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RESEARCH ARTICLE

Fault-tolerant panconnectivity of augmented cubes*

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Abstract The augmented cube AQ_n is a variation of the hypercube Q_n. This paper considers the panconnectivity of AQ_n ($n \geqslant 3$) with at most $2n-5$ faulty vertices and/or edges and shows that, for any two fault-free vertices u and v with distance d in AQ_n, there exist fault-free uv-paths of every length from $d+2$ to $2^n - f - 1$, where f is the number of faulty vertices in AQ_n. The proof is based on an inductive construction.

Keywords Path, pancyclic, hamiltonian connected, panconnectivity, augmented cube, fault tolerance
MSC 05C38, 90B10

1 Introduction

It is well known that a topological structure of an interconnection network can be modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network [19]. One of the central issues in evaluating a network is the embedding problem. A path or cycle structure is suitable for designing simple parallel algorithms with low communication cost.

A graph G of order n is *l-pancyclic* if G contains a cycle of length k for every k with $l \leqslant k \leqslant n$, and G is *pancyclic* if it is g-pancyclic, where g is the girth of G, the length of a shortest cycle in G. A graph is *hamiltonian connected* if for any pair of distinct vertices u and v, there exists a uv-hamiltonian path. A graph is *panconnected* if for any pair of distinct vertices u and v with distance d, there exists a uv-path of length l for every l with $d \leqslant l \leqslant n-1$.

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Since some vertex and/or link faults may happen when a network is put in use, it is practically meaningful and important to consider faulty networks. A graph G is *k-fault-tolerant pancyclic* (resp. *hamiltonian connected, panconnected*) if $G - F$ remains pancyclic (resp. hamiltonian connected, panconnected) for any $F \subset V(G) \cup E(G)$ with $|F| \leqslant k$, and is *k-vertex-fault-tolerant pancyclic* (resp. *hamiltonian connected, panconnected*) if $G - F$ remains pancyclic (resp. panconnected) for any $F \subset V(G)$ with $|F| \leqslant k$, and *k-edge-fault-tolerant pancyclic* (resp. *hamiltonian connected, panconnected*) if $G - F$ remains pancyclic (resp. hamiltonian connected, panconnected) for any $F \subset E(G)$ with $|F| \leqslant k$.

In recent years, cycle embedding and path embedding, fault-tolerant cycle embedding and fault-tolerant path embedding in the hypercube and other networks have been widely investigated in the literature, as, for example, Refs. [1,4–6,9,11,13–17], which all appeared in *Theoretical Computer Science*. Almost all known results on this topic for the hypercube and its variations are stated in a survey article by Xu and Ma [20].

As a variation of the hypercube network Q_n, the augmented cube AQ_n, as proposed by Choudum and Sunitha [2,3], is pancyclic for $n \geqslant 2$. Recently, this result has been generalized by several authors. Hsu et al. [7] showed that AQ_n is $(2n-3)$-fault-tolerant hamiltonian and $(2n-4)$-fault-tolerant hamiltonian connected for $n \geqslant 4$. Ma et al. [10] showed that AQ_n is panconnected for $n \geqslant 1$ and $(2n-3)$-edge-fault-tolerant pancyclic for $n \geqslant 2$. Wang et al. [18] showed that AQ_n is $(2n-3)$-fault-tolerant pancyclic for $n \geqslant 4$. Recently, Ma et al. [12] have showed that the super connectivity is $4n-8$ for $n \geqslant 6$ and the super edge-connectivity is $4n-4$ for $n \geqslant 5$. In this paper, we improve these results by showing the following result.

Theorem 1.1 *If AQ_n ($n \geqslant 3$) contains at most $2n-5$ faulty vertices and/or edges, then for any two distinct non-faulty vertices u and v with distance d in AQ_n, there exist fault-free uv-paths of length l for every l with $d+2 \leqslant l \leqslant 2^n - 1 - f$, where f is the number of faulty vertices in AQ_n.*

The proof is based on an inductive construction of AQ_n and given in Section 4. Section 2 gives the definition of the augmented cube and some propositions. Some lemmas are given in Section 3. In Section 5, we make a conclusion and suggest two questions to investigate further.

2 Definition and preliminaries

Let $G = (V, E)$ be a graph, where V is the vertex-set and E is the edge-set. For two distinct vertices u and v in G, a uv-path P of length k is a sequence of different vertices $(x_0, x_1, \ldots, x_k)$, where $x_0 = u$, $x_k = v$, and $x_{i-1}x_i \in E(G)$ for each $i = 1, 2, \ldots, k$, where k is the number of edges in P, called the *length* of P, denoted by $\varepsilon(P) = k$. The distance between them, denoted by $d_G(u, v)$, is the length of a shortest uv-path in G. Let $P = (u, \ldots, t, x, y, z, \ldots, v)$ be a uv-path of length at least two. An interior vertex x in P partitions P into two sections. We use $P(u,x)$ to denote the subpath $(u,\ldots,t,x)$ of P from u to x and use $P(y,v)$ to denote the subpath $(y,z,\ldots,v)$ of P from y to v. Since xy is an edge in P, we can write the path

$$P = P(u,x) + xy + P(y,v).$$

The *n-dimensional augmented cube* AQ_n ($n \geqslant 1$), can be defined recursively as follows. AQ_1 is a complete graph K_2 with the vertex set $\{0,1\}$. For $n \geqslant 2$, AQ_n is obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and adding $2 \times 2^{n-1}$ edges between AQ^0_{n-1} and AQ^1_{n-1} as follows.

Let

$$V(AQ^0_n) = \{0u_{n-1}\cdots u_2u_1:\ u_i = 0 \text{ or } 1,\ i = 1,2,\ldots,n-1\},$$

$$V(AQ^1_n) = \{1v_{n-1}\cdots v_2v_1:\ v_i = 0 \text{ or } 1,\ i = 1,2,\ldots,n-1\}.$$

A vertex $u = 0u_{n-1}\cdots u_2u_1$ of AQ^0_{n-1} is joined to a vertex $v = 1v_{n-1}\cdots v_2v_1$ of AQ^1_{n-1} if and only if either

(1) $u_i = v_i$ for $1 \leqslant i \leqslant n-1$ (in this case uv is called an *n-dimensional hypercube edge*, setting $v = u^{h_n}$ or $u = v^{h_n}$), or

(2) $u_i = \overline{v}_i$ for $1 \leqslant i \leqslant n-1$ (in this case uv is called an *n-dimensional complement edge*, setting $v = u^{c_n}$ or $u = v^{c_n}$).

And an edge between $u = u_nu_{n-1}\cdots u_2u_1$ and $v = u_nu_{n-1}\cdots u_2\overline{u}_1$ ($u_i =$ 0 or 1, $1 \leqslant i \leqslant n$) is called a 1-*dimensional complement edge*, setting $v = u^{c_1}$ or $u = v^{c_1}$. For example, the graphs shown in Fig. 1 are augmented cubes AQ_1, AQ_2 and AQ_3.

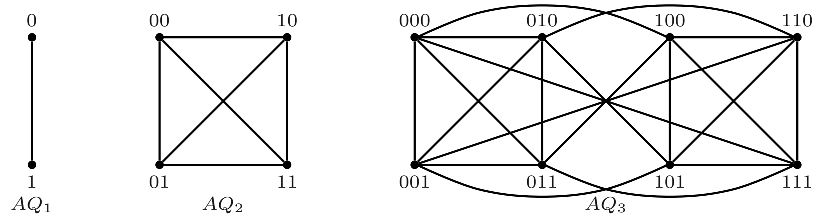

Fig. 1 Three augmented cubes AQ_1, AQ_2 and AQ_3

Obviously, AQ_n is a $(2n-1)$-regular graph with 2^n vertices. It has been shown by Choudum and Sunitha [2,3] that AQ_n is vertex-symmetric, $(2n-1)$-connected for $n \neq 3$ (AQ_3 is 4-connected), and has diameter $\lceil n/2 \rceil$ for $n \geqslant 1$. Some further properties of AQ_n can be found in Refs. [12,21].

For the sake of simplicity, we use $d(x,y)$ to denote the distance between x and y in AQ_n, and write $L = AQ^0_{n-1}$ and $R = AQ^1_{n-1}$. For each vertex $v \in L$ (or R), let $N_L(v)$ (or $N_R(v)$) denote the set of vertices adjacent to v in L (or R).

For a vertex u in AQ_n, we use u^h to denote u^{h_n} and use u^c to denote u^{c_n}. Let $I_n = \{h_2,h_3,\ldots,h_n,c_1,c_2,\ldots,c_n\}$. If $P = (u,x_1,x_2,\ldots,x_t,v)$ is a

uv-path in AQ_n, we use P^b to denote the u^bv^b-path $(u^b, x_1^b, x_2^b, \ldots, x_t^b, v^b)$ in AQ_n for any $b \in I_n$. If $S = \{x_1, x_2, \ldots, x_t\}$ is a subset of vertices in AQ_n, we use S^b to denote the set of vertices $\{x_1^b, x_2^b, \ldots, x_t^b\}$ with $b \in I_n$.

The following two properties can be easily verified from the definition of AQ_n.

Proposition 2.1 *If uv is an edge in AQ_n ($n \geqslant 2$), then so is u^bv^b for any $b \in I_n$.*

Proposition 2.2 *Let u be a vertex in AQ_n ($n \geqslant 2$). Then, for any i with $2 \leqslant i \leqslant n$, u^{h_i} and u^{c_i} are joined by an $(i-1)$-dimensional complement edge; u^{c_i} and $u^{c_{i-1}}$ are joined by an i-dimensional hypercube edge; u^{h_i} and $u^{c_{i-1}}$ are joined by an i-dimensional complement edge; otherwise, u^a and u^b are not adjacent for any two distinct $a, b \in I_n$.*

By Propositions 2.1 and 2.2, we have the following property immediately.

Proposition 2.3 *Let uv be an edge in AQ_n ($n \geqslant 2$). If uv is not an $(n-1)$-dimensional complement edge, then u^h, u^c, v^h and v^c are all distinct. Otherwise $u^h = v^c$, $u^c = v^h$.*

Proposition 2.4 *In AQ_n ($n \geqslant 3$), for any vertex $u \in L$, let $S = N_L(u)$. Then*

$$S^h = N_R(u^h), \quad S^c = N_R(u^c), \quad |S^h \cap S^c| = 2.$$

Proof Let $u = 0u_{n-1} \cdots u_2u_1 \in L$. Then

$$S = N_L(u) = \{u^{h_i}\colon\ 2 \leqslant i \leqslant n-1\} \cup \{u^{c_j}\colon\ 1 \leqslant j \leqslant n-1\},$$

where

$$\begin{aligned} u^{h_i} &= 0u_{n-1} \cdots u_{i+1}\overline{u}_i u_{i-1} \cdots u_1, \quad 2 \leqslant i \leqslant n-1, \\ u^{c_j} &= 0u_{n-1} \cdots u_{j+1}\overline{u}_j \overline{u}_{j-1} \cdots \overline{u}_1, \quad 1 \leqslant j \leqslant n-1. \end{aligned} \tag{1}$$

Thus,

$$S^h = \{(u^{h_i})^h\colon\ 2 \leqslant i \leqslant n-1\} \cup \{(u^{c_j})^h\colon\ 1 \leqslant j \leqslant n-1\},$$

where

$$\begin{aligned} (u^{h_i})^h &= 1u_{n-1} \cdots u_{i+1}\overline{u}_i u_{i-1} \cdots u_1, \quad 2 \leqslant i \leqslant n-1, \\ (u^{c_j})^h &= 1u_{n-1} \cdots u_{j+1}\overline{u}_j \overline{u}_{j-1} \cdots \overline{u}_1, \quad 1 \leqslant j \leqslant n-1; \end{aligned} \tag{2}$$

and

$$S^c = \{(u^{h_i})^c\colon\ 2 \leqslant i \leqslant n-1\} \cup \{(u^{c_j})^c\colon\ 1 \leqslant j \leqslant n-1\},$$

where

$$\begin{aligned} (u^{h_i})^c &= 1\overline{u}_{n-1} \cdots \overline{u}_{i+1} u_i \overline{u}_{i-1} \cdots \overline{u}_1, \quad 2 \leqslant i \leqslant n-1, \\ (u^{c_j})^c &= 1\overline{u}_{n-1} \cdots \overline{u}_{j+1} u_j u_{j-1} \cdots u_1, \quad 1 \leqslant j \leqslant n-1. \end{aligned} \tag{3}$$

Since

$$u^h = 1u_{n-1} \cdots u_2u_1 \in R, \quad u^c = 1\overline{u}_{n-1} \cdots \overline{u}_1 \in R,$$

from (2) and (3), it is easy to verify that

$$S^h = N_R(u^h), \quad S^c = N_R(u^c).$$

Also from (2) and (3), it is easy to see that only two vertices $(u^{h_{n-1}})^h = (u^{c_{n-2}})^c$ and $(u^{c_{n-2}})^h = (u^{h_{n-1}})^c$ in $S^h \cap S^c$, which implies

$$|S^h \cap S^c| = 2. \qquad \square$$

For example, let $u = 00000$ be a vertex in AQ_n. Then 7 vertices in S are

$$u^{h_2} = 00010, \quad u^{h_3} = 00100, \quad u^{h_4} = 01000,$$

$$u^{c_1} = 00001, \quad u^{c_2} = 00011, \quad u^{c_3} = 00111, \quad u^{c_4} = 01111.$$

Thus,

$$S^h = \{10010, 10100, 11000, 10001, 10011, 10111, 11111\},$$

$$S^c = \{11101, 11011, 10111, 11110, 11100, 11000, 10000\},$$

and so

$$S^h \cap S^c = \{11000, 10111\}.$$

Proposition 2.5 *For any edge uv in AQ_n $(n \geqslant 3)$, there exist p internally disjoint uv-paths of length* 3, *where $p = 2n - 4$ if $v = u^{c_i}$ $(2 \leqslant i \leqslant n-1)$, and $p = 2n - 3$ otherwise.*

Proof We prove the proposition by induction on $n \geqslant 3$. For $n = 3$, it is easy to check that the conclusion holds. Now assume that the proposition holds for $n - 1$.

Case 1 uv is not an n-dimensional (complement/hypercube) edge. Without loss of generality, assume that uv is an edge in L.

If $v = u^{c_j}$, $2 \leqslant j \leqslant n-2$, by the induction hypothesis, there exist $2n-6$ internally disjoint uv-paths of length 3 in L. By Proposition 2.3, u^h, u^c, v^h and v^c are all distinct, then $uu^h + u^hv^h + v^hv$ and $uu^c + u^cv^c + v^cv$ are two internally disjoint uv-paths of length 3. Thus, there exist $2n-4$ internally disjoint uv-paths of length 3 in AQ_n.

If $v = u^{c_1}$ or $v = u^{h_j}$, $2 \leqslant j \leqslant n-1$, by the induction hypothesis, there exist $2n-5$ internally disjoint uv-paths of length 3 in L. For the same reason as the above, $uu^h + u^hv^h + v^hv$ and $uu^c + u^cv^c + v^cv$ are two internally disjoint uv-paths of length 3. Thus, there exist $2n-3$ internally disjoint uv-paths of length 3 in AQ_n.

If $v = u^{c_{n-1}}$, by the induction hypothesis, there exist $2n-5$ internally disjoint uv-paths of length 3 in L. By Proposition 2.3, $u^h = v^c$ and $u^c = v^h$, then $uu^h + u^hv^h + v^hv$ and $uu^c + u^cv^c + v^cv$ are two internally joint uv-paths of length 3. Thus, there exist $2n-4$ internally disjoint uv-paths of length 3 in AQ_n.

Case 2 uv is an n-dimensional (complement/hypercube) edge. Without loss of generality, assume $u \in L$ and $v \in R$.

If $b \in I_{n-1}$, then $u^b v^b$ is an n-dimensional edge, and then $uu^b + u^b v^b + v^b v$ is a uv-path of length 3. Since $|I_{n-1}| = 2n-3$, $u^b \in L$, $v^b \in R$, there exist at least $2n-3$ internally uv-paths of length 3. If $v = u^h$, we have $u^c = v^{c_{n-1}}$; if $v = u^c$, we have $u^h = v^{c_{n-1}}$. Since $c_{n-1} \in I_{n-1}$, there exist exactly $2n-4$ internally uv-paths of length 3 in AQ_n.

By the induction principle, the proposition follows. □

Proposition 2.6 *Let u and v be any two distinct vertices in AQ_n. Then $d(u^b, v^b) = d(u, v)$ for any $b \in I_n$.*

Proof Assume $d(u, v) = d_1$ and $d(u^b, v^b) = d_2$. There exist a uv-path P_1 of length d_1 and a $u^b v^b$-path P_2 of length d_2. Assume that $P_1 = (u, x_1, x_2, \ldots, x_{d_1-1}, v)$. Then $P_1^b = (u^b, x_1^b, x_2^b, \ldots, x_{d_1-1}^b, v^b)$ is a $u^b v^b$-path of length d_1. Then we know that $d_2 \leqslant d_1$.

Assume that x and y are two distinct vertices in AQ_n. If $u = v^b$, then $u^b = v$. Assume that $P_2 = (u^h, y_1, y_2, \ldots, y_{d_2-1}, v^h)$. Then $P_2^b = (u, y_1^b, y_2^b, \ldots, y_{d_2-1}^b, v)$ is a uv-path of length d_2. Then we know that $d_1 \leqslant d_2$. So $d_1 = d_2$.

The proof is complete. □

Proposition 2.7 (Choudum and Sunitha [2]) *For any two distinct vertices $u \in L$ and $v \in R$ with distance d in AQ_n $(n \geqslant 2)$, $d(u, v^c) = d-1$ or $d(u, v^h) = d-1$.*

3 Some lemmas

Let F denote the set of faulty vertices and/or faulty edges in AQ_n, f denote the number of faulty vertices in AQ_n, F_L and F_R denote the set of faulty vertices and/or faulty edges in L and R, respectively, and f_L and f_R denote the number of faulty vertices in L and R, respectively. We have $f = f_L + f_R$. A subgraph of AQ_n is *fault-free* if it contains no element in F.

Lemma 3.1 (Hsu, Chiang, Tan and Hsu [7]) *AQ_n $(n \geqslant 2)$ is $(2n-4)$-fault hamiltonian connected for $n \neq 3$, and AQ_3 is 1-fault hamiltonian connected.*

Lemma 3.2 (Wang, Ma and Xu [18]) *AQ_n is $(2n-3)$-fault-tolerant pancyclic for $n \geqslant 4$, and AQ_3 is 2-fault-tolerant pancyclic.*

Lemma 3.3 (Hsu, Chiang, Tan and Hsu [7]) *For any four distinct vertices u, v, x, y in AQ_n $(n \geqslant 2)$, there exist a ux-path P_1 and a vy-path P_2 such that P_1 and P_2 are internally disjoint and $P_1 \cup P_2$ contains all vertices of AQ_n.*

Lemma 3.4 (Hsu, Lai, Tsai [8]) *For any two distinct vertices u and v with distance $d \geqslant 2$ in AQ_n $(n \geqslant 3)$, there exist at least two internally disjoint uv-paths of length l for every l with $d \leqslant l \leqslant 2^{n-1}$ in AQ_n.*

Lemma 3.5 *In AQ_n $(n \geqslant 3)$, if $|F| \leqslant 2n-5$, then for any two fault-free vertices $u \in L$ and $v \in R$ with $d(u, v) = 1$, there exist two fault-free uv-paths of every length 3 and 4, respectively.*

Proof Let $u \in L$ and $v \in R$ with $d(u,v) = 1$. Since $|F| \leqslant 2n-5$, by Proposition 2.5, there exists a fault-free uv-path of length 3 in AQ_n.

We now show that there exists a fault-free uv-path of length 4 in AQ_n. Without loss of generality, assume that $|F_L| \leqslant |F_R|$ and $v = u^h$. Let $S = N_L(u)$. Then $S^h = N_R(v)$ by Proposition 2.4, that is,

$$S = N_L(u) = \{u^{h_i}: \ 2 \leqslant i \leqslant n-1\} \cup \{u^{c_j}: \ 1 \leqslant j \leqslant n-1\},$$

where u^{h_i} and u^{c_j} are defined in (1). By the proof of Proposition 2.4, $(u^{h_{n-1}})^h = (u^{c_{n-2}})^c$ and $(u^{c_{n-2}})^h = (u^{h_{n-1}})^c$ are only two vertices in $S^h \cap S^c$. Let $T = S - \{u^{c_{n-2}}\}$. Then $T^h \cap T^c = \emptyset$.

For the sake of simplicity, let $T = \{x_1, x_2, \ldots, x_{2n-4}\}$, where, $x_1 = u^{c_{n-1}}$. Clearly, $P_1 = (u, x_1, v)$ is a uv-path of length 2 in AQ_n. By Proposition 2.2, for each $i = 2, 3, \ldots, 2n-4$, x_i^h and x_i^c are joined by an $(n-1)$-dimensional complement edge, and so $P_i = (u, x_i, x_i^c, x_i^h, v)$ is a uv-path of length 4 in AQ_n. Since $T \subset L$, $T^h, T^c \subset R$, and $T^h \cap T^c = \emptyset$, the paths $P_1, P_2, \ldots, P_{2n-4}$ are internally disjoint uv-paths, at least one of them is fault-free since $|F| \leqslant 2n-5$.

If P_i is fault-free for some i with $2 \leqslant i \leqslant 2n-4$, we are done. Otherwise, P_1 is fault-free since $|F| \leqslant 2n-5$.

Since $|F_L| \leqslant |F_R|$ by our hypothesis, $|F_L| \leqslant n-3 \leqslant 2(n-1)-4$ for $n \geqslant 3$. By Proposition 2.5, there exists a fault-free ux_1-path P_L of length 3 in L. Clearly, $x_1v \notin F$ since x_1v is not in P_i for each $i = 2, 3, \ldots, 2n-4$. Then $P_L + x_1v$ is a fault-free uv-path of length 4.

The lemma follows. □

Lemma 3.6 *If AQ_3 contains only one faulty element that is a vertex, then for any two distinct fault-free vertices u and v, there exists a fault-free uv-path of length l for every l with $2 \leqslant l \leqslant 6$.*

Proof Since AQ_3 is vertex-symmetric, we can suppose that $w = 000$ is a faulty vertex (see Fig. 2). Let u and v be two distinct vertices in $AQ_3 - w$. We need to prove that $AQ_3 - w$ contains a uv-path of length l for every l with $2 \leqslant l \leqslant 6$. Toward that end, assume that $L = AQ_2^0$ and $R = AQ_2^1$.

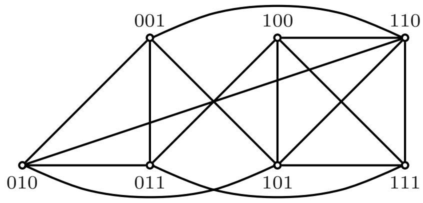

Fig. 2 $AQ_3 - \{000\}$

Case 1 Both u and v in $L - w$.

It is easy to see from Fig. 2 that $L - w$ contains a uv-path of length 2 since $L - w$ is a triangle. Since uv is an edge in L, by Proposition 2.1, $u^h v^h$ is an edge in R. Thus, $uu^h + u^h v^h + v^h v$ is a uv-path of length 3 in $AQ_3 - w$. Let x be the vertex in $L - w$ different from u and v. For $4 \leqslant l \leqslant 6$, let $l_1 = l - 3$. Then $1 \leqslant l_1 \leqslant 3$. Similarly, R contains a $u^h x^h$-path P_R of length l_1, and so the path $uu^h + P_R + x^h x + xv$ is a uv-path of length l in $AQ_3 - w$.

Case 2 Both u and v in R.

For $2 \leqslant l \leqslant 3$, R contains a uv-path of length l since R is a complete graph of order 4. For $4 \leqslant l \leqslant 5$, let $l_1 = l - 3$. Then $1 \leqslant l_1 \leqslant 2$. Assume that x and y are two other vertices in R different from u and v. Then at least two vertices in $\{v^h, v^c, x^h, x^c\}$ or $\{v^h, v^c, y^h, y^c\}$ are fault-free. Without loss of generality, assume that v^h and x^h are fault-free. Since L contains a $v^h x^h$-path P_L of length l_1, the path $ux + xx^h + P_L + v^h v$ is fault-free uv-path of length l, and $uy + yx + xx^h + P_L + v^h v$ is a fault-free uv-path of length 6 when the length of P_L is 2.

Case 3 $u \in L - w$ and $v \in R$.

For $2 \leqslant l \leqslant 4$, let $l_1 = l - 1$. Then $1 \leqslant l_1 \leqslant 3$. Since at least one of u^h and u^c is not v, we can, without loss of generality, assume $u^h \neq v$. Since R contains a $u^h v$-path P_R of length l_1, $uu^h + P_R$ is a uv-path of length l in $AQ_3 - w$. Let x and y be vertices in $L - w$ different from u and, without loss of generality, assume $x^h \neq v$. For $5 \leqslant l \leqslant 6$, let $l_1 = l - 3$. Then $2 \leqslant l_1 \leqslant 3$. Since R contains an $x^h v$-path P'_R of length l_1, $uy + yx + xx^h + P'_R$ is a uv-path of length l in $AQ_3 - w$.

The proof of the lemma is complete. □

Lemma 3.7 *Let w be any vertex in AQ_3. Then for any four distinct vertices u, v, x, y in $AQ_3 - w$, there exist two disjoint either ux-path P_1 and vy-path P_2 or uy-path P_3 and vx-path P_4, such that they contains all vertices of $AQ_3 - w$.*

Proof Since AQ_3 is vertex-symmetric, we can suppose that $w = 000$ is a faulty vertex (see Fig. 2). $L - w$ is a completed graph of 3 vertices and R is a completed graph of 4 vertices.

Case 1 $u, v, x, y \in R$. Without loss of generality, assume that v^h or y^h is fault. We know that vy is an edge in R. And in $L - w$, there exists a hamiltonian path P_L between u^h and x^h. Let $P_1 = uu^h + P_L + x^h x$ and $P_2 = vy$. Then the lemma holds.

Case 2 Three of u, v, x, y are in R, one is in $L - w$. Without loss of generality, assume that $u, v, x \in R$ and $y \in L - w$. Let z_1 and z_2 be two vertices in $L - w$ different from y. Then one of z_1^h and z_2^h is not x, assume $z_1^h \neq x$.

If $z_1^h = u$, then in $R - \{u\}$ there exists a vx-path P_R of length 2. Let $P_3 = uz_2 + z_2 z_1 + z_1 u$ and $P_4 = P_R$. Then the lemma holds.

If $z_1^h = v$, then in $R - \{v\}$ there exists a ux-path P'_R of length 2. Let $P_1 = yz_2 + z_2 z_1 + z_1 v$ and $P_4 = P'_R$. Then the lemma holds.

If $z_1^h \neq u$ and $z_1^h \neq v$, z_1^h is incident with v in R. Let $P_1 = ux$ and $P_2 = vz_1^h + z_1^h z_1 + z_1 z_2 + z_2 y$. Then the lemma holds.

Case 3 Two of u, v, x, y are in R, and two are in $L - w$.

Subcase 3.1 Both u and x are in the same part of R or $L - w$, and both v and y are in the other part. Without loss of generality, assume $u, x \in$

R, $v, y \in L - w$. Since R and $L - w$ are completed graphs, there exist a hamiltonian ux-path P_R in R and a hamiltonian vy-path P_L in $L - w$. Let $P_1 = P_R$ and $P_2 = P_L$. Then the lemma holds.

Subcase 3.2 Both u and y are in the same part of R or $L - w$, and both v and x are in the other part. Without loss of generality, assume $u, y \in R$, $v, x \in L - w$. Since R and $L - w$ are completed graphs, there exist a hamiltonian uy-path P_R in R and a hamiltonian vx-path P_L in $L - w$. Let $P_3 = P_R$ and $P_4 = P_L$. Then the lemma holds.

Subcase 3.3 Both u and v are in the same part of R or $L - w$, and both x and y are in the other part. Without loss of generality, assume $u, v \in L - w$, $x, y \in R$. Let z be a vertex in $L - w$ different from u and v. Then one of uz and vz is not a 2-dimensional complement edge. Assume that uz is not a 2-dimensional complement edge. Then u^h, z^h, u^c, z^h are 4 distinct vertices.

Assume that either u^h or u^c is in $\{x, y\}$, without loss of generality, say $u^h = x$. Since u^h, z^h, u^c, z^h are 4 distinct vertices, one of z^h and z^c is not x and y, say $z^h \neq y$. Since $R - \{x\}$ is a completed graph, there exists a $z^h y$-path P_R of length 2 in $R - \{x\}$. Let $P_1 = ux$ and $P_2 = vz + zz^h + P_R$. Then the lemma holds.

Assume that neither u^h nor u^c is in $\{x, y\}$ below. Since u^h, z^h, u^c, z^h are 4 distinct vertices, $\{z^h, z^c\} = \{x, y\}$, say $z^h = x$ and $z^c = y$. Since $u^h \neq x$, there exists a $u^h x$-path P'_R of length 2 in $R - \{y\}$. Let $P_1 = uu^h + P'_R$ and $P_2 = vz + zy$. Then the lemma holds.

Case 4 One of u, v, x, y is in R, three are in $L - w$. Without loss of generality, assume $u, v, x \in L - w$ and $y \in R$. One of v^h and v^c is not y, say $v^h \neq y$. There exists a hamiltonian $u^h x$-path P_R of length 3 in R. Let $P_1 = ux$ and $P_2 = vv^h + P_R$. Then the lemma holds.

The proof of the lemma is complete. □

Lemma 3.8 *If AQ_n $(n \geqslant 3)$ contains at most $2n - 5$ faulty vertices and no faulty edges, then for any two distinct fault-free vertices u and v with distance d, there exist fault-free uv-paths of length l for each $l = d + 2$, $d + 3$.*

Proof We prove the lemma by induction on $n \geqslant 3$. The induction basis for $n = 3$ holds by Lemma 3.6. Assume that the lemma holds for $n - 1$ with $n \geqslant 4$. Without loss of generality, assume that

$$|F_L| = f_L \leqslant |F_R| = f_R.$$

Then $f_L \leqslant n - 3$. Let u and v be any two distinct fault-free vertices with distance d in AQ_n.

Case 1 Both u and v are in $L - F$.

Since

$$f_L \leqslant n - 3 \leqslant 2(n - 1) - 5 \quad (n \geqslant 4),$$

by the induction hypothesis, there exists a fault-free uv-path of length l for each $l = d + 2$, $d + 3$ in L, and so in AQ_n.

Case 2 Both u and v are in $R - F$.

If $f_R \leqslant 2n-7$, then the conclusion holds by the induction hypothesis. Assume $f_R \geqslant 2n-6$ below. Then $f_L \leqslant 1$.

Subcase 2.1 If uv is not an $(n-1)$-dimensional complementary edge, then $u^h \neq v^c$ and $u^c \neq v^h$. Since at least one of $\{u^h, v^h\}$ and $\{u^c, v^c\}$ is fault-free, without loss of generality, assume that $\{u^h, v^h\}$ is fault-free.

If $d = 1$, then, by Proposition 2.5, there exists a fault-free uv-path of length 3. By Proposition 2.2 and $f_L \leqslant 1$, there exists a fault-free u^hv^h-path P_L of length 2. Then the path $uu^h + P_L + vv^h$ is a fault-free uv-path of length 4. If $d \geqslant 2$, then, since $d(u^h, v^h) = d$ and $f_L \leqslant 1$, by Lemma 3.4, there exist a fault-free u^hv^h-path P'_L of length d and a fault-free u^hv^h-path P''_L of length $d+1$ in L. Then the path $uu^h + P'_L + v^hv$ is a fault-free uv-path of length $d+2$ and the path $uu^h + P''_L + v^hv$ is a fault-free uv-path of length $d+3$.

Subcase 2.2 If uv is an $(n-1)$-dimensional complementary edge, then $u^h = v^c$ and $u^c = v^h$.

Since $|F| \leqslant 2n-5$, there exists a fault-free uv-path of length 3. We assume that this path is uu^bv^bv, where $b \in I_n$. Then u^bv^b is an $(n-1)$-dimensional complementary edge.

If u^h and v^h are fault-free, then, by Lemma 3.4 and $|F_L| \leqslant 1$, there exists a fault-free u^hv^h-path P_L of length 2. Then the path $uu^h + P_L + vv^h$ is a fault-free uv-path of length 4. We assume that one of u^h and v^h is faulty below. Then we know that $u^b \in R$ and $v^b \in R$. Let $x = u^b$ and $y = v^b$. Since xy is an $(n-1)$-dimensional edge, we know that $x^h = y^c$. Since one of u^h and v^h is faulty and $f_R \leqslant 1$, we know that x^h is fault-free. Then uxx^hyv is a fault-free uv-path of length 4.

Case 3 $u \in L - F$ and $v \in R - F$.

By Lemma 3.5, the lemma holds for $d = 1$. Assume $d \geqslant 2$ below. By Proposition 2.7, $d(u, v^h) = d-1$ or $d(u, v^c) = d-1$. Without loss of generality, assume $d(u, v^h) = d-1$.

Subcase 3.1 $f_R \leqslant 2n-7$.

When v^h or u^h is fault-free, without loss of generality, assume that v^h is fault-free. By the induction hypothesis, in L there exist fault-free uv^h-paths P_L of length $d+1$ and P'_L of length $d+2$. Then $P_L + v^hv$ is a fault-free uv-path of length $d+2$ and $P'_L + v^hv$ is a fault-free uv-path of length $d+3$. Assume that v^h and u^h are faulty below.

Let
$$F = \{v^h, x_1, x_2, \ldots, x_{f_L-1}, u^h, y_1, y_2, \ldots, y_{f_R-1}\},$$
where $x_i \in L,\ 1 \leqslant i \leqslant f_L - 1$ and $y_i \in R,\ 1 \leqslant i \leqslant f_R - 1$. Let
$$S = \{x_i\colon\ 1 \leqslant i \leqslant f_L - 1\}, \quad T = \{y_i\colon\ 1 \leqslant i \leqslant f_R - 1\},$$
$$L' = L - S - T^h.$$
Since
$$|S| + |T| = f_L + f_R - 2 \leqslant 2n - 7,$$
by the induction hypothesis, there exist uv^h-path T_L of length $d+1$ and T'_L of length $d+2$ in L'. We use x to denote the vertex incident with v^h in T_L and

use y to denote the vertex incident with v^h in T'_L. Then $T_L(u,x)+xx^h+x^hv$ is a uv-path of length $d+2$ and $T'_L(u,y)+yy^h+y^hv$ is a uv-path of length $d+3$. Since

$$L' \cap (S+T^h) = \emptyset,$$

we know that $T_L(u,x)$, $T'_L(u,y)$, x^h and y^h are fault-free. So $T_L(u,x)+xx^h+x^hv$ and $T'_L(u,y)+yy^h+y^hv$ are fault-free.

Subcase 3.2 $f_R \geqslant 2n-6$, then $f_L \leqslant 1$. In R, there exist $2n-3$ vertices incident with v. Since $|F| \leqslant 2n-5$, there exists a fault-free vertex x incident with v in R, such that $x^h \neq u$, and x^h is fault-free. Since

$$d(u,v^h) = d-1,$$

we know that

$$d-2 \leqslant d(u,x^h) \leqslant d.$$

By Lemma 3.4, Proposition 2.2 and the induction hypothesis, there exist fault-free ux^h-paths P_L of length $d+1$ and P'_L of length $d+2$. Then P_L+x^hx+xv is a fault-free uv-path of length $d+2$ and P'_L+x^hx+xv is a fault-free uv-path of length $d+3$.

The lemma follows. □

4 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. Start with the following lemma.

Lemma 4.1 *If Theorem* 1.1 *holds for any subset* $F \subset V(AQ_n)$ *with* $|F| = 2n-5$, *then Theorem* 1.1 *holds also for*

(i) *any subset* $F' \subset V(AQ_n)$ *with* $|F'| \leqslant 2n-5$, *and*

(ii) *any subset* $F' \subset V(AQ_n) \cup E(AQ_n)$ *with* $|F'| \leqslant 2n-5$.

Proof (i) Let $m = 2n-5-|F|$. Then $0 \leqslant m \leqslant 2n-5$. We prove the lemma by induction on m. For $m=0$, i.e., $|F| = 2n-5$ for any subset $F \subset V(AQ_n)$, the induction basis holds by our hypothesis. Assume that the lemma holds for any m_0 with $0 \leqslant m_0 < 2n-5$, that is, Theorem 1.1 holds for any subset $F' \subset V(AQ_n)$ with $|F'| = 2n-5-m_0$.

Let $m = m_0+1$, and F be any subset of $V(AQ_n)$ with

$$|F| = 2n-6-m_0 < 2n-5.$$

Let u and v be arbitrary two distinct vertices in $AQ_n - F$ with distance $d = d_{AQ_n}(u,v)$, and let x be a vertex in $AQ_n - F$ different from u and v and $F' = F \cup \{x\}$. Then

$$|F'| = 2n-5-m_0 \leqslant 2n-5,$$

that is,

$$m_0 = 2n - 5 - |F'| \geqslant 0.$$

By the induction hypothesis, for every integer l with

$$d + 2 \leqslant l \leqslant 2^n - |F'| - 1,$$

there exists a uv-path of length l in $AQ_n - F'$, so in $AQ_n - F$.

(ii) We now prove the second assertion by induction on k, where k is the number of faulty edges in any subset

$$F \subset V(AQ_n) \cup E(AQ_n) \quad (|F| \leqslant 2n - 5).$$

The induction basis for $k = 0$ holds by (i). Assume that the lemma holds for k with $0 \leqslant k < 2n - 5$.

Assume that

$$F \subset V(AQ_n) \cup E(AQ_n) \quad (|F| \leqslant 2n - 5,\ |F \cap E(AQ_n)| = k + 1).$$

Let u and v be arbitrary two distinct vertices in $AQ_n - F$ with distance d in AQ_n. When $uv \in F$, let $F' = F - \{uv\}$. Then $|F'| \leqslant 2n - 6$ and F' includes k edges. By the induction hypothesis, for any l with

$$d + 2 \leqslant l \leqslant 2^n - f - 1,$$

there exists a uv-path P of length l in $AQ_n - F'$. Clearly, P does not contain the edge uv. Thus, P is a uv-path P of length l in $AQ_n - F$.

Assume that uv is fault-free below. Let xy be an edge in F. Since xy is not uv, we can assume that $x \neq u$ and $x \neq v$. Let

$$F'' = F - \{uv\} \cup \{x\}.$$

Then

$$|F''| = |F| \leqslant 2n - 5$$

and F'' contains at most k edges. By the induction hypothesis, for every integer l with

$$d + 2 \leqslant l \leqslant 2^n - f - 2,$$

there exists a uv-path P of length l in $AQ_n - F''$. Clearly, P does not contain x, and so P is in $AQ_n - F$. For $l = 2^n - f - 1$, by Lemma 3.1 and $|F| \leqslant 2n - 5$, there exists a fault-free uv-path of length l.

The proof of the lemma is complete. □

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1 By Lemma 4.1, we only need to prove the theorem when $|F| = 2n - 5$ and all faulty elements are vertices.

Now, we prove the theorem by induction on $n \geqslant 3$. The induction basis for $n = 3$ holds by Lemma 3.6. Assume that the theorem holds for any k

with $3 \leqslant k < n$. Let u and v be two distinct vertices in $AQ_n - F$. Since all faulty elements are vertices, we have

$$|F| = f = f_L + f_R.$$

Without loss of generality, assume $f_L \leqslant f_R$. For $l = d + 2$ and $d + 3$, by Lemma 3.8, we are done. For $l = 2^n - f - 1$, by Lemma 3.1, we are done. Assume that

$$d + 4 \leqslant l \leqslant 2^n - f - 2$$

below.

Case 1 $f_R \leqslant 2n - 7$. In this case, n cannot be 4.

Subcase 1.1 Both u and v are in either $L - F$ or $R - F$. Without loss of generality, assume $u, v \in L - F$.

For

$$d + 2 \leqslant l \leqslant 2^{n-1} - f_L - 1,$$

by the induction hypothesis, there exists a uv-path of length l in $L - F$. In particular, we use T_L to denote a uv-path of length $2^{n-1} - f_L - 1$ and use T'_L to denote a uv-path of length $2^{n-1} - f_L - 2$. The path T'_L (resp. T_L) contains $2^{n-1} - f_L - 1$ (resp. $2^{n-1} - f_L$) vertices. We have

$$\frac{2^{n-1} - f_L - 1}{2} \geqslant 2n - 5 - f_L + 1 = f_R + 1 \quad (n \geqslant 5),$$

and so there exists an edge xy in T'_L (resp. T_L) with $\{xx^h, yy^h, x^h y^h\}$ that are fault-free. Without loss of generality, assume that x is closer to u than y.

For $l = 2^{n-1} - f_L$, the path $T'_L(u, x) + xx^h + x^h y^h + y^h y + T'_L(y, v)$ is a fault-free uv-path of length l.

For $l = 2^{n-1} - f_L + 1$, the path $T_L(u, x) + xx^h + x^h y^h + y^h y + T_L(y, v)$ is a fault-free uv-path of length l.

For

$$2^{n-1} - f_L + 2 \leqslant l \leqslant 2^n - f - 2,$$

let

$$l_1 = l - 2^{n-1} + f_L + 1.$$

Then

$$3 \leqslant l_1 \leqslant 2^{n-1} - f_R - 1.$$

By the induction hypothesis, there exists an $x^h y^h$-path P_R of length l_1 in $R - F$. Then the path $T'_L(u, x) + xx^h + P_R + y^h y + T'_L(y, v)$ is a fault-free uv-path of length l $(= l_1 + 2 + 2^{n-1} - f_L - 2 - 1$, see Fig. 3 (a)).

Subcase 1.2 $u \in L - F$ and $v \in R - F$. By Proposition 2.7, without loss of generality, we can assume that $d(u, v^h) = d - 1$.

For

$$d + 4 \leqslant l \leqslant 2^{n-1} - f_L + 1,$$

let $l_1 = l - 2$. Then

$$d + 2 \leqslant l_1 \leqslant 2^{n-1} - f_L - 1.$$

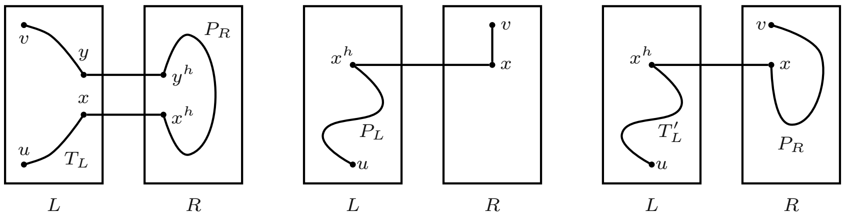

Fig. 3 Illustrations for Case 1

Let $S = N_R(v) - \{u^h\}$. Since $|N_R(v)| = 2n-3$, we have $|S| \geqslant 2n-4$. Since $|F| = 2n-5$, there exists a vertex x in S such that x and x^h are fault-free. Since $d(u, v^h) = d-1$, $d(u, x^h) \leqslant d$. By the induction hypothesis, there exists a fault-free ux^h-path P_L of length l_1. Then the path $P_L + x^hx + xv$ is a fault-free uv-path of length l $(= l_1 + 2$, see Fig. 3 (b)).

Let T'_L be a ux^h-path of length $2^{n-1} - f_L - 2$ in L.

For

$$2^{n-1} - f_L + 2 \leqslant l \leqslant 2^n - f - 2,$$

let

$$l_1 = l - 2^{n-1} + f_L + 1.$$

Then

$$3 \leqslant l_1 \leqslant 2^{n-1} - f_R - 1.$$

By the induction hypothesis, there exists a fault-free vx-path P_R of length l_1 in R. Then the path $T'_L + x^hx + P_R$ is a fault-free uv-path of length l $(= l_1 + 2^{n-1} - f_L - 2 + 1$, see Fig. 3 (c)).

Case 2 $f_R = 2n - 6$. In this case, $f_L = 1$.

Subcase 2.1 Both u and v are in $L - F$.

In this subcase, we have

$$\frac{2^{n-1} - f_L - 1}{2} \geqslant 2n - 5 - f_L + 1 = f_R + 1.$$

For the same reason as Subcase 1.1, for

$$d + 4 \leqslant l \leqslant 2^{n-1} - f_L + 1,$$

there exists a fault-free uv-path of length l in $AQ_n - F$.

When $n \geqslant 5$, for

$$2^{n-1} - f_R + 5 \leqslant l \leqslant 2^n - f - 2,$$

let

$$l_1 = l - 2^{n-1} + f_R - 2, \quad l_2 = l - 2^{n-1} + f_R - 1.$$

Then

$$3 \leqslant l_1 \leqslant 2^{n-1} - f_L - 4, \quad 4 \leqslant l_2 \leqslant 2^{n-1} - f_L - 3.$$

Let $S = N_L(v) - \{u\}$. Since $|N_L(v)| = 2n-3$, we have $|S| \geqslant 2n-4$. Since $|F| = 2n-5$, there exists a vertex x in S such that x and x^h are fault-free. If u^h is fault-free then, since $u^h \neq x^h$, there exists an $x^h u^h$-path T_R of length $2^{n-1} - f_R - 1$ in $R - F$ by Lemma 3.1. Since

$$|F_L + \{u\}| = 2 \leqslant 2n - 7 \quad (n \geqslant 5),$$

by the induction hypothesis, there exists a vx-path P of length l_2 in $L - F - \{u\}$. The path $uu^h + T_R + xx^h + P$ is a fault-free uv-path of length l $(= 1 + 2^{n-1} - f_R - 1 + 1 + l_2$, see Fig. 4 (a)).

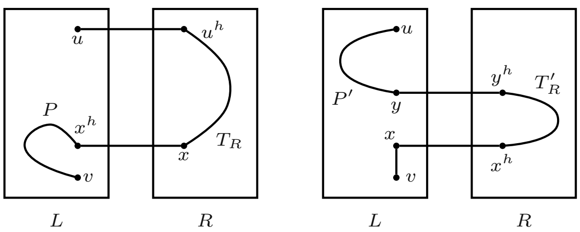

Fig. 4 Illustrations for Subcase 2.1

Assume that u^h is a faulty vertex below.

Let $T = N_L(u) - \{v, x\}$. Since $|N_L(u)| = 2n - 3$, we have $|T| \geqslant 2n - 5$. Since $|F| = 2n - 5$ and u^h is faulty, there exists a vertex y in T such that y and y^h are fault-free. By Lemma 3.1, there exists an $x^h y^h$-path T'_R of length $2^{n-1} - f_R - 1$ in $R - F$. Since

$$|F_L + \{v, x\}| = 3 \leqslant 2n - 7 \quad (n \geqslant 5),$$

by the induction hypothesis, there exists a uy-path P' of length l_1 in $L - F - \{v, x\}$. The path $P' + yy^h + T'_R + x^h x + xv$ is a fault-free uv-path of length l $(= l_1 + 1 + 2^{n-1} - f_R - 1 + 2$, see Fig. 4 (b)).

Since

$$(2^{n-1} - f_R + 5) - (2^{n-1} - f_L + 1) = f_L - f_R + 4 = 11 - 2n \leqslant 1 \quad (n \geqslant 5),$$

we finish the proof of the theorem for this situation.

When $n = 4$, for

$$2^{n-1} - f_L + 2 \leqslant l \leqslant 2^n - f - 2,$$

let

$$l_1 = l - 2^{n-1} + f_L.$$

Then $2 \leqslant l_1 \leqslant 4$. Let w be a faulty vertex in L.

Next, we prove that there exists a fault-free path P_R of length l_1 with end-vertices x and y, such that there exists a vertex x' incident with x and a vertex y' incident with y in R, and $x', y' \notin \{u, v, w\}$, $x' \neq y'$.

Assume that $AQ_2^{10} = L'$ and $AQ_2^{11} = R'$. Let w_1, x_1, y_1 and z_1 be four vertices in L', and assume that

$$w_2 = w_1^{h_2}, \quad x_2 = x_1^{h_2}, \quad y_2 = y_1^{h_2}, \quad z_2 = z_1^{h_2}.$$

Then $w_2, x_2, y_2, z_2 \in R'$. Since there exist exactly two faulty vertices in R, two of $\{w_1, w_2\}$, $\{x_1, x_2\}$, $\{y_1, y_2\}$, $\{z_1, z_2\}$ are fault-free. Without loss of generality, we assume that both $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are fault-free.

And we know that two of w_1, w_2, z_1, z_2 are fault-free. We only need to consider two cases: a) both w_1 and z_1 are fault-free (see Fig. 5 (a)); b) both z_1 and z_2 are fault-free (see Fig. 5 (b)) (We omit some edges in the figure since they are not needed in our proof). The other cases can be considered similarly.

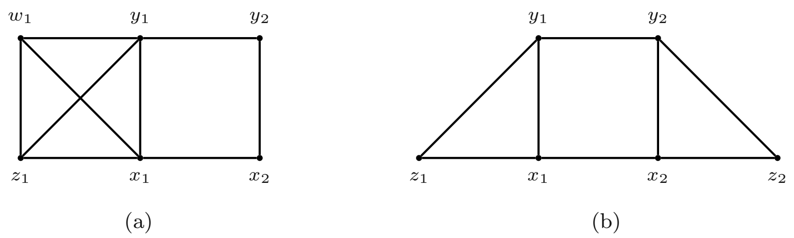

Fig. 5 Illustrations for the situation $n = 4$ of Subcase 2.1

Since x_1y_1 is not a 2-dimensional complement edge and $x_1^h, x_1^c, y_1^h, y_1^c$ are 4 distinct vertices, any one of $x_1^h, x_1^c, y_1^h, y_1^c$ is not in $\{u, v, w\}$. Without loss of generality, assume that x_1^h is not in $\{u, v, w\}$.

In Fig. 5 (a), we enumerate some paths of length 2 with the end-vertex x_1: $x_1z_1y_1$, $x_1y_1w_1$, $x_1y_1z_1$, $x_1y_1y_2$. Since y_1, w_1, z_1, y_2 are all distinct, one of $y_1^h, z_1^h, w_1^h, y_2^h$ is not in $\{u, v, w\}$, say y'. We use x' to denote x_1^h.

Similarly, for the length 3 or 4 and the situation in Fig. 5 (b), there exists a fault-free path P_R of length l_1 end with x and y, such that there exists a vertex x' incident with x and a vertex y' incident with y in R, and $x', y' \notin \{u, v, w\}$, $x' \neq y'$.

Since $L \cong AQ_3$, by Lemma 3.7, there exist ux'-path P_1 and vy'-path P_2 such that P_1 and P_2 are disjoint and $P_1 \cup P_2$ contains all vertices of $L - \{w\}$. Then path $P_1 + x'x + P_R + yy' + P_2$ is a fault-free uv-path of length $l = l_1 + 2^{n-1} - f_L$.

Subcase 2.2 Both u and v are in $R - F$.

In this case, either u^h or u^c is fault-free. Without loss of generality, assume that u^h is fault-free. Let $S = N_R(v) - \{u\}$. Then $|S| \geqslant 2n - 4$. Since $|F| = 2n - 5$, there exists a vertex x in S such that both x and x^h are fault-free. We know that

$$d - 1 \leqslant d(u^h, x^h) \leqslant d + 1.$$

For $l = d + 4$ or $d + 5$, let $l_1 = l - 3$. Then $l_1 = d + 1$ or $d + 2$. Since $f_L = 1$, by Lemma 3.4, Proposition 2.2, and the induction hypothesis, there exists a fault-free u^hx^h-path P_L of length l_1. Then $uu^h + P_L + x^hx + xv$ is a fault-free uv-path of length $l = l_1 + 3$.

For
$$d + 6 \leqslant l \leqslant 2^{n-1} - f_L + 2,$$
let $l_1 = l - 3$. Then
$$d + 3 \leqslant l_1 \leqslant 2^{n-1} - f_L - 1.$$
By the induction hypothesis, there exists a fault-free u^hx^h-path P'_L of length l_1. Then $uu^h + P'_L + x^hx + xv$ is a fault-free uv-path of length $l = l_1 + 3$.

For
$$2^{n-1} - f_R + 3 \leqslant l \leqslant 2^n - f - 2,$$
let
$$l_1 = l - 2^{n-1} + f_R.$$
Then
$$3 \leqslant l_1 \leqslant 2^{n-1} - f_L - 2.$$

When $n \geqslant 5$, by Lemma 3.1, there exists a uv-path T_R of length $2^{n-1} - f_R - 1$ in $R - F$. Since $f_L = 1$, there exists an edge xy in T_R such that x^h and y^h are fault-free. By the induction hypothesis and $d(x^h, y^h) = 1$, there exists a fault-free x^hy^h-path P'_L of length l_1. Without loss of generality, assume that x is closer to u than y. Then $T_R(u, x) + xx^h + P'_L + y^hy + T_R(y, v)$ is a fault-free uv-path of length $l = (l_1 + 2^{n-1} - f_R)$.

When $n = 4$, we have $f_R = 2$ and $f_L = 1$. By Lemma 3.2, there exists a hamiltonian cycle of length 6 in $R-F$. Then there exist two internally disjoint fault-free uv-path P_1 and P_2 in R, and then $\varepsilon(P_1) + \varepsilon(P_2) = 6$. Without loss of generality, assume that $\varepsilon(P_1) \leqslant \varepsilon(P_2)$. Then $1 \leqslant \varepsilon(P_1) \leqslant 3$.

a) When $\varepsilon(P_1) = 1$, P_2 is a hamiltonian uv-path of $R - F$. Since $f_L \leqslant 1$, there exists an edge $x'y'$ in P_2 such that x'^h and y'^h are fault-free. Without loss of generality, assume that x is closer to u than y. Then let $P_3 = P_2(u, x')$ and $P_4 = P_2(v, y)$.

b) When $\varepsilon(P_1) = 2$, let
$$P_1 = ux_1 + x_1v, \quad P_2 = uy_1 + y_1y_2 + y_2y_3 + y_3v.$$
Since $f_L = 1$, x_1^h or x_1^c is fault-free. Without loss of generality, assume that x_1^h is fault-free. And we know that y_1^h or y_3^h is fault-free. Without loss of generality, assume that y_1^h is fault-free. Let
$$x' = x_1, \quad y' = y_1, \quad P_3 = ux_1, \quad P_4 = vy_3 + y_3y_2 + y_2y_1.$$

c) When $\varepsilon(P_1) = 3$, let
$$P_1 = ux_1 + x_1x_2 + x_2v, \quad P_2 = uy_1 + y_1y_2 + y_2v.$$
Since $f_L = 1$, $\{x_1^h, y_2^h\}$ or $\{x_2^h, y_1^h\}$ is fault-free. Without loss of generality, assume that $\{x_2^h, y_1^h\}$ is fault-free. Let
$$x' = x_2, \quad y' = y_1, \quad P_3 = ux_1 + x_1x_2, \quad P_4 = vy_2 + y_2y_1.$$

Since $R \cong AQ_3$, by Lemma 3.6, there exists a fault-free ${x'}^h {y'}^h$-path P'_L of length l_1. Then the path $P_3 + x'{x'}^h + P'_L + {y'}^h y' + P_4$ is a fault-free uv-path of length $l = (l_1 + 6)$.

Since $f_L = 1$ and $f_R = 2$, we finish the proof of the theorem for this subcase.

Subcase 2.3 $u \in L - F$ and $v \in R - F$. By Proposition 2.7, we can assume $d(u, v^h) = d - 1$.

Let $S = N_R(v) - \{u^h\}$. Then $|S| \geqslant 2n - 4$. Since $|F| = 2n - 5$, there exists a vertex w_1 in S such that w_1 and w_1^h are fault-free. We know that

$$d - 2 \leqslant d(u, w_1^h) \leqslant d.$$

In the same sense, there exists a fault-free vertex w_2 incident with u in L such that w_2^h is fault-free and $w_2^h \neq v$.

For

$$d + 4 \leqslant l \leqslant 2^{n-1} - f_L + 1,$$

let $l_1 = l - 2$. Then

$$d + 2 \leqslant l_1 \leqslant 2^{n-1} - f_L - 1.$$

By the induction hypothesis, there exists a uw_1^h-path P_L of length l_1 in $L - F$. Then $P_L + w_1^h w_1 + w_1 v$ is a fault-free uv-path of length $l = l_1 + 2$.

For

$$2^{n-1} - f_R + 3 \leqslant l \leqslant 2^n - f - 2,$$

let

$$l_1 = l - 2^{n-1} + f_R, \quad l_2 = l - 2^{n-1} + f_R + 1.$$

Then

$$3 \leqslant l_1 \leqslant 2^{n-1} - f_L - 2, \quad 2 \leqslant l_2 \leqslant 2^{n-1} - f_L - 1.$$

When $n \geqslant 5$, by Lemma 3.1, there exists a fault-free vw_2^h-path T_R of length $2^{n-1} - f_R - 1$ in R. By the induction hypothesis, there exists a uw_2-path P'_L of length l_1 in $L - F$. Then $P'_L + w_2 w_2^h + T_R$ is a fault-free uv-path of length $l = l_1 + 1 + 2^{n-1} - f_R - 1$.

When $n = 4$, by Lemma 3.2, there exists a fault-free hamiltonian cycle C of length 6 in R. Let

$$C = vx_1 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5v.$$

Since $f_L = 1$, one of x_1^h, x_2^h and x_5^h is fault-free, and not u. If x_2^h is fault-free and $x_2^h \neq u$, then let

$$T'_R = vx_5 + x_5x_4 + x_4x_3 + x_3x_2.$$

Since $L \cong AQ_3$, by Lemma 3.6, there exists a fault-free ux_2^h-path P'_L of length l_2 in L. Then $P'_L + x_2^h x_2 + T'_L$ is a fault-free uv-path of length $l = l_1 + 5$. If x_1 or x_5 is fault-free, and not u, then without loss of generality, assume that x_1^h is fault-free and $x_1^h \neq u$. Let

$$T''_R = vx_5 + x_5x_4 + x_4x_3 + x_3x_2 + x_2x_1.$$

Since $L \cong AQ_3$, by Lemma 3.6, there exists a fault-free ux_1-path P'_L of length l_1. Then $P'_L + x_1x_1^h + T''_R$ is a fault-free uv-path of length $l = l_1 + 6$.

Since $f_L = 1$ and $f_R = 2n - 6$, we finish the proof of the theorem for this subcase.

Case 3 $|F_R| = 2n - 5$. Then $L - F$ is a fault-free $(n-1)$-dimensional augmented cube.

Subcase 3.1 Both u and v are in L.

For $d + 4 \leqslant l \leqslant 2^{n-1} - 1$, by the induction hypothesis, there exists a uv-path of length l in L.

Since there exist $2^{n-1} - 2n + 5$ (> 5) fault-free vertices in R, there exists a fault-free vertex w such that $w \notin \{u^h, u^c, v^h, v^c\}$. By Lemma 3.3, there exist uw^h-path P_1 and vw^c-path P_2 such that P_1 and P_2 are internally disjoint and $P_1 \cup P_2$ contains all vertices of L.

For $l = 2^{n-1}$, the path $P_1 + w^hw + ww^c + P_2$ is a fault-free uv-path of length l.

For

$$2^{n-1} + 1 \leqslant l \leqslant 2^n - f - 2,$$

let $l_1 = l - 2^{n-1}$. Then

$$1 \leqslant l_1 \leqslant 2^{n-1} - f - 2.$$

Assume that w is a fault vertex in R. By Lemma 3.2, there exists a hamiltonian cycle C in $R - F + \{w\}$. Let

$$C = wx_1 + x_1x_2 + \cdots + x_{t-1}x_t + x_tw,$$

where $t = 2^{n-1} - f_R$. Then

$$P_1 = x_1x_2 + \cdots + x_{l_1}x_{l_1+1}, \quad P_2 = x_tx_{t-1} + \cdots + x_{t-l_1+1}x_{t-l_1}$$

are two distinct paths of length l_1. So there exists a fault-free path P_R of length l_1 such that P_R is not a path between u^h and v^h. Assume that P_R is the path between x and y. Then $x \notin \{u^h, v^h\}$ or $y \notin \{u^h, v^h\}$. Without loss of generality, assume $x \notin \{u^h, v^h\}$ below.

If $y = u^h$ or $y = v^h$, then without loss of generality, assume $y = u^h$. Since $x \neq v^h$, by Lemma 3.1, there exists a x^hv-path P_L of length $2^{n-1} - 2$ in $L - \{u\}$. Then $P_L + x^hx + P_R + yu$ is a fault-free uv-path of length $l = l_1 + 2^{n-1}$.

If $y \neq u^h$ and $y \neq v^h$, by Lemma 3.3, there exist uy^h path P_3 and vx^h-path P_4 such that P_3 and P_4 are internally disjoint and $P_3 \cup P_4$ contains all vertices of L. Then $P_3 + y^hy + P_R + xx^h + P_4$ is a fault-free uv-path of length $l = l_1 + 2^{n-1}$.

Subcase 3.2 Both u and v are in $R - F$.

For

$$d + 4 \leqslant l \leqslant 2^{n-1} + 1,$$

let $l_1 = l - 2$. Then

$$d + 2 \leqslant l_1 \leqslant 2^{n-1} - 1.$$

We know that $d(u^h, v^h) = d$. Then there exists a $u^h v^h$-path P_L of length l_1. Then $uu^h + P_L + v^h v$ is a fault-free uv-path of length $l = l_1 + 2$.

For
$$2^{n-1} - f + 4 \leqslant l \leqslant 2^n - f - 2,$$
let
$$l_1 = l - 2^{n-1} + f, \quad l_2 = l - 2^{n-1} + f + 1.$$
Then
$$4 \leqslant l_1 \leqslant 2^{n-1} - 2, \quad 5 \leqslant l_2 \leqslant 2^{n-1} - 1.$$
Assume that w is a faulty vertex in R.

When $n \geqslant 5$, by Lemma 3.1, there exists a hamiltonian uv-path T_R in $R - F + \{w\}$. Assume that x and y are two vertices incident with w in T_R. Without loss of generality, assume that x is closer to u than y. Since
$$d(x^h, y^h) = d(x, y) \leqslant 2,$$
there exists an $x^h y^h$-path P'_L of length l_1. Then $T_R(u, x) + xx^h + P'_L + y^h y + T_R(y, v)$ is a fault-free uv-path of length $l = l_1 + 2^{n-1} - f$.

When $n = 4$, by Lemma 3.2, there exists a hamiltonian cycle C of length 6 in $R - F + \{w\}$. Let
$$C = wx_1 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5w.$$

If v is incident with u in C, then there exist two fault-free disjoint paths P_1 and P_2 in R, such that P_1 ends with u and P_2 ends with v, $P_1 \cup P_2$ contains all vertices in $R - F$. Assume that the other end-vertex of P_1 is x, and the other end-vertex of P_2 is y. By Lemma 3.6, there exists a fault-free $x^h y^h$-path P'_L of length l_1. Then $P_1 + xx^h + P'_L + y^h y + P_2$ is a fault-free uv-path of length $l = l_1 + 5$.

If v is not incident with u in C, then there exist two distinct fault-free vertices x and y, such that x is incident with u and y is incident with v. By Lemma 3.6, there exists a fault-free $x^h y^h$-path P''_L of length l_2. Then the path $ux + xx^h + P''_L + y^h y + yv$ is a fault-free uv-path of length $l = l_2 + 4$.

Since $f_R = 2n - 5$, we finish the proof of the theorem for this subcase.

Subcase 3.3 $u \in L - F$ and $v \in R - F$. By Proposition 2.7, we can assume $d(u, v^h) = d - 1$.

Let $S = N_R(v) - \{u^h\}$. Then $|S| \geqslant 2n - 4$. Since $|F| = 2n - 5$, there exists a vertex w in S such that w and w^h are fault-free. We know that
$$d - 1 \leqslant d(u^h, w^h) \leqslant d + 1.$$

For
$$d + 4 \leqslant l \leqslant 2^{n-1} + 1,$$
let $l_1 = l - 2$. Then
$$d + 2 \leqslant l_1 \leqslant 2^{n-1} - 1.$$

There exists a uw^h-path P_L of length l_1 in L. The path $P_L + w^h w + wv$ is a fault-free uv-path of length $l = l_1 + 2$.

For

$$2^{n-1} + 2 \leqslant l \leqslant 2^n - f - 2,$$

let $l_1 = l - 2^{n-1}$. Then

$$2 \leqslant l_1 \leqslant 2^{n-1} - f - 2.$$

When $n \geqslant 5$, by Lemma 3.2, there exists a cycle C of length $2^{n-1} - f$ in $R - F$. Then there exists a vertex x in $R - F$ such that there exists a fault-free vx-path T_R of length l_1. And we have $x^h \neq u$ or $x^c \neq u$. Without loss of generality, assume $x^h \neq u$. By Lemma 3.1, there exists a ux^h-path T_L of length $2^{n-1} - 1$ in L. The path $T_L + x^h x + T_R$ is a fault-free uv-path of length $l = l_1 + 2^{n-1}$.

When $n = 4$, since

$$f_R = 2n - 5, \quad |N_R(v)| = 2n - 3,$$

there exists a fault-free vy-path P_R of length 2 in $R - F$ for some $y \in R - F$. We know that $y^h \neq u$ or $y^c \neq u$. Without loss of generality, assume $y^h \neq u$. By Lemma 3.6, there exists a uy^h-path T_L of length $2^3 - 1$. Then the path $T_L + y^h y + P_R$ is a fault-free uv-path of length $2^3 + 2$.

We know that there exist two disjoint edges vx_1 and y_1z_1 in $R - F$, such that x_1^h and z_1^h are not u. When $y_1^h \neq u$, by Lemma 3.3, there exist uy_1^h-path P_1 and $x_1^h z_1^h$-path P_2, such that P_1 and P_2 are disjoint and $P_1 \cup P_2$ contains all vertices in L. Then the path $P_1 + y_1^h y_1 + y_1 z_1 + z_1 z_1^h + P_2 + x_1^h x_1 + x_1 v$ is a fault-free uv-path of length $2^3 + 3$. When $y_1^h = u$, there exists an $x_1^h z_1^h$-path P_3 of length 6 in $L - \{u\}$. Then the path $vx_1 + x_1 x_1^h + P_3 + z_1^h z_1 + z_1 y_1 + y_1 u$ is a fault-free uv-path of length $2^3 + 3$.

The proof of the theorem is complete.

5 Conclusion and problems

The augmented cube AQ_n is an important variation of the hypercube Q_n. In this paper, we have shown that if AQ_n ($n \geqslant 3$) has at most $2n - 5$ faulty vertices and/or edges, then for any two fault-free vertices u and v with distance d in AQ_n, there exist fault-free uv-paths of every length from $d + 2$ to $2^n - f - 1$, where f is the number of faulty vertices in AQ_n. Our result is the best possible in the following sense.

Assume that $d(u, v) = 1$ and $u = v^{c_j}$ for some i, where $2 \leqslant j \leqslant n$, by Proposition 2.2,

$$S \cap T = \{u^{h_j}\,(= v^{c_{j-1}}),\ v^{h_j}\,(= u^{c_{j-1}}),\ u^{h_{j+1}}\,(= v^{c_{j+1}}),\ v^{h_{j+1}}(= u^{c_{j+1}})\}.$$

Assume that

$$u = v^b, \quad F = \{u^{h_j}, u^{h_{j+1}}\}.$$

We know that
$$|F| = 2 \leqslant 2n - 5 \quad (n \geqslant 4)$$
and then, there exists no uv-path of length 2.

In AQ_n, if $|F| = 2n-4$, then there exist two distinct fault-free vertices u and v with distance d, such that there exists no fault-free uv-path of length l for some $l \in \{d+2, d+3, \ldots, 2^n - f - 1\}$. We have an instance as follows.

Assume that
$$u = u_1 u_2 u_3 \cdots u_n, \quad v = u^{c_{n-1}} = u_1 \overline{u}_2 \overline{u}_3 \cdots \overline{u}_n.$$
Then uv is an edge in AQ_n. Let
$$x = u^{c_{n-2}} = u_1 u_2 \overline{u}_3 \cdots \overline{u}_n, \quad y = u^h = \overline{u}_1 u_2 u_3 \cdots u_n,$$
and let S be the vertices adjacent to u and $F = S - \{v, x, y\}$. Since $v, x, y \in S$, we have
$$|F| = 2n - 1 - 3 = 2n - 4.$$
We can affirm that there are no fault-free uv-paths of length 3. Assume that
$$A = AQ^{00}_{n-2}, \quad B = AQ^{01}_{n-2}, \quad C = AQ^{10}_{n-2}, \quad D = AQ^{11}_{n-2}.$$
Without loss of generality, assume $u \in A$ since AQ_n is vertex-symmetric. Then
$$v \in B, \quad x \in A, \quad y \in C.$$
We have
$$N(x) \cap V(C) = \{x^h = \overline{u}_1 u_2 \overline{u}_3 \cdots \overline{u}_n\},$$
$$N(x) \cap V(D) = \{x^c = \overline{u}_1 \overline{u}_2 u_3 \cdots u_n\},$$
$$N(x) \cap V(B) = \{v = x^{h_{n-1}}\} \quad (\text{since } x^{c_{n-1}} = u^{h_{n-1}}),$$
$$N(v) \cap V(C) = \{v^c = y = \overline{u}_1 u_2 u_3 \cdots u_n\},$$
$$N(v) \cap V(D) = \emptyset \quad (\text{since } v^h = u^c \text{ is fault}), \quad N(v) \cap V(A) = \{u, x\}.$$
So, there exist no vx-paths of length 2 except xuv. Similarly, there exist no vy-paths of length 2 except vuy. So, there exist no fault-free uv-paths of length 3.

However, these examples are valid only in the case $d = 1$. Excluding this case, for $d \geqslant 2$ or $n \geqslant 4$, it is worthwhile to investigate the following questions suggested by the anonymous referees when they reviewed our manuscript.

First, it is known that AQ_n is pancyclic for $n \geqslant 2$ [2] and panconnected for $n \geqslant 1$ [10]. There are several other generalized results. For example, AQ_n is $(2n-3)$-edge-fault-tolerant pancyclic for $n \geqslant 2$ [10], $(2n-3)$-fault-tolerant pancyclic for $n \geqslant 4$ [18], $(2n-3)$-fault-tolerant hamiltonian, and $(2n-4)$-fault-tolerant hamiltonian connected for $n \geqslant 4$ [7]. The first question is, is AQ_n $(2n-4)$-fault-tolerant panconnected for some large $d \geqslant 2$ or $n \geqslant 4$?

Second, by definition, a graph is panconnected if, for any two vertices u and v, there exists a fault-free uv-path of length l which ranges from d to

$2^n - f - 1$. However, our proof of Theorem 1.1 is not valid for the cases d and $d+1$. What study or comment can we make on these for $d > 2$?

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