 1$; Sung and Wang [21] investigated $D^+(C_m \times C_n)$, etc., and conjectured $D^+(G_1 \times G_2) \geq \max\{D^+(G_1), D^+(G_2)\} + 1$.

In this paper, we first establish a lower bound of λ_k for the Cartesian product $G_1 \times G_2$, that is, $\lambda_{k_1+k_2}(G_1 \times G_2) \geq \lambda_{k_1}(G_1) + \lambda_{k_2}(G_2)$ for $k_i \geq 2$, $i = 1, 2$. As an immediate consequence, we obtain $D^+(G_1 \times G_2) \geq D^+(G_1) + D^+(G_2)$ if $d(G_i) \geq 2$ for $i = 1, 2$. This lower bound is tight, and gives an affirmative answer to the above-mentioned conjecture of Sung and Wang if the diameters of both G_1 and G_2 are at least two. Then we determine $D^+(C_n \times P_m) = 1$ for $n = 3$ and 2 for $n \geq 4$; $D^+(C_n \times C_m) = 2$ if $n = 3$ or $m = 3$ or both n and m are odd, 3 otherwise. Lastly, we determine $D^+(Q_n \times P_m) = n$ for $n \geq 2$ and $m \geq 2$; $D^+(Q_n \times C_m) = n$ for $m = 3$, $n + 1$ for $m \geq 4$. These results correct some inaccurate results on $D^+(C_n \times C_m)$ in [21].

The rest of the paper is organized as follows. In Section 2 we establish the lower bound of $\lambda_{k_1+k_2}(G_1 \times G_2)$. The results on $D^+(G_1 \times G_2)$ for some P_n, C_n and Q_n are presented in Section 3. The conclusions and remarks are in Section 4.

2. Bounded edge-connectivity

For a vertex $x \in V(G_1)$ and a subgraph $H \subseteq G_2$, we use xH to denote the subgraph of $G_1 \times G_2$ induced by $\{x\} \times V(H)$. Similarly, for a vertex $y \in V(G_2)$, a subgraph $H \subseteq G_1$, Hy denotes the subgraph of $G_1 \times G_2$ induced by $V(H) \times \{y\}$. For a path $P = x_1 \cdots x_i \cdots x_j \cdots x_n$ in G, $P(x_i, x_j)$ denotes the section $x_i \cdots x_j$ of P. For the sake of convenience, we will denoted P as

$$P = x_1 \xrightarrow{P(x_1,x_i)} x_i \xrightarrow{P(x_i,x_j)} x_j \xrightarrow{P(x_j,x_n)} x_n.$$

The symbol $\varepsilon(P)$ denotes the length of P, which is the number of edges in P.

Now, we state our main result in this paper.

Theorem 1. *For any connected graphs G_1 and G_2, if $k_i \geq 2$ for $i = 1, 2$, then*

$$\lambda_{k_1+k_2}(G_1 \times G_2) \geq \lambda_{k_1}(G_1) + \lambda_{k_2}(G_2).$$

Proof. Let $G = G_1 \times G_2$, $\lambda_{k_i}(G_i) = n_i$ for $i = 1, 2$. Without loss of generality, assume that $n_1 \geq n_2$. Then $n_1 \geq n_2 \geq 1$ since $k_i \geq d(G_i)$ for $i = 1, 2$. To prove the theorem, we only need to show that $d(G - F) \leq k_1 + k_2$ for any subset $F \subset E(G)$ with $|F| \leq n_1 + n_2 - 1$. To this end, we only need to show that for any two vertices $(x_1, x_2), (y_1, y_2) \in V(G)$ there is an $(x_1, x_2)(y_1, y_2)$-path L of length $\varepsilon(L) \leq k_1 + k_2$ in $G - F$. For any $u \in V(G_1)$ and $v \in V(G_2)$, let

$$F_1(v) = E(G_1 v) \cap F, \qquad F_2(u) = E(uG_2) \cap F,$$
$$S_1 = \{xy \in E(G_1) : x, y \in V(G_1), \exists\, r \in V(G_2) \text{ such that } (x, r)(y, r) \in F\}.$$

We consider the following three cases:

Case 1. $x_1 \neq y_1$, $x_2 \neq y_2$.

Subcase 1.1. $|F_1(v)| \geq n_1$ for some $v \in V(G_2)$.

In this subcase, $|F_1(u)| \leq n_2 - 1$ for any $u \in V(G_2 - v)$ and $|F_2(w)| \leq n_2 - 1$ for any $w \in V(G_1)$ since $|F| \leq n_1 + n_2 - 1$ and $n_1 \geq n_2$. Since x_2 and y_2 are two distinct vertices of G_2, at least one of x_2 and y_2 is not v. So we can, without loss of generality, assume that $y_2 \neq v$. Then $|F_1(y_2)| \leq n_2 - 1 \leq n_1 - 1$.

Since $G_1 y_2 \cong G_1$, $x_1 G_2 \cong G_2$ and $\lambda_{k_1}(G_1) = n_1$, $\lambda_{k_2}(G_2) = n_2$, there exist an $(x_1, y_2)(y_1, y_2)$-path P_1 in $G_1 y_2 - F_1(y_2)$ such that $\varepsilon(P_1) \leq k_1$ and an $(x_1, x_2)(x_1, y_2)$-path P_2 in $x_1 G_2 - F_2(x_1)$ such that $\varepsilon(P_2) \leq k_2$. Thus, the path

$$L_1 = (x_1, x_2) \xrightarrow{P_2} (x_1, y_2) \xrightarrow{P_1} (y_1, y_2)$$

is an $(x_1, x_2)(y_1, y_2)$-path in $G - F$ and $\varepsilon(L_1) = \varepsilon(P_1) + \varepsilon(P_2) \leq k_1 + k_2$.

Subcase 1.2. $|F_1(v)| \leq n_1 - 1$ for any $v \in V(G_2)$.

If $|F_2(x_1)| \leq n_2 - 1$ or $|F_2(y_1)| \leq n_2 - 1$, without loss of generality, assume that $|F_2(x_1)| \leq n_2 - 1$, then there exists an $(x_1, x_2)(x_1, y_2)$-path P_3 in $x_1 G_2 - F_2(x_1)$ such that $\varepsilon(P_3) \leq k_2$. By $|F_1(y_2)| \leq n_1 - 1$, there exists an $(x_1, y_2)(y_1, y_2)$-path P_4 in $G_1 y_2 - F_1(y_2)$ such that $\varepsilon(P_4) \leq k_1$. Thus, the path

$$L_2 = (x_1, x_2) \xrightarrow{P_3} (x_1, y_2) \xrightarrow{P_4} (y_1, y_2)$$

is an $(x_1, x_2)(y_1, y_2)$-path in $G - F$ and $\varepsilon(L_2) = \varepsilon(P_4) + \varepsilon(P_3) \leq k_1 + k_2$.

Now assume that $|F_2(x_1)| \geq n_2$ and $|F_2(y_1)| \geq n_2$. Then $\sum_{u \in V(G_1)} |F_2(u)| \geq 2n_2$. Let M be the set of interior vertices of all $x_1 y_1$-paths of length at most k_1 in $G_1' = G_1 - S_1$.

If $M = \emptyset$, then any $x_1 y_1$-path in G_1' is either a single edge $x_1 y_1$ or of length at least $k_1 + 1$. Since

$$\begin{aligned} |S_1| &\leq \sum_{v \in V(G_2)} |F_1(v)| = |F| - \sum_{u \in V(G_1)} |F_2(u)| \\ &\leq n_1 - n_2 - 1 \leq \lambda_{k_1}(G_1) - 2, \end{aligned}$$

$G_1' - x_1 y_1$ is connected and any $x_1 y_1$-path in $G_1' - x_1 y_1$ is of length at least $k_1 + 1$ by the definition of $\lambda_{k_1}(G_1)$, which implies that $d_{G_1' - x_1 y_1}(x_1, y_1) \geq k_1 + 1$. Hence $\lambda_{k_1}(G_1) \leq |S_1| + 1 \leq \lambda_{k_1}(G_1) - 1$, a contradiction.

Thus, $M \neq \emptyset$. Assume that $|F_2(u)| \geq n_2$ for any $u \in M$. Then

$$\sum_{u \in V(G_1)} |F_2(u)| \geq 2n_2 + \sum_{u \in M} |F_2(u)| \geq (|M| + 2)n_2,$$

and so

$$\begin{aligned} |S_1| &\leq |F| - \sum_{u \in V(G_1)} |F_2(u)| \leq n_1 + n_2 - 1 - (|M| + 2)n_2 \\ &= n_1 - (|M| + 1)n_2 - 1. \end{aligned}$$

This fact and $\lambda_{k_1}(G_1) = n_1$ imply that at least $(|M| + 1)n_2 + 1$ edges must be deleted from G_1' to increase diameter of G_1' to (at least) $k_1 + 1$.

On the other hand, by the definition of M, there are at most $|M| + 1$ edges incident to x_1 in the induced subgraph $G_1'[M \cup \{x_1, y_1\}]$ of G_1' whose deletion results in no $x_1 y_1$-paths of length at most k_1 in G_1'. That is, we can delete $|M| + 1$ edges from G_1' whose diameter can be increased to (at least) $k_1 + 1$, a contradiction.

Thus, $|F_2(u)| \leq n_2 - 1$ for some $u \in M$. There exist an $x_1 y_1$-path P_5 of length at most k_1 in $G_1 - S_1$ with $u \in M$ and an $(u, x_2)(u, y_2)$-path P_6 with length at most k_2 in $uG_2 - F_2(u)$. Thus, the path

$$L_3 = (x_1, x_2) \xrightarrow{P_5(x_1, u)x_2} (u, x_2) \xrightarrow{P_6} (u, y_2) \xrightarrow{P_5(u, y_1)y_2} (y_1, y_2)$$

is an $(x_1, y_1)(x_2, y_2)$-path in $G - F$ with length $\varepsilon(L_3) = \varepsilon(P_5) + \varepsilon(P_6) \leq k_1 + k_2$.

Case 2. $x_1 \neq y_1$, $x_2 = y_2$.

Subcase 2.1. If $|F_1(x_2)| \le n_1 - 1$, then by $\lambda_{k_1}(G_1) = n_1$ and $G_1x_2 \cong G_1$, there exists an $(x_1, x_2)(y_1, y_2)$-path of length at most $k_1(< k_1 + k_2)$ in the subgraph $G_1x_2 - F_1(x_2)$ of $G - F$.

Subcase 2.2. If $|F_1(x_2)| \ge n_1$, then $|F - F_1(x_2)| \le n_2 - 1$. Furthermore, there must exist two vertices $(x_1, y), (y_1, y) \in V(G)$ such that $(x_1, x_2)(x_1, y), (y_1, y)(y_1, y_2) \in E(G - F)$ since $\lambda_{k_2}(G_2) = n_2 \le \delta(G_2)$.

By $|F_1(y)| \le |F - F_1(x_2)| \le n_2 - 1 \le n_1 - 1$ and $G_1y \cong G$, there is an $(x_1, y)(y_1, y)$-path P_7 of length at most k_1 in the subgraph $G_1y - F_1(y)$ of $G - F$. Thus, by $k_2 \ge 2$, the path

$$L_4 = (x_1, x_2) \to (x_1, y) \xrightarrow{P_7} (y_1, y) \to (y_1, y_2)$$

is an $(x_1, x_2)(y_1, y_2)$-path in $G - F$ and $\varepsilon(L_4) \le k_1 + 2 \le k_1 + k_2$.

Case 3. $x_1 = y_1,\ x_2 \ne y_2$

Subcase 3.1. If $|F_2(x_1)| \le n_2 - 1$, then by $\lambda_{k_2}(G_2) = n_2$ and $x_1G_2 \cong G_2$, there exists an $(x_1, x_2) - (y_1, y_2)$-path of length at most $k_2(< k_1 + k_2)$ in the subgraph $x_1G_2 - F_2(x_1)$ of $G - F$.

Subcase 3.2. If $|F_2(x_1)| \ge n_2$, then let $|N_{G_1'}(x_1)| = \delta_{G_1'}(x_1) = \delta_1'$.

We claim that $\delta_1' \ge 1$. Otherwise,

$$|S_1| \ge \delta_{G_1}(x_1) \ge \delta(G_1) \ge \lambda_{k_1}(G_1) = n_1,$$

and so

$$\begin{aligned} n_1 + n_2 &\le |S_1| + |F_2(x_1)| \le \sum_{v\in V(G_2)} |F_1(v)| + \sum_{u\in V(G_1)} |F_2(u)| \\ &= |F| = n_1 + n_2 - 1, \end{aligned}$$

a contradiction.

Assume that $|F_2(u)| \ge n_2$ for any $u \in N_{G_1'}(x_1)$. Then

$$\sum_{u\in V(G_1)} |F_2(u)| \ge |F_2(x_1)| + \sum_{u\in N_{G_1'}(x_1)} |F_2(u)| \ge n_2 + \delta_1' n_2 = (\delta_1' + 1)n_2$$

and

$$|S_1| \le \sum_{v\in V(G_2)} |F_1(v)| = n_1 + n_2 - 1 - \sum_{u\in V(G_1)} |F_2(u)| \le n_1 - \delta_1' n_2 - 1.$$

Thus, we deduce a contradiction as follows.

$$\begin{aligned} \delta_1' &= \delta_{G_1'}(x_1) \ge \lambda_{k_1}(G_1') \ge \lambda_{k_1}(G) - |S_1| \\ &\ge n_1 - (n_1 - \delta_1' n_2 - 1) = \delta_1' n_2 + 1. \end{aligned}$$

Therefore, $|F_2(u)| \le n_2 - 1$ for some $u \in N_{G_1'}(x_1)$. Since $\lambda_{k_2}(G_2) = n_2$ and $uG_2 \cong G_2$, there exists an $(u, x_2)(u, y_2)$-path P_8 of length at most k_2 in the subgraph $uG_2 - F_2(u)$ of $G - F$. Thus, the path

$$L_5 = (x_1, x_2) \to (u, x_2) \xrightarrow{P_8} (u, y_2) \to (y_1, y_2)$$

is an $(x_1, x_2)(y_1, y_2)$-path in $G - F$, and by $k_1 \ge 2$,

$$\varepsilon(L_5) = \varepsilon(P_8) + 2 \le k_1 + k_2.$$

The proof is complete. □

As an immediate consequence of Theorem 1, we obtain a lower bound of the edge-persistence of the Cartesian products of two graphs.

Corollary 2. *$D^+(G_1 \times G_2) \ge D^+(G_1) + D^+(G_2)$ if $d(G_1) \ge 2$ and $d(G_2) \ge 2$.*

3. Edge-persistence

In this section, by Corollary 2 we determine the edge-persistence of the Cartesian products of some simple graphs, such as a path P_m, a cycle C_m and a hypercube Q_n. These examples show that the lower bound given in Corollary 2 is the best possible.

Lemma 3 (*Theorem 2.3.3 in [23]*). *Let $x = (x_1, x_2), y = (y_1, y_2) \in V(G_1 \times G_2)$. Then $d_{G_1\times G_2}(x, y) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$ and, hence, $d(G_1 \times G_2) = d(G_1) + d(G_2)$.*

Theorem 4. *For any $n \ge 3$ and $m \ge 2$,*

$$D^+(C_n \times P_m) = \begin{cases} 1 & \text{for } n = 3; \\ 2 & \text{for } n \ge 4. \end{cases}$$

Proof. Since $d(C_n) = \lfloor \frac{n}{2} \rfloor$ and $d(P_m) = m-1$, we have $d(C_n \times P_m) = \lfloor \frac{n}{2} \rfloor + m - 1$ by Lemma 3. Let $V(C_n) = \{0, 1, 2, \ldots, n-1\}$ and $V(P_m) = \{0, 1, 2, \ldots, m-1\}$.

For $n = 3$, let $e = (0, 0)(0, 1) \in E(C_3 \times P_m)$, and $T = C_3 \times P_m - e$. Then

$$d(T) \geq d_T((0, 0), (0, m-1)) = 2 + m - 1 = m + 1 > m = d(C_3 \times P_m).$$

So $D^+(C_3 \times P_m) \leq 1$. On the other hand, $D^+(C_3 \times P_m) \geq 1$ clearly. Thus, $D^+(C_3 \times P_m) = 1$.

Now we show that $D^+(C_n \times P_m) = 2$ for $n \geq 4$. It is also easy to verify $D^+(C_n \times P_2) = 2$. So assume that $n \geq 4$ and $m \geq 3$ below. By Corollary 2, we have

$$D^+(C_n \times P_m) \geq D^+(C_n) + D^+(P_m) \geq 2.$$

Let $G = C_n \times P_m$, $G' = G - \{(0, 0)(0, 1), (0, 0)(1, 0)\}$, $u = (0, 0)$, $w = (n-1, 0)$ and $v = (\lfloor \frac{n}{2} \rfloor - 1, m-1)$. Then by Lemma 3,

$$\begin{aligned} d(G') &\geq d_{G'}(u, v) = 1 + d_{G'}(w, v) \geq 1 + d_G(w, v) \\ &= 1 + [1 + (\lfloor n/2 \rfloor - 1)] + (m-1) \\ &= \lfloor n/2 \rfloor + m = 1 + d(G). \end{aligned}$$

Hence $D^+(C_n \times P_m) \leq 2$, and so $D^+(C_n \times P_m) = 2$. □

Theorem 5. *For $n \geq 3$ and $m \geq 3$,*

$$D^+(C_n \times C_m) = \begin{cases} 2 & \text{if } n = 3 \text{ or } m = 3 \text{ or both } n \text{ and } m \text{ are odd;} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let $V(C_n) = \{0, 1, \ldots, n-1\}$ and $V(C_m) = \{0, 1, \ldots, m-1\}$, $G = C_n \times C_m$. It is sufficient to prove that the theorem holds for $n \geq 4$ and $m \geq 4$.

Case 1. Both n and m are odd. On the one hand, by Corollary 2, we have

$$D^+(G) \geq D^+(C_n) + D^+(C_m) = 2.$$

On the other hand, since n and m are odd, $d_{C_n}(n-1, \lfloor \frac{n}{2} \rfloor) = \lfloor \frac{n}{2} \rfloor$ and $d_{C_m}(m-1, \lfloor \frac{m}{2} \rfloor) = \lfloor \frac{m}{2} \rfloor$. Let $e_1 = (0, 0)(0, 1)$ and $e_2 = (0, 0)(1, 0)$ be two edges in G, $G' = G - \{e_1, e_2\}$ and $u = (0, 0)$, $v = (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)$. Then, by Lemma 3,

$$\begin{aligned} d(G') &\geq d_{G'}(u, v) \\ &= 1 + \min\{d_{G'}((0, m-1), v), d_{G'}((n-1, 0), v)\} \\ &\geq 1 + \min\{d_G((0, m-1), v), d_G((n-1, 0), v)\} \\ &= 1 + \min\{\lfloor n/2 \rfloor + d_{C_m}(m-1, \lfloor m/2 \rfloor), d_{C_n}(n-1, \lfloor n/2 \rfloor) + \lfloor m/2 \rfloor\} \\ &= 1 + \lfloor n/2 \rfloor + \lfloor m/2 \rfloor \\ &= 1 + d(G), \end{aligned}$$

which implies $D^+(G) \leq 2$. Thus, $D^+(G) = 2$.

Case 2. At most one of m and n is odd.

We first prove that $D^+(G) \geq 3$. It is sufficient to show that there are at least three edge-disjoint uv-paths of length at most $d(G)$ between any two distinct vertices u and v in G, which implies that at least three edges must be removed from G to increase the diameter.

Notice that G is vertex-transitive since C_n is vertex-transitive. Without loss of generality, let $u = (0, 0)$, $v = (x, y) \in V(G)$ and $x \leq \lfloor \frac{n}{2} \rfloor$, $y \leq \lfloor \frac{m}{2} \rfloor$.

If $x = 0$ or $y = 0$, without loss of generality, say $y = 0$, then $x \neq 0$, that is $v = (x, 0)$. Three internally vertex-disjoint uv-paths are constructed as follows.

$$\begin{aligned} P_1 &= (0, 0)(1, 0) \cdots (x, 0), \\ P_2 &= (0, 0)(0, 1)(1, 1) \cdots (x, 1)(x, 0), \\ P_3 &= (0, 0)(0, m-1)(1, m-1) \cdots (x, m-1)(x, 0). \end{aligned}$$

Clearly, $\varepsilon(P_1) = x \leq \lfloor \frac{n}{2} \rfloor$, $\varepsilon(P_2) = \varepsilon(P_3) = 2 + x \leq 2 + \lfloor \frac{n}{2} \rfloor$. Thus, these paths are of length at most $\lfloor \frac{n}{2} \rfloor + 2 \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor = d(G)$ for $m \geq 4$.

If $x \neq 0$ and $y \neq 0$, then there exist four internally vertex-disjoint uv-paths as follows.

$$\begin{aligned} P_4 &= (0, 0)(0, 1) \cdots (0, y)(1, y) \cdots (x, y), \\ P_5 &= (0, 0)(1, 0) \cdots (x, 0)(x, 1) \cdots (x, y), \\ P_6 &= (0, 0)(0, m-1) \cdots (0, y+1)(1, y+1) \cdots (x, y+1)(x, y), \\ P_7 &= (0, 0)(n-1, 0) \cdots (x+1, 0)(x+1, 1) \cdots (x+1, y)(x, y). \end{aligned}$$

Clearly,

$$\varepsilon(P_4) = \varepsilon(P_5) = x + y \le \lfloor n/2 \rfloor + \lfloor m/2 \rfloor = d(G),$$
$$\varepsilon(P_6) = m - y + x, \qquad \varepsilon(P_7) = n - x + y.$$

Since $\varepsilon(P_6) + \varepsilon(P_7) = n + m$ and at most one of n and m is odd, there is at least one of $\varepsilon(P_6)$ and $\varepsilon(P_7)$, without loss of generality, say $\varepsilon(P_6)$, such that

$$\varepsilon(P_6) \le \lfloor (n+m)/2 \rfloor = \lfloor n/2 \rfloor + \lfloor m/2 \rfloor = d(G).$$

Thus, we have $D^+(G) \ge 3$.

Now, we prove $D^+(G) \le 3$. To the end, we only need to find three edges e_1, e_2, e_3 in G such that $d(G - \{e_1, e_2, e_3\}) > d(G)$. Let $e_1 = (0, 0)(0, 1)$, $e_2 = (0, 0)(1, 0)$, $e_3 = (0, 0)(0, n-1)$, $u = (0, 0)$, $v = (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor)$ and $G'' = G - \{e_1, e_2, e_3\}$. Since, by Lemma 3,

$$\begin{aligned} d_{G''}((n-1, 0), v) &\ge d_G((n-1, 0), v) \\ &= d_{C_n}(n-1, \lfloor n/2 \rfloor) + d_{C_m}(0, \lfloor m/2 \rfloor) \\ &= \lfloor n/2 \rfloor + \lfloor m/2 \rfloor = d(G), \end{aligned}$$

we have

$$d(G'') \ge d_{G''}(u, v) = 1 + d_{G''}((n-1, 0), v) \ge 1 + d(G).$$

Thus, $D^+(G) \le 3$, and so $D^+(G) = 3$. The theorem follows. □

We now consider the Cartesian products of n-dimensional hypercube Q_n and a path P_m or a cycle C_m.

The n-dimensional hypercube Q_n has the vertex-set $V = \{x_1 x_2 \cdots x_n : x_i \in \{0, 1\}, i = 1, 2, \ldots, n\}$, and two vertices x and y are linked by an edge if and only if they differ exactly in one coordinate. The n-dimensional hypercube Q_n can also be defined as the Cartesian product $K_2 \times K_2 \times \cdots \times K_2$ of n identical complete graph K_2. The hypercube has many excellent features, and, thus becomes the first choice for the topological structure of parallel processing and computing systems, and have been much studied in network theory, see, for example, [23].

Lemma 6 (*[2,19,23]*). *Let x and y be two vertices in Q_n and $d_{Q_n}(x, y) = d$. Then exist a d-dimensional subcube in Q_n in which there are d internally disjoint xy-paths of length d. Moreover, there exist n internally disjoint xy-paths such that d of which are of length d, otherwise of length $d + 2$.*

Lemma 7 (*[12]*). $D^+(Q_n) = n - 1$.

Theorem 8. *$D^+(Q_n \times P_m) = n$ for $n \ge 2$ and $m \ge 2$.*

Proof. If $m = 2$, then $Q_n \times P_2 = Q_{n+1}$. Thus, $D^+(Q_n \times P_2) = D^+(Q_{n+1}) = n$ by Lemma 7.

Assume that $n \ge 2$ and $m \ge 3$ below. Then $d(Q_n) \ge 2$ and $d(P_m) \ge 2$. By Corollary 2 and Lemma 7,

$$D^+(Q_n \times P_m) \ge D^+(Q_n) + D^+(P_m) = n - 1 + 1 = n.$$

To complete the proof of the theorem, we only need to prove $D^+(Q_n \times P_m) \le n$. To this end, we only need to find a set F of n edges in $Q_n \times P_m$ such that $d(Q_n \times P_m - F) > d(Q_n \times P_m)$.

Let $\{0, 1, \ldots, m-1\}$ be the vertex-set of P_m, $u = (00 \cdots 0, 0)$, $u_1 = (10 \cdots 0, 0)$, $u_2 = (01 \cdots 0, 0), \ldots, u_n = (00 \cdots 1, 0)$, $u_{n+1} = (00 \cdots 0, 1)$, $v = (01 \cdots 1, m-1)$ be $n + 3$ vertices and $F = \{uu_2, \ldots, uu_n, uu_{n+1}\}$ be a set of edges in $Q_n \times P_m$. By Lemma 3, the distance between u and v in $Q_n \times P_m - F$ is

$$\begin{aligned} d_{Q_n \times P_m - F}(u, v) &= 1 + d_{Q_n \times P_m - F}(u_1, v) \ge 1 + d_{Q_n \times P_m}(u_1, v) \\ &= 1 + n + m - 1 = 1 + d(Q_n \times P_m). \end{aligned}$$

This fact shows that $D^+(Q_n \times P_m) \le n$. The theorem follows. □

Theorem 9. *For any $n \ge 1$ and $m \ge 3$,*

$$D^+(Q_n \times C_m) = \begin{cases} n & \text{for } m = 3, \\ n + 1 & \text{for } m \ge 4. \end{cases}$$

Proof. If $n = 1$ then, by Theorem 4, we have

$$D^+(Q_1 \times C_m) = D^+(P_2 \times C_m) = \begin{cases} 1 & \text{for } m = 3, \\ 2 & \text{for } m \ge 4. \end{cases}$$

If $n = 2$ then, by Theorem 5, we have

$$D^+(Q_2 \times C_m) = D^+(C_4 \times C_m) = \begin{cases} 2 & \text{for } m = 3, \\ 3 & \text{for } m \ge 4. \end{cases}$$

Assume that $n \geq 3$ below. For the sake of convenience, let $x_0 = (0 \cdots 0)$ be a vertex in Q_n, x_i be the neighbor of x_0 whose ith position is 1 for $i = 1, 2, \dots, n$, $V(C_m) = \{0, 1, \dots, m-1\}$. Let $G = Q_n \times C_m$. We first show

$$D^+(G) \geq \begin{cases} n & \text{for } m = 3, \\ n+1 & \text{for } m \geq 4. \end{cases} \tag{3.1}$$

To prove this inequality, it is sufficient to find a set $\mathscr{P}$ of at least n for $m = 3$ or at least $n+1$ for $m \geq 4$ internally vertex-disjoint uv-paths of length at most $d(G)$ for any two vertices u and v in G.

Let u and v be any two distinct vertices in G. By vertex-transitivity of Q_n, C_m and G, without loss of generality, we can choose $u = (x_0, 0)$ and $v = (x, y)$, where $x \in V(Q_n)$ and $y \leq \lfloor \frac{m}{2} \rfloor$.

If $x = x_0$, then $y \neq 0$. Then a set $\mathscr{P} = \{P_0, P_1, \dots, P_n\}$ of $n+1$ internally vertex-disjoint uv-paths in G can be constructed as follows.

$$\begin{aligned}
&P_0 = u(x_0, 1)(x_0, 2) \dots (x_0, y-1)v, \\
&P_1 = u(x_1, 0)(x_1, 1) \dots (x_1, y)v, \\
&P_2 = u(x_2, 0)(x_2, 1) \dots (x_2, y)v, \\
&\vdots \\
&P_n = u(x_n, 0)(x_n, 1) \dots (x_n, y)v.
\end{aligned}$$

It is easy to check that their length is at most $y + 2 \leq \lfloor \frac{m}{2} \rfloor + 2 \leq d(G)$ by Lemma 3.

If $y = 0$, then $x \neq x_0$. By Lemma 6, there are n internally vertex-disjoint x_0x-paths $L_1, L_2, \dots, L_n$ in Q_n, where

$$\varepsilon(L_1) \leq n, \ \dots, \varepsilon(L_{n-1}) \leq n, \ \varepsilon(L_n) \leq n+1.$$

Let

$$L_0 = u(0 \cdots 0, 1) \xrightarrow{L_1 1} (x, 1)v.$$

Then $\varepsilon(L_0) \leq n+2$. Hence for $m = 3$, $L_1 0, \dots, L_n 0$ are n internally vertex-disjoint uv-paths of length at most $d(G)$ in G and for $m \geq 4$, $L_0, L_1 0, \dots, L_n 0$ are $n+1$ internally vertex-disjoint uv-paths of length at most $d(G)$ in G.

Now assume that $x \neq x_0$ and $y \neq 0$. By Lemma 6, there are n internally vertex-disjoint x_0x-paths in Q_n denoted by

$$\begin{aligned}
&T_1 = x_0 x_1 \cdots x, \\
&T_2 = x_0 x_2 \cdots x, \\
&\vdots \\
&T_n = x_0 x_n \cdots x
\end{aligned}$$

satisfying

$$\begin{aligned}
&\varepsilon(T_1) \begin{cases} = n+1 & \text{if } d_{Q_n}(x_0, x) = n-1, \\ \leq n & \text{if } d_{Q_n}(x_0, x) \neq n-1, \end{cases} \\
&\varepsilon(T_i) \leq n, \quad i = 2, \dots, n-1, \\
&\varepsilon(T_n) = d_{Q_n}(x_0, x).
\end{aligned}$$

Construct $n+2$ internally vertex-disjoint uv-paths in G as follows.

$$\begin{aligned}
&W_1 = u(x_1, 0) \cdots (x_1, y) \xrightarrow{T_1(x_1,\, x)y} v, \\
&W_2 = u(x_2, 0) \cdots (x_2, y) \xrightarrow{T_2(x_2,\, x)y} v, \\
&\vdots \\
&W_{n-1} = u(x_{n-1}, 0) \cdots (x_{n-1}, y) \xrightarrow{T_{n-1}(x_{n-1},\, x)y} v, \\
&W_n = u(x_0, 1) \cdots (x_0, y) \xrightarrow{T_n y} v, \\
&W_{n+1} = u \xrightarrow{T_n 0} (x, 0)(x, 1) \cdots (x, y-1)v, \\
&W_{n+2} = u(x_0, m-1) \cdots (x_0, y+1) \xrightarrow{T_n(y+1)} (x, y+1)v.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\varepsilon(W_1) &= (1+y) + (\varepsilon(T_1) - 1) = \varepsilon(T_1) + y \\
\varepsilon(W_i) &= \varepsilon(T_i) + y \le n + \lfloor m/2 \rfloor = d(G), \quad i = 2, \ldots, n, \\
\varepsilon(W_{n+1}) &= \varepsilon(T_n) + y \le d(G), \\
\varepsilon(W_{n+2}) &= (m - y - 1) + \varepsilon(T_n) + 1 = (m - y) + \varepsilon(T_n).
\end{aligned}$$

If $d_{Q_n}(x_0, x) \neq n-1$ or $d_{Q_n}(x_0, x) = n-1$ and $y \le \lfloor \frac{m}{2} \rfloor - 1$, then $\varepsilon(W_1) \le d(G)$ and so $W_1, W_2, \ldots, W_{n+1}$ are $n+1$ internally vertex-disjoint uv-paths of length at most $d(G)$ in G.

If $d_{Q_n}(x_0, x) = n-1$ and $y = \lfloor \frac{m}{2} \rfloor$, then

$$\begin{aligned}
\varepsilon(W_{n+2}) &= (m - y) + \varepsilon(T_n) = (m - \lfloor m/2 \rfloor) + d_{Q_n}(x_0, x) \\
&= \lceil m/2 \rceil + (n-1) \le d(G),
\end{aligned}$$

and so $W_2, W_3, \ldots, W_{n+2}$ are $n+1$ internally vertex-disjoint uv-paths of length at most $d(G)$ in G.

Thus, the inequality (3.1) follows. We now show that

$$D^+(G) \le \begin{cases} n & \text{for } m = 3, \\ n+1 & \text{for } m \ge 4. \end{cases} \tag{3.2}$$

To prove this inequality, it is sufficient to find a set $F \subset E(G)$ with $|F| = n$ if $m = 3$ or $n+1$ if $m \ge 4$ such that $d(G - F) > d(G)$.

For $m \ge 4$, let

$$F = \{u(x_2, 0), u(x_3, 0), \ldots, u(x_n, 0), u(x_0, 1), u(x_0, m-1)\}.$$

Then $F \subset E(G)$ and $|F| = n+1$. Let $w = (01 \cdots 1, \lfloor \frac{m}{2} \rfloor)$. Then, by Lemma 3, we have

$$\begin{aligned}
d(G - F) &\ge d_{G-F}(u, w) = 1 + d_{G-F}((x_1, 0), w) \\
&\ge 1 + d_G((x_1, 0), w) \\
&= 1 + n + \lfloor m/2 \rfloor \\
&> d(G)
\end{aligned}$$

For $m = 3$, let $F = \{u(x_1, 0), u(x_2, 0), \ldots, u(x_n, 0)\}$ and $w = (1 \cdots 1, 0)$. Then, by Lemma 3, we have

$$\begin{aligned}
d(G - F) &\ge d_{G-F}(u, w) = 1 + \min\{d_{G-F}((x_0, 1), w), d_{G-F}((x_0, 2), w)\} \\
&\ge 1 + \min\{d_G((x_0, 1), w), d_G((x_0, 2), w)\} \\
&= 1 + n + 1 \\
&> d(G).
\end{aligned}$$

Thus, the inequality (3.2) holds, and so the theorem follows. □

4. Conclusions and remarks

The bounded edge-connectivity λ_k is a generalization of both the edge-persistence D^+ and the classical edge-connectivity λ. The graph-theoretical parameters λ_k and D^+ provide two important measurements for fault tolerance of interconnection networks. We are interested in the two parameters for the Cartesian product of graphs since it is an important method for designing interconnection networks. We establish lower bounds of λ_k and D^+ for the Cartesian product $G_1 \times G_2$, that is, $\lambda_{k_1+k_2}(G_1 \times G_2) \ge \lambda_{k_1}(G_1) + \lambda_{k_2}(G_2)$ for $k_1 \ge 2$ and $k_2 \ge 2$ and $D^+(G_1 \times G_2) \ge D^+(G_1) + D^+(G_2)$ if $d(G_i) \ge 2$ for $i = 1, 2$, and determine $D^+(C_n \times P_m) = 1$ for $n = 3$ and 2 for $n \ge 4$; $D^+(C_n \times C_m) = 2$ if $n = 3$ or $m = 3$ or both n and m are odd, 3 otherwise; $D^+(Q_n \times P_m) = n$ for $n \ge 2$ and $m \ge 2$; $D^+(Q_n \times C_m) = n$ for $m = 3$, $n+1$ for $m \ge 4$. These examples show that the obtained lower bounds of λ_k and D^+ for the Cartesian product $G_1 \times G_2$ are tight.

We have also taken notice of a paper of Sung and Wang [21], in which the authors announced the following results without proofs:

$D^+(C_n \times P_m) = 2$ if n is odd, and 3 if n is even; $D^+(C_n \times C_m) = 2$ if both n and m are odd, 3 if one of n and m is odd, and 4 if both n and m are even; $D^+(Q_n \times C_m) = n+2$ if m is even, and $n+1$ if m is odd.

However, these values are not correct by our results. Also, Sung and Wang proposed a conjecture: $D^+(G_1 \times G_2) \ge \max\{D^+(G_1), D^+(G_2)\} + 1$.

If $d(G_1) \ge 2$ and $d(G_2) \ge 2$, then, by Corollary 2, we have

$$D^+(G) \ge D^+(G_1) + D^+(G_2) \ge \max\{D^+(G_1), D^+(G_2)\} + 1.$$

That is, the conjecture is true for $d(G_1) \ge 2$ and $d(G_2) \ge 2$. However, the result in Theorem 4 for $n = 3$ shows that the conjecture may be incorrect for $d(G_1) = 1$ or $d(G_2) = 1$.

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