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Probabilistic analysis of upper bounds for 2-connected distance k-dominating sets in graphs☆

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ABSTRACT

A small virtual backbone which is modeled as the minimum connected dominating set (CDS) problem has been proposed to alleviate the broadcasting storm for efficiency in wireless ad hoc networks. In this paper, we consider a general fault tolerant CDS problem, called an h-connected distance k-dominating set (HCKDS) to balance high efficiency and fault tolerance, and study the upper bound for HCKDS with a probabilistic method for small h and improve the current best results.

1. Introduction

Wireless ad hoc and sensor networks composed of wireless nodes have been the focus of intense research in recent years and are characterized by a lack of a fixed communication infrastructure. Thus all wireless nodes frequently flood control messages to cause redundancies, contentions and collisions. A virtual backbone has been proposed as a alternative to the fixed routing infrastructure so that local routing messages are exchanged between nodes in a virtual backbone instead of being broadcast to all the nodes.

A connected dominating set (CDS) [1] is a natural candidate for a virtual backbone in wireless networks. A CDS is a connected subset of the network nodes such that any node in the subset is either part of the CDS or has a neighbor in the CDS. Previous studies have focused on finding a minimal CDS for higher efficiencies of the virtual backbone. Unfortunately, a CDS is often vulnerable due to frequent node or link failures. Therefore, how to construct a fault tolerant CDS that continues to function during node or link failures is an important research problem, which has not been studied sufficiently. Dai and Wu [5] addressed the algorithm problem of constructing a k-connected k-tuple dominating set. Weiping Shang et al. [13] and My T. Thai et al. [14] introduced the algorithm problem of constructing a 2-connected k-tuple dominating set, respectively. D. Rautenbach and L. Volkmann [12], V. Zverovich [16] considered the upper bounds for k-tuple dominating sets using a probabilistic method, respectively.

A distance dominating set is also a variation of domination for higher efficiencies in wireless networks which was introduced by Haynes, Hedetniemi and Slater in [6] and has been studied extensively by several authors to consider the distance parameters in many situations and structures which give rise to graphs, see, for example, [6,7,9–11,15].

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In this paper, we assume that the networks are sufficiently dense such that the networks are connected. Consider a combinatorial optimization problem which was formulated from finding an efficient fault tolerant virtual backbone as: Given a graph $G = (V, E)$ and two positive integers h and k, find a subset of vertices $D \subseteq V$ with a minimum size that satisfies: (i) each vertex u in $V - D$ is k-dominated by at least one vertex v in D such that $d_G(u, v) \le k$; (ii) D is h-connected. Every subset D satisfying (i) is called a distance k-dominating set (KDS). Every subset D satisfying (i) and (ii) is called an h-connected distance k-dominating set (HCKDS). The minimum cardinality among all HCKDS of G is called the h-connected distance k-domination number of G and is denoted by $\gamma_k^h(G)$.

Since computing a minimum CDS in G is NP-hard [6], and it is easy to reduce CDS to HCKDS for any fixed positive integers h and k in polynomial time, the HCKDS problem is also NP-hard. Thus it is difficult to determine the value of $\gamma_k^h(G)$ for any given graph G. In this paper, we prove that for any 2-connected graph G with order n and minimum degree δ,

$$\gamma_k^2(G) \le (1 + o_\delta(1))\, n \frac{\ln[m(\delta+1)+1-t]}{[m(\delta+1)+1-t]},$$

where $m = \lceil \frac{k}{3} \rceil$, $t = 3\lceil \frac{k}{3} \rceil - k$. This generalizes the result of the upper bounds for $\gamma_1^2(G) \le (1 + o_\delta(1))n\frac{\ln(\delta+1)}{(\delta+1)}$ in [4] in some sense. The method adopted here is a refinement of [4] and different with [13,14].

The rest of this paper is organized as follows: The proofs of our main results are in Section 3 and some lemmas are given in Section 2.

2. Some lemmas

For every $x \in V(G)$, the k-neighborhood $N_k(x)$ of x is defined as $N_k(x) = \{y \in V(G) : d_G(x, y) \le k, x \ne y\}$, and $N_1(x)$ is usually called the neighborhood of x in G. A vertex which separates two other vertices of the same component is a cut-vertex. A block of a graph G is a maximal subgraph without a cut-vertex. Thus, every block of a graph G is either a maximal 2-connected subgraph, or a bridge, or an isolated vertex. If G is not 2-connected, at least one block of G has exactly one cut-vertex of G, thus we call such a block a leaf block.

A subset $S \subset V(G)$ is called an (ℓ, k)-dominating set of G if for every vertex u in $G - S$, there are ℓ distinct vertices $v_1, v_2, \ldots, v_\ell$ in S such that $d_G(u, v_i) \le k$ and every shortest path P_i between u and v_i satisfies $V(P_i) \cap S = \{v_i\}$ for $i = 1, 2, \ldots, \ell$.

Lemma 2.1. *Let G be a 2-connected graph, k a positive integer and S a $(2, k)$-dominating set of G. If $|S| \ge 3$ and $G[S]$ has λ blocks, then*

$$\gamma_k^2(G) \le |S| + 2k(2k+3)(\lambda - 1).$$

Proof. It is clear from the definition that a $(2, k)$-dominating set is a k-dominating set. If $\lambda = 1$, then $G[S]$ is 2-connected since $|S| \ge 3$, and so S is a 2-connected k-dominating set of G. Thus, $\gamma_k^2(G) \le |S|$, and so the conclusion holds. Assume $\lambda > 1$ below.

Let $\mathscr{S}_G$ be the set of $(2, k)$-dominating sets of G. For any $S \in \mathscr{S}_G$, we use ω_S to denote the number of components of $G[S]$ and λ_S the number of blocks in $G[S]$. Then $1 \le \omega_S \le \lambda_S$. Define a weight function $\mathbf{w}$ on $\mathscr{S}_G$ as follows

$$\mathbf{w}(S) = (2k+2)\omega_S + \lambda_S.$$

Clearly, $\mathbf{w}(S) = 2k + 3$ if and only if $\omega_S = \lambda_S = 1$.

In order to prove the lemma, we only need to construct a $(2, k)$-dominating set D of G from a given $S \in \mathscr{S}_G$ by adding other $2k(2k+3)(\lambda - 1)$ vertices such that $\lambda_D = 1$. To the end, we only need to show the following claim.

Claim. *For any $S \in \mathscr{S}_G$, if $\lambda_S > 1$ then there exists a subset $T' \subset V(G-S)$ with $|T'| \le 2k$ such that $\mathbf{w}(S') = \mathbf{w}(S \cup T') \le \mathbf{w}(S) - 1$, where $S' = S \cup T'$.*

In fact, if the claim is true, then after $(2k+2)\omega_S + \lambda_S - (2k+3)$ times of applying the claim, we can find a subset $T \subset V(G - S)$ with

$$|T| \le 2k\big[(2k+2)\omega_S + \lambda_S - (2k+3)\big] \le 2k(2k+3)(\lambda_S - 1)$$

such that

$$\mathbf{w}(S \cup T) \le \mathbf{w}(S) - \big[(2k+2)\omega_S + \lambda_S - (2k+3)\big] = 2k+3.$$

Thus, we have $\lambda_{S\cup T} = 1$, and so $S \cup T$ is a 2-connected k-dominating set of G and

$$\gamma_k^2(G) \le |S \cup T| \le |S| + 2k(2k+3)(\lambda_S - 1).$$

We now prove the claim. For convenience, let $\omega = \omega_S$ and $\lambda = \lambda_S$. There are two cases to be considered.

Case 1 $\omega > 1$.

Since $\omega > 1$, we can choose two components of $G[S]$, U and V, such that their distance $d_G(U, V) = \min\{d_G(u, v) : u \in U, v \in V\}$ is as small as possible. Let $u \in U$ and $v \in V$ such that $d_G(u, v) = d_G(U, V)$, and let P_{uv} be a shortest path between u and v in G. Then, $V(P_{uv}) \cap S = \{u, v\}$ and $d_G(u, v) \ge 2$. Moreover, $d_G(u, v) \le 2k+1$, otherwise there is an internal vertex x

in P_{uv} whose k-neighbors $N_k(x) \cap S = \emptyset$, a contradiction to the hypothesis that S is a k-dominating set of G. This fact implies $|V(P_{uv}) - \{u, v\}| \leq 2k$. Let $S' = S \cup V(P_{uv})$. Then $|S'| \leq |S| + 2k$.

If $d_G(u, v) = 2$, then let x be the internal vertex in P_{uv} and μ the number of components that contain neighbors of x in $G[S]$. Then $\mu \geq 2$, $\omega_{S'} = \omega - \mu + 1$ and $\lambda_{S'} = \lambda + \mu$. Thus

$$\begin{aligned}\mathbf{w}(S') &= (2k+2)\omega_{S'} + \lambda_{S'} \\ &= (2k+2)(\omega - \mu + 1) + \lambda + \mu \\ &\leq (2k+2)\omega + \lambda - 2k \\ &< \mathbf{w}(S) - 1.\end{aligned}$$

For $d_G(u, v) \geq 3$, we claim $\omega_{S'} = \omega - 1$. Clearly, $\omega_{S'} \leq \omega - 1$. We only need to prove $\omega_{S'} \geq \omega - 1$. Assume to the contrary $\omega_{S'} \leq \omega - 2$. Then there exists another component W different from U and V in $G[S]$ such that W, U and V are in the same component of $G[S']$. Since $S' = S \cup V(P_{uv})$ and $V(W) \cap V(P_{uv}) = \emptyset$, there is a vertex w in W that is adjacent to some internal vertex x in P_{uv}. Since $d_G(u, v) \geq 3$, either $d_G(u, x) \geq 2$ or $d_G(x, v) \geq 2$. If the former holds, then

$$d_G(W, V) \leq d_G(w, x) + d_G(x, v) < d_G(u, x) + d_G(x, v) = d_G(U, V);$$

if the latter holds, then

$$d_G(U, W) \leq d_G(u, x) + d_G(w, x) < d_G(u, x) + d_G(x, v) = d_G(U, V);$$

each of which contradicts the choice of U and V. Thus, $\omega_{S'} = \omega - 1$.

Noting $\lambda_{S'} \leq \lambda + 2k + 1$, we have

$$\begin{aligned}\mathbf{w}(S') &= (2k+2)\omega_{S'} + \lambda_{S'} \\ &\leq (2k+2)(\omega - 1) + (\lambda + 2k + 1) \\ &= (2k+2)\omega + \lambda - 1 \\ &= \mathbf{w}(S) - 1.\end{aligned}$$

Case 2 $\omega = 1$.

If $\lambda = 1$, then we complete the proof. Otherwise, G has at least one leaf block which has exactly one cut-vertex. Let B be the vertex set of some leaf block of $G[S]$, and let $b \in B$ be the unique cut-vertex of $G[S]$.

Let $u \in B - \{b\}$ and $v \in S - B$ be two vertices whose distance in $G - \{b\}$ is as small as possible and P_{uv} denote a shortest path between u and v in $G - \{b\}$, that is, $V(P_{uv}) \cap S = \{u, v\}$. Clearly, $G - \{b\}$ is connected since G is 2-connected. Also, $d_{G-\{b\}}(u, v) \leq 2k + 1$. Otherwise, there exists an internal vertex x in P_{uv} whose k-neighbors $N_k(x) \cap (S - \{b\}) = \emptyset$, a contradiction to the hypothesis that S is a $(2, k)$-dominating set of G. Thus, $|V(P_{uv}) - \{u, v\}| \leq 2k$. Since $\omega = 1$, there exists a path P'_{uv} between u and v in $G[S]$. Then $P'_{uv} \cap P_{uv} = \{u, v\}$, and $P'_{uv} \cup P_{uv}$ generates a cycle. Let $S' = S \cup V(P_{uv})$. Thus, we have $|S'| \leq |S| + 2k$, $\omega_{S'} = 1$, and $\lambda_{S'} \leq \lambda - 1$. It follows that

$$\begin{aligned}\mathbf{w}(S') &= (2k+2)\omega_{S'} + \lambda_{S'} \\ &\leq (2k+2)\omega + \lambda - 1 \\ &= \mathbf{w}(S) - 1.\end{aligned}$$

The claim follows and the proof of the lemma is complete. ■

Lemma 2.2. *Let $0 < \varepsilon < \frac{k}{3k-1}$, $\delta \geq 1$, and G be a graph with n vertices. If at most εn vertices in G have degree less than δ, then G has an induced subgraph with order at least $\left[1 - (4k-1)\varepsilon\right]n$ and minimum degree greater than $\frac{2k-1}{6k-2}\delta$.*

Proof. Let $\delta' = \frac{2k-1}{6k-2}\delta$. For $0 < \varepsilon < \frac{k}{3k-1}$, if there are at most εn vertices in G with degree less than δ, then by $2|E(G)| = \sum_{v \in V(G)} \deg(v)$, G has more than

$$\frac{1}{2}\left(1 - \frac{k}{3k-1}\right) n\,\delta = n\,\delta'$$

edges. Since every graph with n vertices and ρn edges has a subgraph with minimum degree at least ρ (see, a result of p. xvii in [2]), G has a subgraph with minimum degree greater than δ'.

Let F be the vertex set of a maximal subgraph of G with minimum degree greater than δ', let $f = |F|$ and $W = V(G) - F$.

On one hand, by the maximality of F, every vertex in W has at most δ' neighbors in F, and $G[W]$ has at most $(n-f)\delta'$ edges, otherwise $G[W]$ contains a subgraph T with minimum degree greater than δ' and $F \cup T \supset F$, a contradiction to the maximality of F.

On the other hand, since $G[W]$ contains at least $n-f-\varepsilon n$ vertices whose degrees in G are all at least δ. Hence, the sum of the degrees in $G[W]$ is at least

$$(n-f-\varepsilon n)(\delta-\delta').$$

Hence, we have

$$2(n-f)\delta' \geq (n-f-\varepsilon n)\frac{4k-1}{6k-2}\,\delta,$$

that is,

$$f \geq \left[1-(4k-1)\varepsilon\right] n.$$

This completes the proof. ■

The following lemma which belongs to Kouider and Lonc in [8] is also used in the proof of our results.

Lemma 2.3 (*Kouider and Lonc [8]*). *Let G be a graph with order n and minimum degree δ. Then $V(G)$ can be covered by at most n/δ subgraphs such that each of them is a vertex, an edge or a cycle.*

3. Main results

Let $f(x)$ and $g(x)$ be two functions. If $f(x)/g(x) \to 0$ as $x \to \infty$, we write $f(x) = o(g(x))$. So $f(x) = o(1)$ signify that $f(x) \to 0$ as $x \to \infty$. We write $f(x) = O(g(x))$ if there exists a positive constant c such that $f(x) \leq cg(x)$ for large enough x. If G is a graph with n vertices and degrees $d_1 \leq \cdots \leq d_n$, then the n-tuple $(d_1, \ldots, d_n)$ is called the degree sequence of G. We use a probabilistic method to give an upper bound of $\gamma_k^2(G)$ in terms of the minimum degree $\delta = \delta(G)$ below.

For an event A and for a random variable Z of an arbitrary probability space $(\Omega, \mathscr{F}, P)$, $P[A]$ and $E[Z]$ denote the probability of A, the expectation of Z, respectively.

Theorem 3.1. *Let G be a 2-connected graph with order n and minimum degree δ. Then*

$$\gamma_k^2(G) \leq (1+o_\delta(1))n\frac{\ln[m(\delta+1)+1-t]}{m(\delta+1)+1-t}, \tag{1}$$

where $m = \left\lceil \frac{k}{3} \right\rceil$, $t = 3\left\lceil \frac{k}{3} \right\rceil - k$, and $o_\delta(1)$ denotes a function in δ that tends to 0 as δ tends to ∞.

Proof. Let $0 < \varepsilon < \frac{k}{3k-1}$ be a fixed rational number and $p = (1+\varepsilon)\frac{\ln q}{q}$, where $q = m(\delta+1)+1-t$, $m = \left\lceil \frac{k}{3} \right\rceil$ and $t = 3\left\lceil \frac{k}{3} \right\rceil - k$. Let us pick, randomly and independently, each vertex of V with probability p. Let $X \subset V(G)$ be the set of vertices picked. Let $Y_1 \subset V(G-X)$ such that $N_k(Y_1) \cap X = \emptyset$. For any $v \in V(G-X-Y_1)$, let P_{vX} denote one shortest path between v and X, and u denote the second-last vertex on P_{vX} from v to X. Assuming an arbitrary ordering of the neighbors of u from 1 to $|N_1(u)|$ and taking δ lexicographically smaller ones, if there are at most one of them belongs to X, then $v \in Y_2$. Therefore, from the choices of Y_1 and Y_2, $X \cup Y_1 \cup Y_2$ is a $(2, k)$-dominating set of G.

Let $Q = G[X \cup Y_1 \cup Y_2]$, and let λ be the number of blocks of Q. By Lemma 2.1, there exists a 2-connected k-dominating set of G by adding at most $2k(2k+3)(\lambda-1)$ vertices to $X \cup Y_1 \cup Y_2$. Let $|X| = \alpha$, $|Y_1| = \beta_1$ and $|Y_2| = \beta_2$. Thus

$$\gamma_k^2(G) < \alpha + \beta_1 + \beta_2 + 2k(2k+3)\lambda \tag{2}$$

First, we establish an upper bound on λ. Let θ denote a positive integer such that $\theta - 1$ is the exact term at the position $\lfloor \varepsilon\alpha \rfloor$ in the degree sequence of $G[X]$.

Claim 1. $\lambda < \frac{4(3k-1)}{2k-1}\frac{\alpha}{\theta} + 2(4k-1)\varepsilon\alpha + 2(\beta_1+\beta_2)$.

Proof of Claim 1. By the definition of θ, Lemmas 2.2 and 2.3, there exists a subset $F \subset X$ such that $|F| \geq \left[1-(4k-1)\varepsilon\right]\alpha$ and $\delta(G[F]) > \frac{2k-1}{6k-2}\theta$, furthermore, F can be covered by at most $\frac{6k-2}{2k-1}\frac{|F|}{\theta}$ subgraphs of $G[F]$ that are all blocks. Add the vertices in $(X-F) \cup Y_1 \cup Y_2$ to the above covers for F, then we obtain a covering of $V(Q)$ using only blocks. Let ξ be the minimum number of blocks to cover $V(Q)$, thus

$$\xi \leq \frac{6k-2}{2k-1}\frac{\alpha}{\theta} + (4k-1)\varepsilon\alpha + \beta_1 + \beta_2.$$

Since we can add at most $\xi - 1$ edges to connect the above ξ blocks, thus $\lambda < 2\xi$. Hence,

$$\lambda < \frac{4(3k-1)}{2k-1}\frac{\alpha}{\theta} + 2(4k-1)\varepsilon\alpha + 2(\beta_1 + \beta_2). \quad \blacksquare$$

It follows from Claim 1 and (2) that

$$\gamma_k^2(G) < \left[1 + 4k(2k+3)(4k-1)\varepsilon\right]\alpha + \left[1 + 4k(2k+3)\right](\beta_1 + \beta_2) + \frac{8k(2k+3)(3k-1)}{2k-1}\frac{\alpha}{\theta} \tag{3}$$

and

$$E\left[\gamma_k^2(G)\right] < \left[1 + 4k(2k+3)(4k-1)\varepsilon\right]E[\alpha] + \left[1 + 4k(2k+3)\right]E[\beta_1 + \beta_2] + \frac{8k(2k+3)(3k-1)}{2k-1}E\left[\frac{\alpha}{\theta}\right]. \tag{4}$$

We will prove the upper bounds for $E[\alpha]$, $E[\beta_1 + \beta_2]$ and $E\left[\frac{\alpha}{\theta}\right]$, respectively.

Claim 2. $E[\alpha] = (1+\varepsilon)n\frac{\ln q}{q}$.

Proof of Claim 2. Since α can be written as a sum of n indicator random variables χ_v, where $\chi_v = 1$ if $v \in X$ and $\chi_v = 0$ otherwise, it follows that the expectation of α satisfies $E[\alpha] = (1+\varepsilon)n\frac{\ln q}{q}$. $\blacksquare$

To get the upper bound for $E[\beta_1 + \beta_2]$, we first give two claims below.

Claim 3. $d_G(X, Y_1) = k + 1$.

Proof of Claim 3. It is clear from the choice of Y_1 that $d_G(X, Y_1) \geq k + 1$. Let $a \in X$, $b \in Y_1$ be two vertices such that $d_G(a, b) = d_G(X, Y_1)$. Let P be any shortest path from a to b and let v be the second-last vertex on P. If $d_G(a, b) \geq k+2$, then v has no k-neighbors in X. By definition of Y_1, we get $v \in Y_1$, thus $d_G(a, v) < d_G(a, b) = d_G(X, Y_1)$, a contradiction. $\blacksquare$

Claim 4. *$|N_k(v)| \geq q$ for any $v \in V(G - X)$.*

Proof of Claim 4. Let $v \in V(G - X)$ and $X_i(v) = \{u \in V(G) : d_G(u, v) = i\}$.

If $v \in Y_1$, then by Claim 3 and G is connected, we have $X_i(v) \neq \emptyset$ for $i = 1, \ldots, k$. Clearly, $|X_1(v)| \geq \delta$. For $2 \leq i \leq k-2$, we have that $|X_i(v)| + |X_{i+1}(v)| + |X_{i+2}(v)| \geq \delta + 1$. In fact, for any $u \in X_{i+1}(v)$, $N_1(u) \subseteq X_i(v) \cup X_{i+1}(v) \cup X_{i+2}(v)$, thus, $|X_i(v)| + |X_{i+1}(v)| - 1 + |X_{i+2}(v)| \geq \delta$. So, we have

$$\begin{aligned}
|N_k(v)| &= |X_1(v)| + |X_2(v)| + \cdots + |X_k(v)| \\
&\geq \delta + \left\lfloor \frac{k-1}{3} \right\rfloor (\delta+1) + \left(k - 1 - 3\left\lfloor \frac{k-1}{3} \right\rfloor\right) \\
&= \delta + (m-1)(\delta+1) + (2-t) \\
&= m(\delta+1) + 1 - t.
\end{aligned}$$

Let $v \in V(G) - (X \cup Y_1)$. Thus $d_G(v, X) \leq k$. If $d_G(v, X) = k$ or $d_G(v, Y_1) \geq k$, using the same discussions as above we get $|N_k(v)| \geq m(\delta+1) + 1 - t$. Now suppose that $d_G(v, Y_1) < k$ and $d_G(v, X) < k$. Since $d_G(X, Y_1) = k + 1$, there must exist a shortest path between a vertex $a \in X$ and a vertex $b \in Y_1$ through v such that $d_G(a, b) \geq k + 1$, $d_G(v, b) < k$ and $d_G(a, v) < k$. We only consider the worst case $d_G(a, b) = k + 1$, and let P_{ab} denote the shortest path from a to b passing through v.

Let v_1 and v_2 be two neighbors of v on P_{ab} from b to v and from a to v, respectively. Let $d_G(b, v_1) = \ell_1$, $d_G(a, v_2) = \ell_2$. Thus, $\ell_1 + \ell_2 = k - 1$. By symmetry, we consider the following three cases.

If $\ell_1 \equiv 1 \pmod 3$, $\ell_2 \equiv 1 \pmod 3$, then $k \equiv 0 \pmod 3$, that is, $k = 3m$, $t = 0$.

$$\begin{aligned}
|N_k(v)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + 2 \\
&= \delta + \frac{\ell_1 + \ell_2 - 2}{3}(\delta+1) + 2 \\
&= \delta + \frac{k-3}{3}(\delta+1) + 2 \\
&= \delta + (m-1)(\delta+1) + 2 \\
&= m(\delta+1) + 1 - t.
\end{aligned}$$

If $\ell_1 \equiv 1 (\bmod 3)$, $\ell_2 \equiv 2 (\bmod 3)$, then $k \equiv 1 (\bmod 3)$, that is, $k = 3m-2$, $t = 2$. Notice $\ell_2 \equiv 2(\bmod 3)$ and $d_G(v, a) < k$, then $N_1(a) \subseteq N_k(v)$, thus $|X_{\ell_2-1}(v_2)| + |X_{\ell_2}(v_2)| + |X_{\ell_2+1}(v_2)| \geq \delta + 1$. So we have

$$\begin{aligned}
|N_k(v)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + 1 + (\delta+1) \\
&= \delta + \frac{\ell_1 - 1 + \ell_2 - 2}{3}(\delta+1) + \delta + 2 \\
&= \delta + \frac{k-4}{3}(\delta+1) + \delta + 2 \\
&= m(\delta+1) \\
&> m(\delta+1) + 1 - t.
\end{aligned}$$

If $\ell_1 \equiv 2 (\bmod 3)$, $\ell_2 \equiv 2 (\bmod 3)$, then $k \equiv 2 (\bmod 3)$, that is, $k = 3m-1$, $t = 1$. By the discussions as above, we also get $|X_{\ell_1-1}(v_1)| + |X_{\ell_1}(v_1)| + |X_{\ell_1+1}(v_1)| \geq \delta + 1$. Thus, we have,

$$\begin{aligned}
|N_k(v)| &\geq \delta + \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor + \left\lfloor \frac{\ell_2}{3} \right\rfloor\right)(\delta+1) + 2(\delta+1) \\
&= \delta + \frac{\ell_1 - 2 + \ell_2 - 2}{3}(\delta+1) + 2\delta + 2 \\
&= \delta + \frac{k-5}{3}(\delta+1) + 2\delta + 2 \\
&= m(\delta+1) + \delta + 1 - t \\
&> m(\delta+1) + 1 - t.
\end{aligned}$$

The Claim 4 follows. ■

By Claim 4, we could prove the following claim.

Claim 5. $E\left[\beta_1 + \beta_2\right] < o\left(\frac{n}{q}\right)$.

Proof of Claim 5. Let v be a vertex in G. By the total probability law, we have

$$P[v \in Y_1 \cup Y_2] = P\left[v \in Y_1 \cup Y_2 | v \in V(G) - X\right] P\left[v \in V(G) - X\right].$$

By the definition of Y_1, we have $P[v \in Y_1 | v \in V(G) - X] = (1-p)^{|N_k(v)|} = \binom{\delta}{0} p^0 (1-p)^{N_k(v)-0}$. For any $v \in V(G) - X$, let P_{vX} denote one shortest path between v and X, and u denote the second-last vertex on P_{vX} from v to X. Assuming an arbitrary ordering of the neighbors of u from 1 to $|N_1(u)|$ and taking δ lexicographically smaller ones, by the definition of Y_2, $P\left[v \in Y_2 | v \in V(G) - X)\right] < \binom{\delta}{0}(1-p)^{N_k(v)} + \binom{\delta}{1} p^1 (1-p)^{N_k(v)-1}$. Since $p = (1+\varepsilon)\frac{\ln q}{q}$, $\binom{\delta}{i} < \delta^i$, the inequality $(1-x) < \exp(-x)$ for $x > 0$, it follows that, for sufficiently large δ,

$$\begin{aligned}
P\left[v \in Y_1 \cup Y_2\right] &< P[v \in Y_1 \cup Y_2 | v \in V(G) - X] \\
&< 2 \sum_{i=0}^{1} \binom{\delta}{i} p^i (1-p)^{N_k(v)-i} \\
&\leq 2 \sum_{i=0}^{1} \binom{\delta}{i} p^i (1-p)^{q-i} \\
&< 2 \sum_{i=0}^{1} (\delta p)^i \exp(-pq + p) \\
&< 4 \left((1+\varepsilon)\frac{\ln q}{m}\right) \exp(-pq) \exp(p) \\
&< 4 \left(\frac{4k-1}{3k-1} \frac{\ln q}{m}\right) q^{-(1+\varepsilon)} \exp(p) \\
&= O\left(q^{-(1+\frac{3\varepsilon}{4})} \ln q\right) \\
&= o\left(\frac{1}{q}\right).
\end{aligned}$$

The second-last equality comes from the fact that for sufficiently large δ, $\exp(p) = q^{\frac{1+\varepsilon}{q}} = O\left(q^{\frac{\varepsilon}{4}}\right)$. The last equality comes from the fact that $\lim_{\delta\to\infty} q^{-\frac{3\varepsilon}{4}} \ln q = 0$.

Since $\beta_1 + \beta_2$ can also be written as a sum of n indicator random variables χ_v, where $\chi_v = 1$ if $v \in Y_1 \cup Y_2$ and $\chi_v = 0$ otherwise, it follows that the expectation of $\beta_1 + \beta_2$ satisfies $E[\beta_1 + \beta_2] < no\left(\frac{1}{q}\right) = o\left(\frac{n}{q}\right)$. ■

Lastly, we only prove the upper bound for $E\left[\frac{\alpha}{\theta}\right]$. By conditional expectation, we have

$$E\left[\frac{\alpha}{\theta}\right] = E\left[\frac{\alpha}{\theta}\middle|\Lambda_1\right]P\left[\Lambda_1\right] + E\left[\frac{\alpha}{\theta}\middle|\Lambda_2\right]P\left[\Lambda_2\right] + E\left[\frac{\alpha}{\theta}\middle|\Lambda_3\right]P\left[\Lambda_3\right]$$

in which Λ_1 denotes the event $\alpha > (3k+1)np$, Λ_2 denotes the event $\alpha \le (3k+1)np$ and $\theta \le \lfloor(\ln q)^{\frac{1}{4}}\rfloor$, and Λ_3 denotes the event $\alpha \le (3k+1)np$ and $\theta > \lfloor(\ln q)^{\frac{1}{4}}\rfloor$. Hence, we have

$$E\left[\frac{\alpha}{\theta}\right] < nP\left[\alpha > (3k+1)np\right] + (3k+1)npP\left[\theta \le \lfloor(\ln q)^{\frac{1}{4}}\rfloor\right] + \frac{3k+1}{\lfloor(\ln q)^{\frac{1}{4}}\rfloor}np. \tag{5}$$

In order to obtain the upper bound for $E[\frac{\alpha}{\theta}]$, we will get the upper bounds for $P\left[\alpha > (3k+1)np\right]$ and $P[\theta \le \lfloor(\ln q)^{\frac{1}{4}}\rfloor]$, respectively.

Claim 6. $P\left[\alpha > (3k+1)np\right] < \frac{1}{q^{2.25}}$.

Proof of Claim 6. Since $E[\alpha] = np$, we use an inequality attributed to Chernoff in [3], that is, for any $s \ge 0$,

$$P[\alpha > E[\alpha] + s] \le \exp\left\{\frac{-s^2}{2(E[\alpha] + \frac{s}{3})}\right\}.$$

Take $s = 3knp$ to this inequality and notice that $k + 1 \le 2k$ and $n \ge q$, we have

$$\begin{aligned} P\left[\alpha > (3k+1)np\ \right] &\le \exp\left(-\frac{9k^2}{2(k+1)}np\right) \\ &< \exp\left(-\frac{9k}{4}(1+\varepsilon)\ln q\right) \\ &< \exp\left(-\frac{9k}{4}\ln q\right) = q^{-\frac{9k}{4}} \\ &\le \frac{1}{q^{2.25}}. \end{aligned}$$

The Claim 6 follows. ■

Claim 7. $P\left[\alpha < \frac{k}{k+1}np + \frac{1}{\varepsilon}\right] < \frac{\exp\left(\frac{1}{\varepsilon(k+1)}\right)}{q^{\frac{1}{2(k+1)^2}}}$.

Proof of Claim 7. By using another inequality of Chernoff [3] and $E[\alpha] = np$, that is, for any $s \ge 0$,

$$P\left[\alpha < E[\alpha] - s\right] \le \exp\left(-\frac{s^2}{2E[\alpha]}\right),$$

we have, for sufficiently large δ,

$$\begin{aligned} P\left[\alpha < \frac{k}{k+1}np + \frac{1}{\varepsilon}\right] &= P\left[\alpha - E\left[\alpha\right] < -\left(\frac{1}{k+1}np - \frac{1}{\varepsilon}\right)\right] \\ &\le \exp\left(-\frac{\left(\frac{1}{k+1}np - \frac{1}{\varepsilon}\right)^2}{2np}\right) \\ &< \exp\left(-\frac{1}{2(k+1)^2}np + \frac{1}{\varepsilon(k+1)}\right) \\ &< \exp\left(-\frac{1}{2(k+1)^2}(1+\varepsilon)\ln q + \frac{1}{\varepsilon(k+1)}\right) \\ &< \exp\left(-\frac{1}{2(k+1)^2}\ln q + \frac{1}{\varepsilon(k+1)}\right) \\ &= \exp\left(\frac{1}{\varepsilon(k+1)}\right)q^{-\frac{1}{2(k+1)^2}}. \quad ■ \end{aligned}$$

Claim 8. $P\left[\theta \le \lfloor(\ln q)^{\frac{1}{4}}\rfloor\right] < \frac{k+1}{k\varepsilon q^{\frac{1}{2m+1}}} + \frac{\exp\left(\frac{1}{\varepsilon(k+1)}\right)}{q^{\frac{1}{2(k+1)^2}}}$.

Proof of Claim 8. For $v \in V(G)$, take δ neighbors of v. Let $\deg^{\delta}_{G[X]}(v)$ denote the number of these δ neighbors in the induced subgraph $G[X]$. By similar discussions with Claim 5, we have, for sufficiently large δ,

$$\begin{aligned}
P\left[\deg^{\delta}_{G[X]}(v) \le \lfloor(\ln q)^{\frac{1}{4}}\rfloor\right] &\le \sum_{i=0}^{\lfloor(\ln q)^{\frac{1}{4}}\rfloor} \binom{\delta}{i} p^i(1-p)^{\delta-i} \\
&< \sum_{i=0}^{(\ln q)^{\frac{1}{4}}} (\delta p)^i \exp\left(-p\left(\delta - (\ln q)^{\frac{1}{4}}\right)\right) \\
&< 2\,(\ln q)^{\frac{1}{4}} \left(\delta(1+\varepsilon)\frac{\ln q}{q}\right)^{(\ln q)^{\frac{1}{4}}} \exp(-p\delta)\exp\left((\ln q)^{\frac{1}{4}}\,p\right) \\
&< 2\,(\ln q)^{\frac{1}{4}} \left(\frac{2}{m}\ln q\right)^{(\ln q)^{\frac{1}{4}}} q^{-\frac{(1+\varepsilon)}{2m+1}} \exp\left((\ln q)^{\frac{1}{4}}\,p\right) \\
&= O\left(q^{-(\frac{1}{2m+1}+\frac{\varepsilon}{4m+2})}\,(\ln q)^{\frac{1}{4}} \left(\frac{2}{m}\ln q\right)^{(\ln q)^{\frac{1}{4}}}\right) \\
&= o\left(\frac{1}{q^{\frac{1}{2m+1}}}\right) < \frac{1}{q^{\frac{1}{2m+1}}}.
\end{aligned}$$

The second-last equality comes from the fact that $\exp\left((\ln q)^{\frac{1}{4}}\,p\right) = O\left(q^{\frac{\varepsilon}{4m+2}}\right)$ which is derived by $(\ln q)^{\frac{1}{4}}\,p = O\left(\ln q^{\frac{\varepsilon}{4m+2}}\right)$. The last equality comes from the fact that

$$\lim_{\delta\to\infty} \frac{(\ln q)^{\frac{1}{4}}\left(\frac{2}{m}\ln q\right)^{(\ln q)^{\frac{1}{4}}}}{q^{\frac{\varepsilon}{4m+2}}} = 0.$$

Since the event that a vertex v is picked into X is independent of the event that $\deg^{\delta}_{G[X]}(v) \le \lfloor(\ln q)^{\frac{1}{4}}\rfloor$, we have

$$\begin{aligned}
P[v \in X;\ \deg^{\delta}_{G[X]}(v) < \lfloor(\ln q)^{\frac{1}{4}}\rfloor] &= P\,[v \in X] \cdot P[\deg^{\delta}_{G[X]}(v) < \lfloor(\ln q)^{\frac{1}{4}}\rfloor] \\
&< p\frac{1}{q^{\frac{1}{2m+1}}}.
\end{aligned}$$

Let ℓ denote the number of vertices in $G[X]$ satisfying $\deg^{\delta}_{G[X]}(v) \le \lfloor(\ln q)^{\frac{1}{4}}\rfloor$, then we have

$$E\,[\ell] < np\frac{1}{q^{\frac{1}{2m+1}}}.$$

By Markov's inequality, that is, for any $s > 0$, $P[\ell > s] < \frac{E[\ell]}{s}$, it follows that

$$P\left[\ell > \frac{\varepsilon k}{k+1}np\right] = P\left[\ell > \left(\frac{\varepsilon k}{k+1}q^{\frac{1}{2m+1}}\right)\frac{np}{q^{\frac{1}{2m+1}}}\right] < \frac{k+1}{\varepsilon k q^{\frac{1}{2m+1}}}.$$

Let

$$\tau = 1 - \frac{k+1}{k\varepsilon q^{\frac{1}{2m+1}}} - P\left[\alpha < \frac{k}{k+1}np + \frac{1}{\varepsilon}\right].$$

Thus,

$$P\left[\ell \le \varepsilon\frac{k}{k+1}np, \alpha \ge \frac{k}{k+1}np + \frac{1}{\varepsilon}\right] \ge \tau$$

in which we notice that $\ell \leq \lfloor \varepsilon\alpha \rfloor - 1$. Thus, by the definitions of θ and ℓ,

$$P\left[\theta - 1 \geq \lfloor (\ln q)^{\frac{1}{4}} \rfloor\right] = P\left[\theta \geq \left\lfloor (\ln q)^{\frac{1}{4}} \right\rfloor + 1\right] \geq \tau.$$

Finally, we have

$$\begin{aligned}
P\left[\theta \leq \lfloor (\ln q)^{\frac{1}{4}} \rfloor\right] &\leq 1 - \tau \\
&= \frac{k+1}{k\varepsilon q^{\frac{1}{2m+1}}} + P\left[\alpha < \frac{k}{k+1}np + \frac{1}{\varepsilon}\right] \\
&< \frac{k+1}{k\varepsilon q^{\frac{1}{2m+1}}} + \frac{\exp\left(\frac{1}{\varepsilon(k+1)}\right)}{q^{\frac{1}{2(k+1)^2}}}
\end{aligned}$$

as required. Here, the last inequality comes from Claim 7. ■

From Claim 6, Claim 8 and (5), we find that

$$\begin{aligned}
E\left[\frac{\alpha}{\theta}\right] &< nP\left[\alpha > (3k+1)np\right] + (3k+1)npP\left[\theta \leq \lfloor (\ln q)^{\frac{1}{4}} \rfloor\right] + \frac{3k+1}{\lfloor (\ln q)^{\frac{1}{4}} \rfloor}np \\
&< n\frac{1}{q^{2.25}} + (3k+1)np\left(\frac{k+1}{k\varepsilon q^{\frac{1}{2m+1}}} + \frac{\exp\left(\frac{1}{\varepsilon(k+1)}\right)}{q^{\frac{1}{2(k+1)^2}}}\right) + \frac{3k+1}{\lfloor (\ln q)^{\frac{1}{4}} \rfloor}np \\
&= n\frac{\ln q}{q}\left[\frac{1}{q^{1.25}\ln q} + (3k+1)(1+\varepsilon)\left(\frac{k+1}{k\varepsilon q^{\frac{1}{2m+1}}} + \frac{\exp\left(\frac{1}{\varepsilon(k+1)}\right)}{q^{\frac{1}{2(k+1)^2}}}\right) + \frac{3k+1}{\lfloor (\ln q)^{\frac{1}{4}} \rfloor}(1+\varepsilon)\right] \\
&= o\left(n\frac{\ln q}{q}\right).
\end{aligned} \tag{6}$$

Hence, by (3) and (6), Claims 2 and 5, for sufficiently large δ, we have

$$\begin{aligned}
\gamma_k^2(G) &\leq E\left[\gamma_k^2(G)\right] \\
&< \left[1 + 4k(2k+3)(4k-1)\varepsilon\right]E[\alpha] + \left[1 + 4k(2k+3)\right]E[\beta_1 + \beta_2] + \frac{8k(2k+3)(3k-1)}{2k-1}E\left[\frac{\alpha}{\theta}\right] \\
&< \left[1 + 4k(2k+3)(4k-1)\varepsilon\right](1+\varepsilon)n\frac{\ln q}{q} + \left[1 + 4k(2k+3)\right]o\left(\frac{n}{q}\right) + \frac{8k(2k+3)(3k-1)}{2k-1}o\left(n\frac{\ln q}{q}\right) \\
&< \left(1 + 32k^2(2k+3)\varepsilon\right)n\frac{\ln q}{q} \\
&= (1 + o_\delta(1))\,n\frac{\ln q}{q}.
\end{aligned}$$

Here, the second-last inequality comes from the facts,

$$\lim_{\delta\to\infty}\frac{\left[1 + 4k(2k+3)\right]o\left(\frac{n}{q}\right)}{\frac{n\ln q}{q}} = 0 \quad \text{and} \quad \lim_{\delta\to\infty}\frac{\frac{8k(2k+3)(3k-1)}{2k-1}o\left(n\frac{\ln q}{q}\right)}{\frac{n\ln q}{q}} = 0.$$

Thus, for sufficiently large δ, we have

$$\frac{\left[1 + 4k(2k+3)\right]o\left(\frac{n}{q}\right)}{\frac{n\ln q}{q}} < \varepsilon \quad \text{and} \quad \frac{\frac{8k(2k+3)(3k-1)}{2k-1}o\left(n\frac{\ln q}{q}\right)}{\frac{n\ln q}{q}} < \varepsilon.$$

This completes the proof of the theorem. ■

Remarks: This paper gives probabilistic analysis of the upper bounds for 2-connected distance k-domination numbers whose method adopted here is different with [13,14] and a refinement of [4]. For $k = 1$ in (1), we get $\gamma_1^2(G) < (1 + o_\delta(1))n\frac{\ln\delta}{\delta}$, which is the result of Caro et al. in [4].

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