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A new class of transitive graphs☆

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ABSTRACT

Let n and k be integers with $n \geq k \geq 0$. This paper presents a new class of graphs $H(n, k)$, which contains hypercubes and some well-known graphs, such as Johnson graphs, Kneser graphs and Petersen graph, as its subgraphs. The authors present some results of algebraic and topological properties of $H(n, k)$. For example, $H(n, k)$ is a Cayley graph, the automorphism group of $H(n, k)$ contains a subgroup of order $2^n n!$ and $H(n, k)$ has a maximal connectivity $\binom{n}{k}$ and is hamiltonian if k is odd; it consists of two isomorphic connected components if k is even. Moreover, the diameter of $H(n, k)$ is determined if k is odd.

1. Introduction

In this paper, a graph $G = (V, E)$ is considered as an undirected graph where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. The symbol $\mathrm{Aut}(G)$ denotes the automorphism group of G. A graph G is *vertex-transitive* (resp. *edge-transitive*) if $\mathrm{Aut}(G)$ acts transitively on $V(G)$ (resp. on $E(G)$). An arc in G is an ordered pair of adjacent vertices, and G is *arc-transitive* if $\mathrm{Aut}(G)$ acts transitively on the set of its arcs. It is well known that a connected arc-transitive graph is necessarily vertex- and edge-transitive.

Let Γ be a finite group and S a subset of Γ that is closed under taking inverses and does not contain the identity. A *Cayley graph* $C_\Gamma(S)$ is a graph with vertex-set Γ and edge-set $E(C_\Gamma(S)) = \{gh: hg^{-1} \in S\}$. It is well known that every Cayley graph is vertex-transitive.

Let n, m and i be fixed integers with $n \geq m \geq i \geq 0$, and let Ω_n be the power set of the set $A = \{x_1, x_2, \ldots, x_n\}$. Let $\Omega_n^m = \{X \in \Omega_n : |X| = m\}$. In [7], a class of graphs $J(n, m, i)$ is defined as follows. The vertex-set of $J(n, m, i)$ is Ω_n^m, where two subsets X and Y are adjacent if $|X \cap Y| = i$. For $n \geq 2m$, the graphs $J(n, m, m-1)$ are the *Johnson graphs*, $J(n, m, 0)$ are the *Kneser graphs* and $J(5, 2, 0)$ is the *Petersen graph*. Johnson graphs and Kneser graphs are important classes of graphs in algebraic graph theory, and have received much research attention; see for example [1–7,9–12,14–16,18].

The *n-dimensional hypercube* Q_n is the graph with vertex-set V consisting of all binary sequences of length n on the set $\{0, 1\}$, two vertices x and y being linked by an edge if and only if they differ in exactly one coordinate. The hypercube is one of the most popular, versatile and efficient topological structures of interconnection networks; see for example [13,19].

In this paper, we present a new class of graphs $H(n, k) (n \geq k \geq 0)$. We will prove that $H(n, k)$ contains hypercubes and some well-known graphs, such as Johnson graphs and Kneser graphs, as its subgraphs. We present some results about

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algebraic and topological properties of $H(n,k)$. For example, $H(n,k)$ is an arc-transitive Cayley graph; the automorphism group of $H(n,k)$ contains a subgroup of order $2^n n!$; $H(n,k)$ has maximal connectivity $\binom{n}{k}$ and is hamiltonian if k is odd; $H(n,k)$ consists of two isomorphic connected components if k is even. Moreover, when k is odd, the diameter of $H(n,k)$ is equal to $\lceil \frac{n-1}{k} \rceil + 1$ if $n \geq 2k-1$, and equal to $\lceil \frac{n-1}{n-k} \rceil + 1$ if $n \leq 2k-2$.

The rest of the paper is organized as follows. Section 2 gives the definition of the graph $H(n,k)$ and some preliminaries. Section 3 investigates some properties of subgraphs of $H(n,k)$. Section 4 presents some algebraic properties of $H(n,k)$. Section 5 considers the connectivity of $H(n,k)$. Section 6 considers hamiltonian properties of $H(n,k)$. The diameter of $H(n,k)$ is determined in Section 7.

The other concepts not defined here can be found in [7,20].

2. Definitions and preliminaries

Recall that Ω_n denotes the power set of the set $A = \{x_1, x_2, \ldots, x_n\}$ and $\Omega_n^m = \{X \in \Omega_n : |X| = m\}$. Let $\Omega_n' = \{X \in \Omega_n : |X| \text{ is odd}\}$ and $\Omega_n'' = \{X \in \Omega_n : |X| \text{ is even}\}$. Then $|\Omega_n| = 2^n$, $|\Omega_n^m| = \binom{n}{m}$ and $|\Omega_n'| = |\Omega_n''| = 2^{n-1}$.

For any two sets $X, Y \in \Omega_n$, the *symmetric difference* $X \Delta Y$ of X and Y is the set $X \cup Y - X \cap Y$. It is easy to verify that Ω_n is an *Abelian group* about the operation Δ. We now give the definition of the graph $H(n,k)$.

Definition 2.1. Let n and k be fixed integers with $n \geq k \geq 0$. The graph $H(n,k)$ is defined as follows. $H(n,k)$ has the vertex-set Ω_n and there is an edge between two distinct vertices X and Y if and only if $|X \Delta Y| = k$.

Clearly, $H(n,0)$ is an empty graph, containing no edges. For any two distinct vertices X and Y in $H(n,n)$, there is an edge between them if and only if they are mutually complementary, and so $H(n,n) = 2^{n-1}K_2$. Thus, we always suppose that $0 < k < n$ in our discussion.

Proposition 2.1. *$H(n,k)$ is an $\binom{n}{k}$-regular graph with 2^n vertices.*

Proof. We only need to prove that $H(n,k)$ is $\binom{n}{k}$-regular. Let X be a vertex in $H(n,k)$ with $|X| = m$. If Y is a neighbor of X and $|X \cap Y| = j$, then $|Y| = k - m + 2j$. Thus, the neighbors of X can be partitioned into $m+1$ parts $V_0, V_1, \ldots, V_m$ such that $|X \cap Y| = j$ for any $Y \in V_j$ and each $j = 0, 1, \ldots, m$. By Vandermonde's identity, it follows that the degree of X is equal to

$$\sum_{j=0}^{m} |V_j| = \sum_{j=0}^{m} \binom{m}{j} \binom{n-m}{k-m+j} = \binom{n}{k},$$

which is independent of m, and so $H(n,k)$ is $\binom{n}{k}$-regular. ■

Proposition 2.2. *$H(n,k)$ has $\binom{n}{k}$ edge-disjoint perfect matchings, and hence it is 1-factorable.*

Proof. For a fixed $X \in \Omega_n^k$ and any $Y \in \Omega_n$, since $|(X \Delta Y) \Delta Y| = |X| = k$, $H(n,k)$ has an edge, denoted by e_{XY}, between $X \Delta Y$ and Y. Then, $E_X = \{e_{XY} : Y \in \Omega_n\}$ is a perfect matching of $H(n,k)$, and $|E_X| = 2^{n-1}$. There are $\binom{n}{k}$ such perfect matchings in $H(n,k)$ since $|\Omega_n^k| = \binom{n}{k}$. Moreover, if $X_1, X_2 \in \Omega_n^k$ and $X_1 \neq X_2$ then $E_{X_1} \cap E_{X_2} = \emptyset$. Thus, $H(n,k)$ has $\binom{n}{k}$ edge-disjoint perfect matchings. Since $\binom{n}{k} |E_X| = \binom{n}{k} 2^{n-1} = |E(H(n,k))|$ by Proposition 2.1, the $\binom{n}{k}$ perfect matchings contain all edges of $H(n,k)$. Thus, $H(n,k)$ is 1-factorable. ■

Proposition 2.3. *Let X and Y be two adjacent vertices in $H(n,k)$. Then the parity of $|X|$ and $|Y|$ is the same if k is even, and is different if k is odd.*

Proof. Immediate. ■

Proposition 2.4. *If n is even and k is odd, then $H(n,k) \cong H(n,n-k)$.*

Proof. Since n is even and k is odd, $n-k$ is odd. For any $X \in \Omega_n$, we use $\overline{X}$ to denote the complement of X in $\{x_1, x_2, \ldots, x_n\}$. Then $\overline{X} \neq \emptyset$ for any $X \in \Omega_n'$ since n is even. A mapping $\varphi : \Omega_n \to \Omega_n$ is defined by

$$\varphi(X) = \begin{cases} \overline{X}, & \text{if } X \in \Omega_n' \\ X, & \text{if } X \in \Omega_n''. \end{cases}$$

Clearly, if $X \in \Omega_n'$, then $\overline{X} \in \Omega_n'$ since n is even. Thus, the mapping φ is a permutation on Ω_n. Since k is odd, for any two vertices X and Y in $H(n,k)$, if $|X \Delta Y| = k$, then $|X|$ and $|Y|$ have different parity by Proposition 2.3. Assume, without loss of generality, $X \in \Omega_n'$ and $Y \in \Omega_n''$.

$$\begin{aligned} XY \in E(H(n,k)) &\Leftrightarrow |X \Delta Y| = k \\ &\Leftrightarrow |\overline{X} \Delta Y| = n-k \\ &\Leftrightarrow |\varphi(X) \Delta \varphi(Y)| = n-k \\ &\Leftrightarrow \varphi(X)\varphi(Y) \in E(H(n,n-k)), \end{aligned}$$

which implies that φ is an isomorphism between $H(n,k)$ and $H(n,n-k)$. ■

3. Properties of subgraphs

We use $H^m(n,k)$, $H'(n,k)$ and $H''(n,k)$ to denote the subgraphs of $H(n,k)$ induced by Ω_n^m, Ω_n' and Ω_n'', respectively.

The following theorem shows that the graphs $J(n,m,i)$ defined in the Introduction are found as subgraphs of $H(n,k)$.

Theorem 3.1. *For any integers n, m and i with $n \geq m \geq i \geq 0$, if $n \geq 2m-2i$ then $H^m(n, 2m-2i) = J(n,m,i)$.*

Proof. Clearly, $V(H^m(n,2m-2i)) = \Omega_n^m = V(J(n,m,i))$. Let X and Y be any two distinct sets in Ω_n^m. Then

$$\begin{aligned} XY \in E(H^m(n,2m-2i)) &\Leftrightarrow |X\Delta Y| = 2m-2i \\ &\Leftrightarrow |X|+|Y|-2|X\cap Y| = 2m-2i \\ &\Leftrightarrow |X\cap Y| = i \Leftrightarrow XY \in E(J(n,m,i)). \end{aligned}$$

It follows that $H^m(n,2m-2i) = J(n,m,i)$. ■

By Theorem 3.1, $H^m(n,2)$ is the Johnson graphs $J(n,m,m-1)$ and $H^m(n,2m)$ is the Kneser graphs $J(n,m,0)$ when $n \geq 2m$.

The following result shows that the hypercubes Q_n appear as special cases among $H(n,k)$ for $k=1$.

Theorem 3.2. *$H(n,1) \cong Q_n$ for any positive integer n.*

Proof. To prove the theorem, it is sufficient to give an isomorphism between $H(n,1)$ and Q_n. Let $X = \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$ be a vertex in $H(n,1)$ for some $s \leq n$ and let $Y = y_1y_2\cdots y_n$ be a vertex in Q_n. Define a mapping $\phi : V(H(n,1)) \to V(Q_n)$ such that $\phi(X) = Y$, where the i_1th, i_2th, …, i_sth coordinates of Y are 1 and all the others are 0. It is easy to see that ϕ is a bijection and preservers adjacency. Thus, the mapping ϕ is an isomorphism between $H(n,1)$ and Q_n. ■

Let X be an arbitrary fixed element in Ω_n. Define the mapping $\rho_X : \Omega_n \to \Omega_n$ as follows.

$$\rho_X : Y \mapsto X\Delta Y, \quad \forall Y \in \Omega_n. \tag{3.1}$$

Clearly, ρ_X is a permutation on Ω_n.

Theorem 3.3. *If k is even, then $H'(n,k) \cong H''(n,k)$.*

Proof. It is clear that if $X, Y \in \Omega_n'$ then $X\Delta Y \in \Omega_n''$. Thus, the mapping ρ_X defined as (3.1) is a mapping from Ω_n' to Ω_n''. For $Y, Z \in \Omega_n'$, if $Y \neq Z$ then $X\Delta Y \neq X\Delta Z$. It follows that the mapping ρ_X is injective. Conversely, for any $Y \in \Omega_n''$, we have $X\Delta Y \in \Omega_n'$, and so $\rho_X(X\Delta Y) = Y$. It follows that ρ_X is surjective. Thus ρ_X is a bijective mapping from Ω_n' to Ω_n''.

We now show that ρ_X is an isomorphism between $H'(n,k)$ and $H''(n,k)$. It is sufficient to prove that ρ_X preserves adjacency. For any two distinct vertices $Y, Z \in V(H'(n,k))$ and $|Y\Delta Z| = k$, then $|\rho_X(Y)\Delta\rho_X(Z)| = |(X\Delta Y)\Delta(X\Delta Z)| = |Y\Delta Z| = k$ and vice versa, which implies that $\rho_X(Y)\rho_X(Z)$ is an edge in $H''(n,k)$ whenever YZ is an edge in $H'(n,k)$. It follows that $H'(n,k) \cong H''(n,k)$. ■

Theorem 3.4. *If k is odd, then $H(n,k)$ is isomorphic to a spanning subgraph of $H''(n+1,k+1)$.*

Proof. Define a mapping $\varphi : \Omega_n \to \Omega_{n+1}''$ as follows.

$$\varphi : X \mapsto \begin{cases} X \cup \{x_{n+1}\}, & \text{if } X \in \Omega_n'; \\ X, & \text{if } X \in \Omega_n''. \end{cases} \quad \forall X \in \Omega_n$$

Since $\varphi(X) \neq \varphi(Y)$ if $X \neq Y$, the mapping φ is injective, and so $|\Omega_n| \leq |\Omega_{n+1}''|$. Since $|\Omega_n| = 2^n = |\Omega_{n+1}''|$, the mapping φ is surjective. It follows that the mapping φ is bijective.

Let $X, Y \in \Omega_n$. If $XY \in E(H(n,k))$, then $|X\Delta Y| = k$ is odd. Thus, $|X|$ and $|Y|$ are of different parity by Proposition 2.3. Without loss of generality, suppose that $X \in \Omega_n'$ and $Y \in \Omega_n''$. Then

$$|\varphi(X)\Delta\varphi(Y)| = |(X\cup\{x_{n+1}\})\Delta Y| = |X\Delta Y| + 1 = k+1.$$

It follows that if X and Y are adjacent in $H(n,k)$ then $\varphi(X)$ and $\varphi(Y)$ are adjacent in $H''(n+1,k+1)$. Thus, $\varphi(H(n,k))$ is a spanning subgraph of $H''(n+1,k+1)$. ■

4. Algebraic properties

In this section, we will investigate some algebraic properties of $H(n, k)$. We start with the following theorem.

Theorem 4.1. *Let Γ be the Abelian group for the operation Δ on the set Ω_n. Then $H(n, k)$ is a Cayley graph $C_\Gamma(S)$ with $S = \Omega_n^k$ for $k \geq 1$.*

Proof. It is clear that the empty set $\emptyset$ is the identity of Γ, and so the $X^{-1} = X$ for any $X \in \Omega_n$. Let X and Y be two any vertices in $H(n, k)$. Then

$$XY \in E(H(n, k)) \Leftrightarrow |X\Delta Y| = k \Leftrightarrow |Y\Delta X^{-1}| = k \Leftrightarrow Y\Delta X^{-1} \in S.$$

It follows that $H(n, k)$ is a Cayley graph $C_\Gamma(S)$. ■

Lemma 4.1. *Let S_n be the symmetric group on the set $A = \{x_1, x_2, \ldots, x_n\}$. Then S_n is a subgroup of $\mathrm{Aut}(H(n, k))$. In particular, if k is even, then S_n is a subgroup of $\mathrm{Aut}(H''(n, k))$.*

Proof. Let $\sigma \in S_n$ and $X = \{x_1, x_2, \ldots, x_m\} \in \Omega_n$. Define

$$\sigma(X) = \begin{cases} \emptyset & \text{if } X = \emptyset; \\ \{\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_m)\} & \text{if } X \neq \emptyset. \end{cases} \tag{4.1}$$

Clearly, σ is a permutation on Ω_n and $|\sigma(X)| = |X|$ for each $X \in \Omega_n$. Let $Y \in \Omega_n$, since

$$|X\Delta Y| = k \Leftrightarrow |\sigma(X)\Delta\sigma(Y)| = k, \tag{4.2}$$

the permutation σ is an automorphism on $H(n, k)$. Thus, S_n is a subgroup of $\mathrm{Aut}(H(n, k))$.

If k is even then, by Proposition 2.3, $|X|$ and $|Y|$ have the same parity for any two adjacent vertices X and Y in $H(n, k)$. For any $X, Y \in \Omega_n''$ and any $\sigma \in S_n$, both Eqs. (4.1) and (4.2) hold, and so S_n is a subgroup of $\mathrm{Aut}(H''(n, k))$. ■

Theorem 4.2. *$H(n, k)$ is arc-transitive.*

Proof. It is clear that $H(n, k)$ is vertex-transitive since $H(n, k)$ is a Cayley graph by Theorem 4.1. To prove that $H(n, k)$ is arc-transitive, it is sufficient to prove that there is an automorphism of $H(n, k)$ such that it maps an arc $(\emptyset, \{x_1, x_2, \ldots, x_k\})$ to any arc with the initial vertex $\emptyset$ since $H(n, k)$ is vertex-transitive. Let $(\emptyset, \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})$ is any arc with the initial vertex $\emptyset$. By Lemma 4.1, there is an automorphism of $H(n, k)$ that maps arc $(\emptyset, \{x_1, x_2, \ldots, x_k\})$ to $(\emptyset, \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})$. Thus, $H(n, k)$ is arc-transitive. ■

Lemma 4.2. *Let $H_n = \{\rho_X : X \in \Omega_n\}$, where ρ_X is defined in Eq. (3.1). Then H_n is a subgroup of $\mathrm{Aut}(H(n, k))$.*

Proof. By Theorem 4.1, $H(n, k)$ is a Cayley graph and then by the definition of Cayley graphs, H_n is a subgroup of $\mathrm{Aut}(H(n, k))$. ■

Corollary 4.1. *If k is even, $H''(n, k)$ is arc-transitive.*

Proof. For any $X, Y \in \Omega_n''$, there is an automorphism $\rho_{X\Delta Y} \in H_n''$ such that $\rho_{X\Delta Y}(X) = (X\Delta Y)\Delta X = Y$. It follows that $H''(n, k)$ is vertex-transitive.

Let $(\emptyset, \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})$ is an any arc in $H''(n, k)$. By Lemma 4.1, there is an automorphism of $H''(n, k)$ that maps arc $(\emptyset, \{x_1, x_2, \ldots, x_k\})$ to $(\emptyset, \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})$. Thus, $H''(n, k)$ is arc-transitive. ■

Theorem 4.3. *$\mathrm{Aut}(H(n, k))$ contains a subgroup H_nS_n with order $2^n n!$.*

Proof. By Lemma 4.1, S_n is a subgroup of $\mathrm{Aut}(H(n, k))$ with order $n!$. By Lemma 4.2, H_n is a subgroup of $\mathrm{Aut}(H(n, k))$ with order 2^n. Thus, H_nS_n is a subgroup of $\mathrm{Aut}(H(n, k))$, and $|H_nS_n| = \frac{|H_n||S_n|}{|H_n \cap S_n|}$. Since $H_n \cap S_n$ is the identity subgroup, $|H_n \cap S_n| = 1$. Thus, $|H_nS_n| = 2^n n!$. ■

It has been shown by Harary [8] that $\mathrm{Aut}(Q_n) = [S_2]^{S_n}$ and $|\mathrm{Aut}(Q_n)| = 2^n n!$. Since $H(n, 1) \cong Q_n$, by Theorem 4.3 we have $\mathrm{Aut}(Q_n) \supseteq H_nS_n$ with order $2^n n!$. The following theorem shows that $\mathrm{Aut}(H(n, 1)) = H_nS_n$, which is another form of $\mathrm{Aut}(Q_n)$.

Theorem 4.4. *$\mathrm{Aut}(H(n, 1)) = H_nS_n$.*

Proof. Let $\Gamma = \mathrm{Aut}(H(n, 1))$. By Theorem 4.3, H_nS_n is a subgroup of Γ. Thus, if we can prove $|\Gamma| \leq |H_nS_n| = 2^n n!$, then $\Gamma = H_nS_n$. To this end, let $\Gamma_\emptyset$ be the stabilizer of the element $\emptyset \in \Omega_n$, that is, $\Gamma_\emptyset = \{\rho \in \Gamma : \rho(\emptyset) = \emptyset\}$. By the proof of Lemma 4.1, S_n is a subgroup of $\Gamma_\emptyset$. We now show that $\Gamma_\emptyset$ is a subgroup of S_n.

Consider a distance partition $\{\Omega_n^0, \Omega_n^1, \ldots, \Omega_n^n\}$ of $H(n, 1)$, where $\Omega_n^0 = \emptyset$, that is, $d(\emptyset, X) = s$ for any $X \in \Omega_n^s$ for each $s = 1, 2, \ldots, n$. For any $\rho \in \Gamma_\emptyset$, $\rho(\Omega_n^i) = \Omega_n^i$ for each $i = 0, 1, \ldots, n$ since $H(n, 1)$ is vertex-transitive. Let $X = \{x_1, x_2, \ldots, x_s\} \in \Omega_n^s$, where $1 \leq s \leq n$. We want to prove that $\rho(X) = \{\rho(x_1), \rho(x_2), \ldots, \rho(x_s)\}$ by induction on s.

There is nothing to do if $s = 1$. Suppose that the conclusion is true for any integer $s - 1 \geq 1$. For each $j = 1, 2, \ldots, s$, let $X_j = X \setminus \{x_j\}$. Clearly, $|X_j \Delta X| = 1$ and so $X_jX \in E(H(n, 1))$ for each $j = 1, 2, \ldots, s$. Moreover, $X_1, X_2, \ldots, X_s$ are all neighbors of X in Ω_n^{s-1}. By the induction hypothesis, $\rho(X_j) = \{\rho(x_1), \ldots, \rho(x_{j-1}), \rho(x_{j+1}), \ldots, \rho(x_s)\}$ for each $j = 1, 2, \ldots, s$. Since ρ preserves adjacency, $\rho(X_j)\rho(X) \in E(H(n, 1))$ for each $j = 1, 2, \ldots, s$. Thus, $\rho(X_1), \rho(X_2), \ldots, \rho(X_s)$ must have a common neighbor in Ω_n^s. Since $s \geq 2$, such a common neighbor must be $\{\rho(x_1), \rho(x_2), \ldots, \rho(x_s)\}$. It follows that $\rho(X) = \{\rho(x_1), \rho(x_2), \ldots, \rho(x_s)\}$. By the principle of induction, we have $\rho(X) = \{\rho(x_1), \rho(x_2), \ldots, \rho(x_s)\}$ for each $s = 1, 2, \ldots, n$. In particular, if $X = \{x_1, x_2, \ldots, x_n\}$, then $\rho(X) = \{\rho(x_1), \rho(x_2), \ldots, \rho(x_n)\}$. It follows that

$$\rho = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ \rho(x_1) & \rho(x_2) & \ldots & \rho(x_n) \end{pmatrix} \in S_n.$$

Thus, $\Gamma_\emptyset$ is a subgroup S_n.

Let $\emptyset^\Gamma$ be the orbit of Γ with respect to $\emptyset$, that is, $\emptyset^\Gamma = \{\rho(\emptyset) : \rho \in \Gamma\}$. Then $\emptyset^\Gamma = \Omega_n$ since $H(n, 1)$ is vertex-transitive. By the orbit-stabilizer theorem (see Lemma 2.2.2 in [7]), we have $|\Gamma| = |\emptyset^\Gamma|\,|\Gamma_\emptyset| \leq |\Omega_n|\,|S_n| = 2^n n!$. The theorem follows. ■

5. Connectivity

In this section, we will show that if k is odd then $H(n, k)$ is a connected bipartite graph with connectivity $\binom{n}{k}$, and if k is even then $H(n, k)$ consists of two isomorphic connected components with connectivity $\binom{n}{k}$.

We use $H^+_{x_{n+1}}(n + 1, k)$ to denote the subgraph of $H(n + 1, k)$ induced by all sets in Ω_{n+1} that contain x_{n+1}, and $H^-_{x_{n+1}}(n + 1, k)$ to denote the subgraph of $H(n + 1, k)$ induced by all sets in Ω_{n+1} that do not contain x_{n+1}.

Lemma 5.1. $H^+_{x_{n+1}}(n + 1, k) \cong H(n, k) \cong H^-_{x_{n+1}}(n + 1, k)$.

Proof. Clearly, $H^-_{x_{n+1}}(n+1, k) \cong H(n, k)$. In order to prove $H(n, k) \cong H^+_{x_{n+1}}(n+1, k)$, we define a mapping φ from $V(H(n, k))$ to $V(H^+_{x_{n+1}}(n + 1, k))$ as follows.

$$\varphi : X \mapsto X \cup \{x_{n+1}\}, \quad \forall X \in \Omega_n.$$

Clearly, the mapping φ is bijective. Since for any two distinct vertices X and Y in $H(n, k)$,

$$\begin{aligned} XY \in E(H(n, k)) &\Leftrightarrow |X \Delta Y| = k \\ &\Leftrightarrow |(X \cup \{x_{n+1}\})\Delta(Y \cup \{x_{n+1}\})| = k \\ &\Leftrightarrow |\varphi(X)\Delta\varphi(Y)| = k \\ &\Leftrightarrow \varphi(X)\varphi(Y) \in E(H^+_{x_{n+1}}(n + 1, k)), \end{aligned}$$

the mapping φ is an isomorphism between $H(n, k)$ and $H^+_{x_{n+1}}(n + 1, k)$, and so $H^+_{x_{n+1}}(n + 1, k) \cong H(n, k)$. ■

Theorem 5.1. *If k is odd then $H(n, k)$ is bipartite and connected, and if k is even then $H(n, k)$ consists of two isomorphic connected components.*

Proof. Suppose that k is odd. Then for any two vertices X and Y in $H(n, k)$, $|X\Delta Y| = k$ means that $|X|$ and $|Y|$ have different parity. Thus, $\{\Omega_n', \Omega_n''\}$ is a bipartition of $V(H(n, k))$. We now prove that $H(n, k)$ is connected by induction on $n \geq 2$ and $k \geq 1$. Since $H(2, 1)$ is a 4-cycle, it is connected. Suppose that $H(n, k)$ is connected.

We first prove that $H(n + 1, k)$ is connected for a fixed k. By Lemma 5.1, $H^+_{x_{n+1}}(n + 1, k) \cong H(n, k) \cong H^-_{x_{n+1}}(n + 1, k)$. By the induction hypothesis, $H(n, k)$ is connected, and so both $H^+_{x_{n+1}}(n + 1, k)$ and $H^-_{x_{n+1}}(n + 1, k)$ are connected. The empty set $\emptyset$ is a vertex in $H^-_{x_{n+1}}(n + 1, k)$, and not in $H^+_{x_{n+1}}(n + 1, k)$. Let $X = \{x_1, \ldots, x_{k-1}, x_{n+1}\}$. Then, $X \in \Omega^k_{n+1}$ is a vertex in $H^+_{x_{n+1}}(n + 1, k)$, and not in $H^-_{x_{n+1}}(n + 1, k)$. Since two vertices $\emptyset$ and X are adjacent in $H(n + 1, k)$, $H(n + 1, k)$ is connected.

We now prove that $H(n, k + 2)$ is connected. To this end, it suffices to see that $H(k + 3, k + 2)$ is connected since we have just proved in the previous paragraph that the connectedness of $H(n, k)$ implies that $H(n + 1, k)$ is connected, for a fixed odd k. Thus, it follows from Proposition 2.4 that $H(k + 3, k + 2) \cong H(k + 3, 1)$ which is a hypercube and connected. Hence, by induction, $H(n, k)$ is connected for every odd k and for all n.

Suppose now that k is even. If XY is an edge, then $|X|$ and $|Y|$ are of the same parity, so there is no edge between $H'(n, k)$ and $H''(n, k)$ and these two are disconnected from each other. By Theorem 3.3, we only need to prove that $H''(n, k)$ is connected. By Theorem 3.4, $H''(n, k)$ contains a spanning subgraph isomorphic to $H(n-1, k-1)$. Since $k-1$ is odd, by the first conclusion, $H(n-1, k-1)$ is connected, and so is $H''(n, k)$.

The theorem follows. ■

Theorem 5.2. *The connectivity of $H(n, k)$ is equal to $\binom{n}{k}$ if k is odd; the connectivities of $H''(n, k)$ and $H'(n, k)$ are both equal to $\binom{n}{k}$ if k is even.*

Proof. It has been shown by Watkins [17] that the connectivity of a connected edge-transitive graph is equal to its minimum degree. By Theorem 4.2, $H(n, k)$ is edge-transitive. By Theorem 5.1 and Proposition 2.1, if k is odd, then the connectivity of $H(n, k)$ is equal to $\binom{n}{k}$.

By Corollary 4.1 both $H'(n, k)$ and $H''(n, k)$ are edge-transitive for an even k. Similarly as in the previous paragraph, their connectivity is equal to $\binom{n}{k}$. ■

6. Hamiltonian property

A cycle in a graph is called a *Hamilton cycle* if it contains all vertices of the graph. A graph is called *hamiltonian* if it contains a Hamilton cycle. In the preceding section, we have shown that $H(n, k)$ is connected if k is odd, and consists of two isomorphic connected components if k is even. In this section, we will prove that $H(n, k)$ is hamiltonian if k is odd, and two isomorphic connected components are hamiltonian if k is even.

Theorem 6.1. *$H(n, k)$ is hamiltonian when k is odd.*

Proof. Note that if k is odd then $H(n, k)$ is bipartite and connected by Theorem 5.1. We prove the theorem by induction on $n \geq 2$ and $k \geq 1$. $H(2, 1)$ is a 4-cycle and so it is hamiltonian. Suppose that $H(n, k)$ is hamiltonian. We need to prove that both $H(n+1, k)$ and $H(n, k+2)$ are hamiltonian.

We first prove that $H(n+1, k)$ is hamiltonian for a fixed k. By Lemma 5.1, $H^{+}_{x_{n+1}}(n+1, k) \cong H(n, k) \cong H^{-}_{x_{n+1}}(n+1, k)$. By the induction hypothesis, both $H^{-}_{x_{n+1}}(n+1, k)$ and $H^{+}_{x_{n+1}}(n+1, k)$ are hamiltonian. Let C^- and C^+ be Hamilton cycles of $H^{-}_{x_{n+1}}(n+1, k)$ and $H^{+}_{x_{n+1}}(n+1, k)$, respectively. Since $H(n, k)$ is edge-transitive by Theorem 4.2, we can assume that the edge $e_1 = \emptyset\{x_1, \ldots, x_k\}$ is in C^- and the edge $e_2 = \{x_k, x_{n+1}\}\{x_1, \ldots, x_{k-1}, x_{n+1}\}$ is in C^+. Clearly, $e_3 = \emptyset\{x_1, \ldots, x_{k-1}, x_{n+1}\}$ and $e_4 = \{x_k, x_{n+1}\}\{x_1, \ldots, x_k\}$ are edges of $H(n+1, k)$. Then $C^- \cup C^+ - e_1 - e_2 + e_3 + e_4$ is a Hamilton cycle in $H(n+1, k)$.

We now prove that $H(n, k+2)$ is hamiltonian. To this end, it suffices to see that $H(k+3, k+2)$ is hamiltonian since we have just proved in the previous paragraph that the hamiltonicity of $H(n, k)$ implies that $H(n+1, k)$ is hamiltonian. Thus, it follows from Proposition 2.4 that $H(k+3, k+2) \cong H(k+3, 1)$ which is a hypercube and hamiltonian. Hence, by induction, $H(n, k)$ is hamiltonian for every odd k and for all n.

By the principle of induction, $H(n, k)$ is hamiltonian when k is odd. The theorem follows. ■

Corollary 6.1. *If k is even, then $H''(n, k)$ is hamiltonian.*

Proof. By Theorem 3.4, $H''(n, k)$ contains a spanning subgraph isomorphic to $H(n-1, k-1)$. By Theorem 6.1, $H(n-1, k-1)$ is hamiltonian, and so is $H''(n, k)$. ■

7. Diameter

A *path* P from x_0 to x_k in G is a sequence of pairwise distinct vertices $P = (x_0, x_1, \ldots, x_k)$, where $x_{i-1}x_i \in E(G)$ for each $i = 1, 2 \ldots, k$. The *length* of P, denoted by $\varepsilon(P)$, is the number of edges in P. For two distinct vertices x and y in G, the minimum length among all paths from x to y is called the *distance* from x to y, denoted by $d_G(x, y)$. The maximum distance among all pairs of vertices in G is called the *diameter* of G, denoted by $d(G)$, that is, $d(G) = \max\{d_G(x, y) : \forall x, y \in V(G)\}$.

In this section, the symbol $d(X, Y)$ denotes the distance $d_{H(n,k)}(X, Y)$ between two vertices X and Y in $H(n, k)$. We will prove that for any odd integer k, $d(H(n, k)) = \lceil \frac{n-1}{k} \rceil + 1$ if $n \geq 2k-1$ and $d(H(n, k)) = \lceil \frac{n-1}{n-k} \rceil + 1$ if $n \leq 2k-2$.

We first prove that $d(H(n, k)) = \lceil \frac{n-1}{k} \rceil + 1$ for $n \geq 2k-1$ in Lemma 7.4. The basal outline of the proof is as follows. To compute $d(H(n, k))$, we only need to consider $d(\emptyset, X)$ for any vertex X since $H(n, k)$ is vertex-transitive by Theorem 4.2. It is easy to find a vertex X such that $d(\emptyset, X) \geq \lceil \frac{n-1}{k} \rceil + 1$, and not easy to prove that $d(\emptyset, X) \leq \lceil \frac{n-1}{k} \rceil + 1$ for every vertex X. In fact, we prove the latter by constructing a path of required length from $\emptyset$ to X by Lemma 7.1. In the same idea, we prove $d(H(n, k)) = \lceil \frac{n-1}{n-k} \rceil + 1$ for $n \leq 2k-2$ in Lemma 7.8.

Lemma 7.1. *Let X and Y be any two distinct elements in Ω_n. If $XY \in E(H(n, k))$, then $k \leq |X|+|Y| \leq 2n-k$ and $||X|-|Y|| \leq k$. For any $s \in \{0, 1, \ldots, n\}$ and any $X \in \Omega_n$, if $k \leq s+|X| \leq 2n-k$ and $||X|-s| \leq k$, k and $|X|+s$ have the same parity, then there exists $Y \in \Omega_n^s$ such that $XY \in E(H(n, k))$.*

Proof. For any two distinct vertices X and Y in $H(n,k)$,

$$XY \in E(H(n,k)) \Leftrightarrow |X\Delta Y| = |X| + |Y| - 2|X\cap Y| = k$$

and hence $|X| + |Y| \geq k$. Also

$$|X\Delta Y| = |X\cup Y| - |X\cap Y| \leq n - |X\cap Y|$$

and hence $|X\cap Y| \geq |X| + |Y| - n$. It follows that

$$k \leq n - |X\cap Y| \leq 2n - |X| - |Y|,$$

that is, $|X| + |Y| \leq 2n - k$. Since $|X\cap Y| \leq \min\{|X|,|Y|\}$, we have that $||X| - |Y|| \leq k$.

Let $X = \{x_1, x_2, \ldots, x_m\}$. For any integer s satisfying our hypothesis, let

$$Y = \left\{x_{m+1-\frac{m+s-k}{2}}, x_{m+1-\frac{m+s-k}{2}+1}, \ldots, x_{k+\frac{m+s-k}{2}}\right\}.$$

Then $|Y| = s$. Since k and $m+s$ are the same parity, $m+s-k$ is even. Since $|m-s| \leq k$, we have that $m+1-\frac{m+s-k}{2} \geq 1$. Also since $s+m \leq 2n-k$, we have that $k + \frac{m+s-k}{2} \leq n$. It follows that Y is a vertex in $H(n,k)$. Since $|X\cap Y| = \frac{m+s-k}{2}$, we have that $|X\Delta Y| = |X| + |Y| - 2|X\cap Y| = m+s-(m+s-k) = k$, which implies that Y is adjacent to X in $H(n,k)$. ■

Lemma 7.2. *Let $P = (\emptyset, X_1, X_2, \ldots, X_{s-1}, X_s)$ is a path from $\emptyset$ to $X = X_s$ in $H(n,k)$, then $|X_s| \leq sk$.*

Proof. By Lemma 7.1, if $XY \in E(H(n,k))$ then $||X| - |Y|| \leq k$ and then $|X_{i+1} - X_i| \leq k$ for $i = 0, 1, \ldots, s-1$ where $X_0 = \emptyset$, and hence $\sum_{i=0}^{s-1} |X_{i+1} - X_i| \leq sk$, and then $|X_s| \leq sk$. ■

Lemma 7.3. *Let $X \in \Omega_n^m$ with $m \geq 1$. Suppose that $k \geq 2$, $n \geq 2k-1$ and $m \leq \min\{n, 2k+1\}$. Then $d(\emptyset, X) = 1$ if $m = k$. For $m \neq k$, $d(\emptyset, X) = 2$ if m is even, and $d(\emptyset, X) = 3$ if both k and m are odd.*

Proof. Without loss of generality, let $X = \{x_1, x_2, \ldots, x_m\}$. If $m = k$, then two vertices $\emptyset$ and X are adjacent in $H(n,k)$ since $|\emptyset\Delta X| = k$, and so $d(\emptyset, X) = 1$. Assume $m \neq k$. Then $d(\emptyset, X) \geq 2$ since $\emptyset$ and X are not adjacent in $H(n,k)$.

Assume m is even. Then $m \leq \min\{n, 2k\}$. Let $Y = \{x_{\frac{m}{2}+1}, x_{\frac{m}{2}+2}, \ldots, x_{\frac{m}{2}+k}\}$. Then $|Y| = k$ and $Y \in \Omega_n$ since $\frac{m}{2} + k \leq n$. Moreover, $|X\Delta Y| = \frac{m}{2} + \left(k - \frac{m}{2}\right) = k$. Thus, $(\emptyset, Y, X)$ is a path from $\emptyset$ to X in $H(n,k)$, and so $d(\emptyset, X) = 2$.

Assume that both k and m are odd. Let $P = (\emptyset, X_1, X_2, \ldots, X)$ be a shortest path from $\emptyset$ to X in $H(n,k)$. Then $X \neq X_1$ since any neighbor of $\emptyset$ in $H(n,k)$ is a k-set in Ω_n and $m \neq k$. Since both k and m are odd, by Proposition 2.3, X_1 and X are not adjacent. It follows that the length of P is at least three, and so $d(\emptyset, X) \geq 3$. To prove that $d(\emptyset, X) = 3$, we only need to construct a path $P = (\emptyset, X_1, X_2, X)$ of length three from $\emptyset$ to X in $H(n,k)$.

If $m < k$, then let $i = \frac{1}{2}(k+m)$, which is an integer with $m < i < k$. Moreover, $\max\{i+k, m+k\} \leq 2k-1 \leq n$. Let

$$X_1 = \{x_{i+1}, x_{i+2}, \ldots, x_{i+k}\} \quad \text{and} \quad X_2 = \{x_1, \ldots, x_i, \ldots, x_{m+k}\}.$$

Then

$$|X_1\Delta X_2| = |\{x_1, x_2, \ldots, x_i\} \cup \{x_{m+k+1}, \ldots, x_{i+k}\}| = i + (i-m) = k$$

and

$$|X_2\Delta X| = |\{x_{m+1}, x_{m+2}, \ldots, x_{m+k}\}| = k,$$

and so $X_1X_2, X_2X \in E(H(n,k))$. Thus, $P = (\emptyset, X_1, X_2, X)$ is a path from $\emptyset$ to X in $H(n,k)$ with length three.

If $m > k$, then let $j = \frac{1}{2}(m-k)$ and let

$$X_1 = \{x_{j+1}, x_{j+2}, \ldots, x_{j+k}\} \quad \text{and} \quad X_2 = \{x_1, \ldots, x_j, \ldots, x_{m-k}\}.$$

Then

$$|X_1\Delta X_2| = |\{x_1, x_2, \ldots, x_j\} \cup \{x_{m-k+1}, \ldots, x_{j+k}\}| = j + (2k + j - m) = k$$

and

$$|X_2\Delta X| = |\{x_{m-k+1}, x_{m-k+2}, \ldots, x_m\}| = k,$$

and so $X_1X_2, X_2X \in E(H(n,k))$. Thus, $P = (\emptyset, X_1, X_2, X)$ is a path from $\emptyset$ to X in $H(n,k)$ with length three. The lemma follows. ■

Lemma 7.4. *If k is odd and $n \geq 2k-1$, then $d(H(n,k)) = \lceil\frac{n-1}{k}\rceil + 1$.*

Proof. If k is odd, then $H(n, k)$ is connected by Theorem 5.1, and so the diameter is well defined. To compute the diameter of $H(n, k)$, we only need to consider the distance from the vertex $\emptyset$ to any other vertex since $H(n, k)$ is vertex-transitive by Theorem 4.2. For $k = 1$ and $X \in V(H(n, 1))$ it is obvious that $d(\emptyset, X) = |X|$ in $H(n, 1)$, so $d(H(n, 1)) = d(\emptyset, \{x_1, \dots, x_n\}) = n = \lceil \frac{n-1}{1} \rceil + 1$. Now suppose that $k \geq 3$.

Let $i = \lceil \frac{n-1}{k} \rceil$. Then $i \geq 2$ and $(i-1)k + 2 \leq n \leq ik + 1$. Let $X = \{x_1, x_2, \dots, x_{(i-1)k+2}\}$. By Lemma 7.2, we have $d(\emptyset, X) \geq i$ in $H(n, k)$. Let P be a shortest path from $\emptyset$ to X in $H(n, k)$. Then $\varepsilon(P) \geq i$. Suppose that $\varepsilon(P) = i$. Since k is odd, by Proposition 2.3, any two adjacent vertices in P have different parity. Since $\emptyset$ is even, $|X|$ and i have the same parity. However, since k is odd, i and $(i-1)k + 2 = |X|$ have different parity, a contradiction. It follows that

$$d(H(n, k)) \geq d(\emptyset, X) = \varepsilon(P) \geq i + 1 = \left\lceil \frac{n-1}{k} \right\rceil + 1.$$

To complete the proof of the lemma, we only need to prove $d(H(n, k)) \leq \lceil \frac{n-1}{k} \rceil + 1$.

When $2k - 1 \leq n \leq 2k + 1$, we have that $d(H(n, k)) \leq 3 = \lceil \frac{n-1}{k} \rceil + 1$ by Lemma 7.3. Assume $n \geq 2k + 2$ and let $X \in \Omega_n^m$. By Lemma 7.3 it suffices to consider those m with $m > 2k + 1$ only. Thus, there is some integer j $(3 \leq j \leq i)$ such that $(j-1)k + 2 \leq m \leq jk + 1$. Without loss of generality, let $X = \{x_1, x_2, \dots, x_m\}$. For each $\ell = 1, \dots, j$, let $X_\ell = \{x_1, \dots, x_{m-(j-\ell)k}\}$. Then $|X_\ell \Delta X_{\ell+1}| = k$ for each $\ell = 1, \dots, j-1$, and $X_j = X$.

If $m - (j-1)k$ is even, then substituting $s = k$ into Lemma 7.1 yields $k \leq k + m - (j-1)k \leq 2n - k$. It follows from $(j-1)k + 2 \leq m \leq jk + 1$ that $2 - k \leq m - (j-1)k - k \leq 1$, so $|m - (j-1)k - k| \leq k$, and k and $m - (j-1)k + k$ are of the same parity. Then, by Lemma 7.1, there exists such an $X_0 \in V(H(n, k))$ of size k that is adjacent to X_1. Thus, $(\emptyset, X_0, X_1, \dots, X_\ell, \dots, X_j)$ is a path of length $j + 1$ from $\emptyset$ to X. Thus, $d(H(n, k)) \leq i + 1 = \lceil \frac{n-1}{k} \rceil + 1$.

If $m - (j-1)k$ is odd, then substituting $s = k$ into Lemma 7.1 yields $k \leq k + m - (j-2)k \leq 2n - k$. It follows from $(j-1)k + 2 \leq m \leq k + 1$ that $2 \leq m - (j-2)k - k \leq k + 1$, and then $3 \leq m - (j-2)k - k \leq k$ since $m - (j-1)k$ is odd, so $|m - (j-2)k - k| \leq k$, and k and $m - (j-2)k + k$ are of the same parity. Then, by Lemma 7.1, there exists such an $X_0 \in V(H(n, k))$ of size k that is adjacent to X_2. Thus, $(\emptyset, X_0, X_2, \dots, X_\ell, \dots, X_j)$ is a path of length j from $\emptyset$ to X, that is, $d(H(n, k)) \leq i = \lceil \frac{n-1}{k} \rceil$.

The Lemma follows. ■

Lemma 7.5. *Let k be even and let $d''(n, k)$ denote the diameter of $H''(n, k)$. If $n \geq 2k - 1$, then $d''(n, k) = \lceil \frac{n-1}{k} \rceil$.*

Proof. By Theorem 5.1, $H''(n, k)$ is connected, and so $d''(n, k)$ is well defined.

We first consider $n \in \{2k - 1, 2k, 2k + 1\}$. If $k = 2$, then $n \in \{3, 4, 5\}$. It is easy to verify that $d''(3, 2) = 1$, $d''(4, 2) = d''(5, 2) = 2$, and so the lemma holds for $n \in \{3, 4, 5\}$. Assume $k \geq 4$.

By Lemma 7.3, $d''(2k - 1, k) = d''(2k, k) = d''(2k + 1, k) = 2$, which satisfy $\lceil \frac{n-1}{k} \rceil$.

Assume that $n \geq 2k + 2$ and let $i = \lceil \frac{n-1}{k} \rceil$. Then $i \geq 3$, and $(i-1)k + 2 \leq n \leq ik + 1$. Let $Z = \{x_1, x_2, \dots, x_{(i-1)k+2}\}$. By Lemma 7.2, we have $d''(\emptyset, Z) \geq i = \lceil \frac{n-1}{k} \rceil$.

To complete our proof, we only need to prove that $d''(\emptyset, X) \leq \lceil \frac{n-1}{k} \rceil$ for any vertex X different from $\emptyset$ since $H''(n, k)$ is vertex-transitive by Corollary 4.1.

Without loss of generality, let $X = \{x_1, x_2, \dots, x_m\}$ be any vertex in $H''(n, k)$, where m is even. Since k is even and $m \leq n \leq ik + 1$, $m \leq ik$. If $m \leq 2k$, then $k \leq k + m \leq 2n - k$, $|m - k| \leq k$, k and $k + m$ are both even. By Lemma 7.1, there is some $X_1 \in \Omega_n^k$ such that it is adjacent to X in $H''(n, k)$. Thus, $(\emptyset, X_1, X)$ is a path of length two from $\emptyset$ to X in $H''(n, k)$.

Assume $m \geq 2k + 2$. Then there is an integer j $(2 \leq j \leq i - 1)$ such that $jk + 2 \leq m \leq (j+1)k$. Let

$$X_\ell = \{x_1, x_2, \dots, x_{m-(j-\ell)k}\}, \quad \ell = 1, \dots, j.$$

Clearly, for each $\ell = 1, 2, \dots, j$, X_ℓ is a vertex in $H''(n, k)$ since $m - (j - \ell)k$ is even. Moreover, $|X_{\ell-1} \Delta X_\ell| = |\{x_{m-(j-\ell+1)k+1}, \dots, x_{m-(j-\ell)k}\}| = k$, that is, $X_{\ell-1}$ and X_ℓ are adjacent in $H''(n, k)$, where $X = X_j$. By Lemma 7.1, there is some $X_0 \in \Omega_n^k$ such that X_0 and X_1 are adjacent in $H''(n, k)$. Thus, $P = (\emptyset, X_0, X_1, \dots, X_j)$ is a path from $\emptyset$ to X in $H''(n, k)$, its length is equal to $j + 1 \leq i = \lceil \frac{n-1}{k} \rceil$. Thus, $d''(n, k) = \lceil \frac{n-1}{k} \rceil$. ■

Lemma 7.6. *If both n and k are odd, then $d(H(n, k)) \geq d(H''(n, n-k)) + 1$.*

Proof. Define a mapping φ from $V(H(n, k))$ to $V(H''(n, n-k))$ as follows.

$$\varphi : X \mapsto \begin{cases} \overline{X}, & \text{if } X \in \Omega_n'; \\ X, & \text{if } X \in \Omega_n'', \end{cases} \quad \forall X \in V(H(n, k)).$$

It is easy to see that φ is a surjective mapping from $V(H(n, k))$ to $V(H''(n, n-k))$. For any two vertices X and Y in $H(n, k)$, if $|X \Delta Y| = k$ then $|X|$ and $|Y|$ have different parity by Proposition 2.3 since k is odd. Without loss of generality, suppose that $|X|$ is odd and $|Y|$ is even. Then, $|\overline{X} \Delta Y| = n - |X \Delta Y| = n - k$. Thus, $\varphi(X)\varphi(Y) = \overline{X}Y$ is an edge in $H(n, n-k)$, which implies that φ is a surjective homomorphism from $H(n, k)$ to $H''(n, n-k)$. Thus, it is not hard to see that $d(H(n, k)) \geq d(H''(n, n-k))$. Use $d'(X, Y)$ to denote the distance from X to Y in $H(n, k)$ and $d''(X, Y)$ to denote the distance from X to Y in $H''(n, n-k)$.

Let X and Y be two vertices in $H''(n, n-k)$ such that $d''(X, Y) = d(H''(n, n-k))$.

Suppose that $d''(X, Y)$ is odd. Since $|X|$ and $|Y|$ are even, both X and Y are in $H(n, k)$, and so $\varphi(X) = X$ and $\varphi(Y) = Y$. Then $d'(X, Y)$ is even by Proposition 2.3. Thus, $d'(X, Y) \geq d''(X, Y) + 1$, and so $d(H(n, k)) \geq d'(X, Y) \geq d''(X, Y) + 1 = d(H''(n, n-k)) + 1$.

Suppose that $d''(X, Y)$ is even. As $|Y|$ is even, $|\bar{Y}|$ is odd. By Proposition 2.3, $d'(X, \bar{Y})$ is odd. Thus, $d(H(n, k)) \geq d'(X, \overline{Y}) \geq d''(X, Y) + 1 = d(H''(n, n-k)) + 1$. ■

Lemma 7.7. *Suppose that k is odd and $n \leq 2k-2$. Then for any $X \in \Omega_n^m$, $d(\emptyset, X) \leq 2$ if m is even and $m \in \{0, 2, 4, \ldots, 2(n-k)\}$, and $d(\emptyset, X) \leq 3$ if m is odd and $m \in \{3k-2n, 3k-2n+1, \ldots, n\}$.*

Proof. If $m = k$, then $d(\emptyset, X) = 1$, clearly. If $m = 0$, then $X = \emptyset$ and $d(\emptyset, X) = 0$. Assume $m \neq 0, k$.

Without loss of generality, let $X = \{x_1, x_2, \ldots, x_m\}$. Suppose that m is even and $m \in \{2, 4, \ldots, 2(n-k)\}$. Let $X_0 = \{x_1, x_2, \ldots, x_{\frac{m}{2}}, x_{m+1}, x_{m+2}, \ldots, x_{k+\frac{m}{2}}\}$. Then $X_0 \in \Omega_n$ since $k + \frac{m}{2} \leq k + (n-k) = n$. Moreover, $|X_0| = k$ and $|X_0 \Delta X| = k$. It follows that $(\emptyset, X_0, X)$ is a path of length two from $\emptyset$ to X in $H(n, k)$, and so $d(\emptyset, X) \leq 2$.

Suppose that m is odd and $m \in \{3k-2n, 3k-2n+1, \ldots, n\}$. If $m > k$, let $X_0 = \{x_1, x_2, \ldots, x_{\frac{m-k}{2}}, x_{m-k+1}, x_{m-k+2}, \ldots, x_{k+\frac{m-k}{2}}\}$ and $X_1 = \{x_1, x_2, \ldots, x_{m-k}\}$; if $m < k$, let $X_0 = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+\frac{k-m}{2}}, x_{k+1}, \ldots, x_{k+\frac{k-m}{2}}\}$, $X_1 = \{x_{m+1}, x_{m+2}, \ldots, x_k\}$. Then $|X_0| = k$ and $|X_0 \Delta X_1| = |X_1 \Delta X| = k$. Thus, $(\emptyset, X_0, X_1, X)$ is a path of length three from $\emptyset$ to X, and so $d(\emptyset, X) \leq 3$. ■

Lemma 7.8. *If k is odd and $n \leq 2k-2$, then $d(H(n, k)) = \lceil \frac{n-1}{n-k} \rceil + 1$.*

Proof. We first note that $k \geq 3$ since $n \leq 2k-2$. If n is even, then $H(n, k) \cong H(n, n-k)$ by Proposition 2.4. Since $n \leq 2k-2$, we have $n \geq 2(n-k)-1$. Since n is even and k is odd, $n-k$ is odd. By Lemma 7.4, we have $d(H(n, k)) = d(H(n, n-k)) = \lceil \frac{n-1}{n-k} \rceil + 1$.

Suppose that n is odd. Since k is odd, $n-k$ is even. By Lemma 7.5, we have $d''(n, n-k) = \lceil \frac{n-1}{n-k} \rceil$. By Lemma 7.6, we have $d(H(n, k)) \geq d(H''(n, n-k)) + 1$. Thus, $d(H(n, k)) \geq \lceil \frac{n-1}{n-k} \rceil + 1$.

We now need to prove that $d(H(n, k)) \leq \lceil \frac{n-1}{n-k} \rceil + 1$. To complete the proof, it is sufficient to show that the distance from $\emptyset$ to any other vertex in $H(n, k)$ is at most $\lceil \frac{n-1}{n-k} \rceil + 1$ since $H(n, k)$ is vertex-transitive.

Let $i = \lceil \frac{n-1}{n-k} \rceil$. Then $3 \leq i \leq k$ since $k+1 \leq n \leq 2k-2$. Then $(i-1)(n-k)+2 \leq n \leq i(n-k)+1$. Let X be any vertex different from $\emptyset$ in $H(n, k)$. Without loss of generality, let $|X| = m$ and $X = \{x_1, x_2, \ldots, x_m\}$.

Case 1. m is odd.

Since $n \leq i(n-k)+1$, we have $3k-2n \leq (i-3)(n-k)+1$. There is an integer j with $0 \leq 2j \leq i-2$ such that $(i-3)(n-k)+1 \leq m+2j(n-k) \leq n$ since $(i-1)(n-k)+2 \leq n \leq i(n-k)+1$. Let

$$X_\ell = \{x_1, \ldots, x_{m+2\ell(n-k)}\}, \qquad \ell = 0, 1, \ldots, j;$$
$$X'_\ell = \{x_{m+(2\ell+1)(n-k)+1}, \ldots, x_n\}, \quad \ell = 0, 1, \ldots, j-1.$$

Then $X_\ell, X'_\ell \in \Omega_n$ and $|X_\ell \Delta X'_\ell| = |X'_\ell \Delta X_{\ell+1}| = k$. Thus, the sequence $(X_0, X'_0, X_1, \ldots, X_l, X'_l, \ldots, X_{j-1}, X'_{j-1}, X_j)$ is a path of length $2j$ from X to X_j. Since $|X_j| = m + 2j(n-k) \geq (i-3)(n-k)+1 \geq 3k-2n$, by Lemma 7.7, we have $d(\emptyset, X_j) \leq 3$. Thus, $d(\emptyset, X) \leq 2j+3 \leq i+1 = \lceil \frac{n-1}{n-k} \rceil + 1$.

Case 2. m is even.

There is an integer j with $0 \leq 2j \leq i-2$ such that $0 \leq m-2j(n-k) \leq 2(n-k)$ since $(i-1)(n-k)+2 \leq n \leq i(n-k)+1$. Let

$$X_\ell = \{x_1, \ldots, x_{m-2\ell(n-k)}\}, \qquad \ell = 0, 1, \ldots, j;$$
$$X'_\ell = \{x_{m-(2\ell+1)(n-k)+1}, \ldots, x_n\}, \quad \ell = 0, 1, \ldots, j-1.$$

Then $X_\ell, X'_\ell \in \Omega_n$ and $|X_\ell \Delta X'_\ell| = |X'_\ell \Delta X_{\ell+1}| = k$. Thus, the sequence $(X_0, X'_0, X_1, \ldots, X_l, X'_l, \ldots, X_{j-1}, X'_{j-1}, X_j)$ is a path of length $2j$ from X to X_j. Since $|X_j| = m - 2\ell(n-k) \leq 2(n-k)$, by Lemma 7.7, we have $d(\emptyset, X_j) \leq 2$. Thus, $d(\emptyset, X) \leq 2j+2 \leq i = \lceil \frac{n-1}{n-k} \rceil$.

The proof of the lemma is complete. ■

By Lemmas 7.4 and 7.8, we immediately have the following result.

Theorem 7.1. *For any odd integer k, $d(H(n, k)) = \lceil \frac{n-1}{k} \rceil + 1$ if $n \geq 2k-1$; $d(H(n, k)) = \lceil \frac{n-1}{n-k} \rceil + 1$ if $n \leq 2k-2$.*

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