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Bipanconnectivity of Cartesian product networks*

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Abstract

This paper is concerned with the bipanconnectivity of Cartesian product networks. It provides sufficient conditions for a Cartesian product network to be bipanconnected and uses this general result to re-prove existing results as regards the k-ary n-cube, due to Stewart and Xiang and to Hsieh and Lin, as well as to extend these results to multidimensional toroidal meshes.

1 Introduction

It is well-known that the topological structure of an *interconnection network* (*network* for short) is usually represented by a graph in which vertices represent processors and edges represent communication links between processors. For notation and terminology on graph theory not defined here we follow [22]. Let $G = (V, E)$ be a simple connected graph with vertex-set $V = V(G)$ and edge-set $E = E(G)$. For any $x, y \in V(G)$, the distance $d_G(x, y)$ between u and v is the length of a shortest xy-path in G.

The study of topological properties of a network is important for parallel or distributed systems. The problem of finding paths of various lengths in networks has recently received much attention because this is an important measurement for determining whether the topology of a network is suitable for an application in which mapping paths of various lengths into the topology is required. In particular, panconnectivity and bipanconnectivity are two important properties, and have been described in many articles [5, 7, 11, 13, 14, 15, 16, 17, 18, 19]. A survey of results on this topic can be found in [23].

A graph G of order n is said to be *panconnected* (respectively, *bipanconnected*) if for any pair of distinct vertices x and y, there exists an xy-path in G with every

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length ℓ satisfying $d_G(x,y) \le \ell \le n-1$ (respectively, $d_G(x,y) \le \ell \le n-1$ and $\ell \equiv d_G(x,y) \pmod 2$).

Among various designs of large-scale networks, the Cartesian product method is a very effective method of building larger networks from several specified small-scale networks. For graphs G_1 and G_2, the *Cartesian product* $G_1 \times G_2$ is the graph with vertex-set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge-set $E(G_1 \times G_2) = \{(x_1,x_2)(y_1,y_2) \mid x_1 = y_1$ and $x_2y_2 \in E(G_2)$ or $x_2 = y_2$ and $x_1y_1 \in E(G_1)\}$.

As a graph operation, it is easy to observe that the Cartesian product satisfies the commutative law and the associative law if we identify isomorphic graphs. By the commutative and associative laws of the Cartesian product, we may write $G_1 \times \cdots \times G_n$ for the Cartesian product of $G_1, \ldots, G_n$, where $V(G_1 \times \cdots \times G_n) = V(G_1) \times \cdots \times V(G_n)$. Two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are linked by an edge in $G_1 \times \cdots \times G_n$ if and only if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ differ in exactly one coordinate, say the ith, and there is an edge $x_iy_i \in E(G_i)$.

Many popular networks can be constructed by Cartesian product, such as the hypercube and undirected toroidal mesh. The *n-dimensional hypercube* Q_n is $K_2 \times \cdots \times K_2$ where K_i is the complete graph on i vertices. The bipanconnectivity of the hypercube Q_n has been investigated in depth by several authors (for example, [12, 13, 14, 20]). A quite natural generalization of Q_n is the undirected toroidal mesh. The *n-dimensional undirected toroidal mesh* $C_n(k_1, \ldots, k_n)$ can be defined as $C_{k_1} \times \cdots \times C_{k_n}$, where C_{k_i} is a cycle of length k_i for $i = 1, \ldots, n$. Note that $C_n(k, \ldots, k)$ is a usual *k-ary n-cube* Q_n^k, which has received much attention [1, 2, 3, 4, 6, 8, 9, 10, 11, 21, 24]. In particular, Hsieh and Lin [10] showed that Q_n^k is bipanconnected if k is even, and Stewart and Xiang [17] improved this result by proving that Q_n^k is also bipanconnected when k is odd.

In this paper, we study the bipanconnectivity of Cartesian product graphs, and show that $G_1 \times G_2$ is bipanconnected if, for $i = 1, 2$, G_i is either a bipanconnected hamiltonian graph with odd order or a bipanconnected bipartite graph of order at least four. As a consequence, we prove that the undirected toroidal mesh $C_n(k_1, \ldots, k_n)$ is bipanconnected for any integers $n \ge 2$ and $k_i \ge 3$ $(i = 1, \ldots, n)$, which generalizes the above-mentioned result of Stewart and Xiang [17] that Q_n^k is bipanconnected.

The rest of the paper is organized as follows. Section 2 presents some preliminary results. Section 3 investigates bipanconnectivity of Cartesian product graphs. Section 4 investigates bipanconnectivity of the undirected toroidal mesh. Our conclusions are in Section 5.

2 Some preliminary results

For a vertex x in G_1 and a subgraph $H \subseteq G_2$, we use xH to denote the subgraph $\{x\} \times H$ of $G_1 \times G_2$. Similarly, for a vertex y in G_2 and a subgraph $H \subseteq G_1$, Hy denotes the subgraph $H \times \{y\}$ of $G_1 \times G_2$. For a path $P = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$, $P(x_i, x_j)$ denotes the section $(x_i, \ldots, x_j)$ of P. For the sake of convenience, we will express P as

$$P = x_1 \xrightarrow{P(x_1,x_i)} x_i \xrightarrow{P(x_i,x_j)} x_j \xrightarrow{P(x_j,x_n)} x_n.$$

The symbol $\varepsilon(P)$ denotes the length of P, which is the number of edges in P.

Lemma 1 *(Theorem 2.3.3 in Xu [22]) For any two distinct vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $G_1 \times G_2$, where $x_i, y_i \in V(G_i)$ for $i = 1, 2$, we have $d_{G_1 \times G_2}(x, y) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$.*

Observation 2 *For any integer $m \geq 2$, let $P_m = (0, 1, \ldots, m-1)$ be a path. For every vertex (i, j) in $P_2 \times P_m$ different to $(0, 0)$, there exists a path of length ℓ between $(0, 0)$ and (i, j) in $P_2 \times P_m$ for every ℓ with $i + j \leq \ell \leq 2m - 1$ and $\ell \equiv i + j \pmod 2$.*

In fact, the result of Observation 2 is also valid more generally.

Theorem 3 *For any integers $n \geq 3$ and $m \geq 3$, let $P_n = (0, 1, \ldots, n-1)$ and $P_m = (0, 1, \ldots, m-1)$ be two paths. For every vertex (i, j) in $P_n \times P_m$ different to $(0, 0)$, there is a path of length ℓ between $(0, 0)$ and (i, j) in $P_n \times P_m$ for every ℓ with $i + j \leq \ell \leq nm - 1$ and $\ell \equiv i + j \pmod 2$.*

Proof. Let (i, j) be a vertex in $P_n \times P_m$ different from $(0, 0)$, and let ℓ be an integer with $i + j \leq \ell \leq nm - 1$ and $\ell \equiv i + j \pmod 2$. We need to show that there exists a path L of length ℓ between $(0, 0)$ and (i, j) in $P_n \times P_m$. We distinguish the following five cases according to the position of (i, j) in $P_n \times P_m$.

Case 1. $i = n - 1$ and $j = 0$ or $i = 0$ and $j = m - 1$. Without loss of generality, say $i = n - 1$ and $j = 0$. Let ℓ be such that $n - 1 \leq \ell \leq nm - 1$ and $\ell \equiv n - 1 \pmod 2$.

Subcase 1.1. n is even.

Since $0 \leq \ell - (n-1) \leq n(m-1)$ and $\ell - (n-1)$ is even, there are n integers $k_0, \ldots, k_{n-1}$ in $[0, m-1]$ such that $\sum_{t=0}^{n-1} k_t = \ell - (n-1)$ and $k_t = k_{t+1}$ for every even integer $t \in [0, n-1]$. Thus the path

$$\begin{aligned} L = \ & (0,0) \xrightarrow{0P_m(0,k_0)} (0,k_0) \to (1,k_1) \xrightarrow{1P_m(k_1,0)} (1,0) \to \ldots \\ & \to (n-2,0) \xrightarrow{(n-2)P_m(0,k_{n-2})} (n-2,k_{n-2}) \\ & \to (n-1,k_{n-1}) \xrightarrow{(n-1)P_m(k_{n-1},0)} (n-1,0) \end{aligned}$$

is a path between $(0, 0)$ and $(n - 1, 0)$ in $P_n \times P_m$ and

$$\varepsilon(L) = (n-1) + \sum_{t=0}^{n-1} k_t = (n-1) + [\ell - (n-1)] = \ell.$$

Subcase 1.2. n is odd.

Then $n-1$ and ℓ are even. For any integer $\ell' \in [n-2,(n-1)m-1]$ with $\ell' \equiv n-2 \pmod 2$, by Subcase 1.1, there is a path $L_{\ell'}$ of length ℓ' between $(0,0)$ and $(n-2,0)$ in $P_n(0,n-2)\times P_m$.

If $n-1 \le \ell \le (n-1)m$, then the path

$$L=(0,0)\xrightarrow{L_{\ell-1}}(0,n-2)\to(0,n-1)$$

is a path of length ℓ between $(0,0)$ and $(n-1,0)$ in $P_n\times P_m$.

If $(n-1)m+1 \le \ell \le nm-1$, then we have $m+1 \le \ell-(n-2)m \le 2m-1$. Since n is odd and ℓ is even, $\ell-(n-2)m \equiv m \pmod 2$. By Observation 2, there exists a path L' between $(n-2,m-1)$ and $(n-1,0)$ in $P_n(n-2,n-1)\times P_m$ and $\varepsilon(L')=\ell-(n-2)m$. By the structure of $L_{(n-1)m-1}$ in Subcase 1.1, the section from $(0,0)$ to $(n-2,m-1)$ in $L_{(n-1)m-1}$, denoted by L'', is a path between $(0,0)$ and $(n-2,m-1)$ in $P_n(0,n-2)\times P_m$ and $\varepsilon(L'')=[(n-1)m-1]-(m-1)=(n-2)m$. It follows that

$$L=(0,0)\xrightarrow{L''}(n-2,m-1)\xrightarrow{L'}(n-1,0)$$

is a path between $(0,0)$ and $(n-1,0)$ in $P_n\times P_m$ and

$$\varepsilon(L)=\varepsilon(L'')+\varepsilon(L')=(n-2)m+[\ell-(n-2)m]=\ell.$$

Case 2. $i=n-1$ and $j=m-1$.

Since $n+m-2 \le \ell \le nm-1$ and $\ell \equiv n+m-2 \pmod 2$, $n-2 \le \ell-m \le (n-1)m-1$ and $\ell-m \equiv n-2$. By Case 1, there is a path L' between $(0,0$ and $(n-2,0)$ in $P_n(0,n-2)\times P_m$ and $\varepsilon(L')=\ell-m$. Thus, the path

$$L=(0,0)\xrightarrow{L'}(n-2,0)\to(n-1,0)\xrightarrow{(n-1)P_m(0,m-1)}(n-1,m-1)$$

is a path between $(0,0)$ and $(n-1,m-1)$ in $P_n\times P_m$ and $\varepsilon(L)=\varepsilon(L')+1+(m-1)=\ell$.

Case 3. $i\in[1,n-2]$ and $j=0$ or $i=0$ and $j\in[1,m-2]$. Without loss of generality, assume $i\in[1,n-2]$ and $j=0$. Then $i\le \ell\le nm-1$ and $\ell\equiv i \pmod 2$.

Case 3.1. If $i\le \ell\le (i+1)m-1$, then, by Case 1, there is a path of length ℓ between $(0,0)$ and $(i,0)$ in $P_n(0,i)\times P_m$ $(\subseteq P_n\times P_m)$.

Case 3.2. If $(i+1)m\le \ell\le nm-1$, then, by Case 2, there is a path L' between $(0,0)$ and $(i-1,m-1)$ in $P_n(0,i-1)\times P_m$ with length $\ell'\in\{im-2,im-1\}$ and $\ell'\equiv i+m-2 \pmod 2$. We first show that $m\le \ell-\ell'-1\le (n-i)m-1$ and $\ell-\ell'-1\equiv m-1 \pmod 2$.

If $\ell'=im-2$, since $\ell'\equiv i+m-2 \pmod 2$, both i and m are even. By $\ell\equiv i \pmod 2$, ℓ is also even, and so $(i+1)m\le \ell\le nm-2$. Hence $\ell-\ell'-1\equiv 1\equiv m-1 \pmod 2$ and $m+1\le \ell-\ell'-1\le (n-i)m-1$.

If $\ell'=im-1$, it is easy to see that $m\le \ell-\ell'-1\le (n-i)m-1$. Since $\ell\equiv i \pmod 2$, $\ell-\ell'-1\equiv i-im\equiv i(m-1) \pmod 2$. When $m-1$ is even, it is clear that $\ell-\ell'-1\equiv m-1 \pmod 2$. When $m-1$ is odd, we have $\ell'\equiv i \pmod 2 \equiv \ell \pmod 2$ by $\ell'\equiv i+m-2 \pmod 2$, and so $\ell-\ell'-1\equiv m-1 \pmod 2$.

By Case 1, there is a path L'' between $(i, m-1)$ and $(i, 0)$ in $P_n(i, n-1) \times P_m$ with length $\ell - \ell' - 1$. Thus, the path

$$L = (0,0) \xrightarrow{L'} (i-1, m-1) \to (i, m-1) \xrightarrow{L''} (i, 0)$$

is a path between $(0,0)$ and $(i,0)$ in $P_n \times P_m$ and

$$\varepsilon(L) = \varepsilon(L') + 1 + \varepsilon(L'') = \ell' + 1 + (\ell - \ell' - 1) = \ell.$$

Case 4. $i \in [1, n-2]$ and $j = m-1$ or $i = n-1$ and $j \in [1, m-2]$. Without loss of generality, assume $i \in [1, n-2]$ and $j = m-1$. Then $i + m - 1 \le \ell \le nm - 1$ and $\ell \equiv i + m - 1 \pmod 2$.

Subcase 4.1. If $i + m - 1 \le \ell \le 2n + m - i - 3$, then, by $\ell \equiv i + m - 1 \pmod 2$, $t = \frac{\ell - (i+m-1)}{2}$ is an integer in $[0, n-i-1]$. Thus, the path

$$L = (0,0) \xrightarrow{P_n(0,i+t)0} (i+t, 0) \xrightarrow{(i+t)P_m(0,m-1)} (i+t, m-1) \xrightarrow{P_n(i+t,i)(m-1)} (i, m-1)$$

is a path between $(0,0)$ and $(i, m-1)$ in $P_n \times P_m$ and $\varepsilon(L) = (i+t) + (m-1) + t = \ell$.

Subcase 4.2. If $2n + m - i - 2 \le \ell \le nm - 1$, then, by $\ell \equiv i + m - 1 \pmod 2$, we have $\ell - 1 \equiv i + m \equiv (n-i) + (n+m-4) \pmod 2$.

If we can choose two integers $\ell' \in [n-i, 2n-1]$ with $\ell' \equiv n - i \pmod 2$ and $\ell'' \in [n+m-4, n(m-2)-1]$ with $\ell'' \equiv n + m - 4 \pmod 2$ such that $\ell' + \ell'' = \ell - 1$, then there is a path L' of length ℓ' between $(n-1, m-2)$ and $(i, m-1)$ in $P_n \times P_m(m-2, m-1)$ by Observation 2, and a path L'' of length ℓ'' between $(0,0)$ and $(n-1, m-3)$ in $P_n \times P_m(0, m-3)$ by Case 2. Thus, the path

$$L = (0,0) \xrightarrow{L''} (n-1, m-3) \to (n-1, m-2) \xrightarrow{L'} (i, m-1)$$

is a path between $(0,0)$ and $(i, m-1)$ in $P_n \times P_m$ and

$$\varepsilon(L) = \varepsilon(L'') + 1 + \varepsilon(L') = \ell'' + 1 + \ell' = \ell.$$

If we can not choose such two integers ℓ' and ℓ'', then we can check easily that $\ell = nm - 1$, $n - i \equiv 2n - 2 \pmod 2$ and $n + m - 4 \equiv n(m-2) - 2 \pmod 2$. By $n + m - 4 \equiv n(m-2) - 2 \equiv n(m-4) \pmod 2$, both n and m are even. So i is also even. Since $m - 2 \ge 2$ is even, there is a path L' between $(0,0)$ and $(0, m-3)$ in $P_n \times P_m(0, m-3)$ such that $\varepsilon(L') = n(m-2) - 1$ by Subcase 1.1. Since $1 + i \equiv 1 \equiv 2n - 1 \pmod 2$, by Observation 2, there is a path L'' between $(0, m-2)$ and $(i, m-1)$ in $P_n \times P_m(m-2, m-1)$ such that $\varepsilon(L'') = 2n - 1$. Thus, the path

$$L = (0,0) \xrightarrow{L'} (0, m-3) \to (0, m-2) \xrightarrow{L''} (i, m-1)$$

is a path between $(0,0)$ and $(i, m-1)$ in $P_n \times P_m$ and

$$\varepsilon(L) = \varepsilon(L') + 1 + \varepsilon(L'') = [n(m-2) - 1] + 1 + (2n-1) = nm - 1 = \ell.$$

Case 5. $i \in [1, n-2]$ and $j \in [1, m-2]$. Then $i+j \le \ell \le nm-1$ and $\ell \equiv i+j \pmod 2$.

Subcase 5.1. If $i+j \le \ell \le 2n+j-i-2$, then, by $\ell \equiv i+j \pmod 2$, $h = \frac{\ell-(i+j)}{2}$ is an integer in $[0, n-i-1]$. Thus, the path

$$L = (0,0) \xrightarrow{P_n(0,i+h)0} (i+h, 0) \xrightarrow{(i+h)P_m(0,j)} (i+h, j) \xrightarrow{P_n(i+h,i)j} (i,j)$$

is a path between $(0,0)$ and (i,j) in $P_n \times P_m$ and $\varepsilon(L) = (i+h)+j+h = \ell$.

Subcase 5.2. If $2n+j-i-1 \le \ell \le nm-1$, then, by $\ell \equiv i+j \pmod 2$, we have $\ell - 1 \equiv i+j-1 \equiv (n-i-1)+(n+j-2) \pmod 2$.

If we can choose two integers $\ell' \in [n-i-1, n(m-j)-1]$ and $\ell'' \in [n+j-2, nj-1]$ such that $\ell' + \ell'' = \ell - 1$, $\ell' \equiv n-i-1 \pmod 2$ and $\ell'' \equiv n+j-2 \pmod 2$, then there is a path L' between $(n-1,j)$ and (i,j) in $P_n \times P_m(j, m-1)$ with $\varepsilon(L') = \ell'$ by Case 3, and there exists a path L'' between $(0,0)$ and $(n-1, j-1)$ with $\varepsilon(L'') = \ell''$ in $P_n \times P_m(0, j-1)$ by Case 2. Thus, the path

$$L = (0,0) \xrightarrow{L''} (n-1, j-1) \to (n-1, j) \xrightarrow{L'} (i,j)$$

is a path between $(0,0)$ and (i,j) in $P_n \times P_m$ and

$$\varepsilon(L) = \varepsilon(L'') + 1 + \varepsilon(L') = \ell'' + 1 + \ell' = \ell.$$

If we can not choose such two integers ℓ' and ℓ'', then we can prove easily that $\ell = nm-1$, $n-i-1 \equiv n(m-j)-2 \pmod 2$ and $n+j-2 \equiv nj-2 \pmod 2$. Thus both n and j are even, and so ℓ and i are odd. Since $j \ge 2$ is even, there is a path L' between $(0,0)$ and $(0, j-1)$ in $P_n \times P_m(0, j-1)$ such that $\varepsilon(L') = nj-1$ by Subcase 1.1. Since $n(m-j)-1 \equiv 1 \equiv i \pmod 2$, by Case 3, there exists a path L'' between $(0,j)$ and (i,j) in $P_n \times P_m(j, m-1)$ with $\varepsilon(L'') = n(m-j)-1$. Thus, the path

$$L = (0,0) \xrightarrow{L'} (0, j-1) \to (0,j) \xrightarrow{L''} (i,j)$$

is a path between $(0,0)$ and (i,j) in $P_n \times P_m$ and

$$\varepsilon(L) = \varepsilon(L') + 1 + \varepsilon(L'') = (nj-1) + 1 + [n(m-j)-1] = nm-1 = \ell.$$

The theorem follows. □

3 The Cartesian product of bipanconnected graphs

In this section, we discuss bipanconnectivity of the Cartesian product of two bipanconnected graphs. A cycle $(x_1, \ldots, x_n, x_1)$ is a sequence of vertices in which any two consecutive vertices are adjacent, where $x_1, \ldots, x_n$ are all distinct. A cycle (respectively, path) in a graph G is called a *hamiltonian cycle* (respectively, *hamiltonian path*) if it contains every vertex of G. A graph G is said to be *hamiltonian* if it contains a hamiltonian cycle. A graph $G = (X, Y)$ is *bipartite* if its vertex set can be partitioned into two disjoint sets X and Y such that every edge joins a vertex of X and a vertex of Y.

Lemma 4 *Let $G = (X, Y)$ be a bipanconnected bipartite graph with order $n \geq 4$. Then $|X| = |Y|$. Furthermore, G is hamiltonian.*

Proof. We first show $|X| = |Y|$. Suppose that $|V(G)| = n$ is odd. Then $n \geq 5$, and at least one of $|X|$ and $|Y|$ is at least 3, without loss of generality, say $|X| \geq 3$. Let x_1 and x_2 be two arbitrary vertices in X, so $d_G(x_1, x_2)$ is even. By the bipanconnectivity of G, there exists an x_1x_2-path P of length $n-1$ in G. Since G is bipartite and n is odd, we have

$$|X| = |V(P) \cap X| = |V(P) \cap Y| + 1 = |Y| + 1. \tag{3.1}$$

Thus $|Y| = |X| - 1 \geq 2$ and so we can choose two distinct vertices from Y. By proceeding as we did above for Y rather than X we get that $|Y| = |X|$, a contradiction. So n is even.

Let x and y be two adjacent vertices in G such that $x \in X$ and $y \in Y$. Then $d_G(x, y) = 1$. Since $n - 1$ is odd, by bipanconnectivity of G, there exists an xy-path U of length $n - 1$ in G. Since G is bipartite and n is even,

$$|X| = |V(U) \cap X| = |V(U) \cap Y| = |Y|.$$

Secondly, we now show that G is hamiltonian. Let $x \in X$ and $y \in Y$ be two adjacent vertices in G, so $d_G(x, y) = 1$. Since n is even, by the bipanconnectivity of G, there exists an xy-path W of length $n-1$ in G. Thus $x \xrightarrow{W} y \rightarrow x$ is a hamiltonian cycle of G. Therefore, the lemma follows. □

Theorem 5 *For $i = 1, 2$, let G_i be a bipanconnected hamiltonian graph with odd order or a bipanconnected bipartite graph with order at least four. Then $G_1 \times G_2$ is bipanconnected.*

Proof. Let $G = G_1 \times G_2$, $n = |V(G_1)|$ and $m = |V(G_2)|$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any two different vertices in G and let ℓ be an integer with $d_G(x, y) \leq \ell \leq nm - 1$ and $\ell \equiv d_G(x, y) \pmod 2$. To prove the theorem, we only need to find an xy-path of length ℓ in G. Let $d_1 = d_{G_1}(x_1, y_1)$ and $d_2 = d_{G_2}(x_2, y_2)$. By Lemma 1, $\ell \equiv d_1 + d_2 \pmod 2$.

By Lemma 7 and the conditions on G_1 and G_2, G_1 and G_2 are hamiltonian, and so we can choose a hamiltonian path U_i with starting vertex x_i in G_i for $i = 1, 2$. At least one of the paths from x_i to y_i in U_i has length equal to d_i mod 2, for $i = 1, 2$. For convenience, let the length of this path be d'_i. Clearly, $d_i \leq d'_i$.

If $d'_1 + d'_2 \leq \ell \leq nm - 1$, then since $\ell \equiv d_1 + d_2 \equiv d'_1 + d'_2 \pmod 2$, there is an xy-path of length ℓ in $U_1 \times U_2$ ($\subseteq G_1 \times G_2$) by Theorem 3.

If $d_1 + d_2 \leq \ell \leq d'_1 + d'_2$, then since $\ell \equiv d_1 + d_2 \equiv d'_1 + d'_2 \pmod 2$, there exist two integers $h_1 \in [d_1, d'_1]$ and $h_2 \in [d_2, d'_2]$ such that $h_1 \equiv d_1 \pmod 2$, $h_2 \equiv d_2 \pmod 2$ and $h_1 + h_2 = \ell$. For $i = 1, 2$, by the bipanconnectivity of G_i, there is an x_iy_i-path W_i of length h_i in G_i. Thus, the path

$$L = (x_1, x_2) \xrightarrow{W_1 x_2} (y_1, x_2) \xrightarrow{y_1 W_2} (y_1, y_2)$$

is an xy-path in $G_1 \times G_2$ and $\varepsilon(L) = \varepsilon(W_1) + \varepsilon(W_2) = h_1 + h_2 = \ell$.

The theorem follows. □

4 The undirected toroidal mesh $C_n(k_1, \ldots, k_n)$

Lemma 6 *(Theorem 2.3.5 in Xu [22])* $G_1 \times G_2 \ldots G_n$ *is vertex-transitive if* G_i *is vertex-transitive for each* $i = 1, 2, \ldots, n$.

We first consider the case $n = 2$.

Lemma 7 *Let* $k_1 \geq 3$ *and* $k_2 \geq 3$ *be two integers. Then* $C_2(k_1, k_2)$ *is bipanconnected.*

Proof. Let $C_{k_1} = (0, 1, \ldots, k_1, 0)$ and $C_{k_2} = (0, 1, \ldots, k_2, 0)$ be two cycles. Then $C_2(k_1, k_2) = C_{k_1} \times C_{k_2}$. Since C_{k_1} and C_{k_2} are vertex-transitive, $C_2(k_1, k_2)$ is also vertex-transitive by Lemma 6. For any two different vertices, without loss of generality, say $x = (0, 0)$ and $y = (i, j)$ with $0 \leq i \leq \lfloor \frac{k_1}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{k_2}{2} \rfloor$, we only need to find an xy-path of length ℓ in $C_2(k_1, k_2)$ such that $i + j \leq \ell \leq k_1 k_2 - 1$ and $\ell \equiv i + j \pmod 2$. Since $P_{k_1} = (0, 1, \ldots, k_1) \subseteq C_{k_1}$ and $P_{k_2} = (0, 1, \ldots, k_2) \subseteq C_{k_2}$, by Theorem 3, there exists obviously a desired xy-path in $P_{k_1} \times P_{k_2}$ $(\subseteq C_2(k_1, k_2))$. □

Lemma 8 *Let* G *be a bipanconnected hamiltonian graph with odd order or a bipanconnected bipartite graph with order at least four, and let* C_m *be a cycle of order* m. *Then* $G \times C_m$ *is bipanconnected.*

Proof. Let $n = |V(G)|$ and denote the vertex set of G by $\{0, 1, \ldots, n\}$. Let $C_m = (0, 1, \ldots, m, 0)$. Note that C_m is vertex-transitive. For any two different vertices, without loss of generality, say $x = (0, 0)$ and $y = (i, j)$ with $0 \leq j \leq \lfloor \frac{m}{2} \rfloor$, in $G \times C_m$, we only need to find an xy-path of length ℓ in G satisfying $d + j \leq \ell \leq nm - 1$ and $\ell \equiv d + j \pmod 2$, where d is the distance from 0 to i in G.

By Lemma 4 and the condition of G, G is hamiltonian, and so we can choose a hamiltonian path U with start vertex 0 in G. At least one of the paths from 0 to i in U has length equal to d mod 2. For convenience, let the length of this path be d'. Clearly, $d \leq d'$. Let $P = (0, 1, \ldots, m)$, so $P \subseteq C_m$ and $d_P(0, j) = d_{C_m}(0, j) = j$.

If $d' + j \leq \ell \leq nm - 1$, since $\ell \equiv d + j \equiv d' + j \pmod 2$, there is an xy-path of length ℓ in $U \times P$ $(\subseteq G \times C_m)$ by Theorem 3.

If $d + j \leq \ell \leq d' + j$, by the bipanconnectivity of G and $\ell - j \equiv d \pmod 2$, there is a $0i$-path W of length $\ell - j$ in G. Hence the path

$$L = (0, 0) \xrightarrow{W0} (i, 0) \xrightarrow{iP(0,j)} (i, j)$$

is an xy-path in $G \times P$ $(\subseteq G \times C_m)$ and $\varepsilon(L) = \varepsilon(W) + j = (\ell - j) + j = \ell$.

The lemma follows. □

Theorem 9 *For any integer $n \geq 2$ and integer $k_i \geq 3$ $(i = 1, \ldots, n)$, $C_n(k_1, \ldots, k_n)$ is bipanconnected.*

Proof. Clearly, $C_n(k_1, \ldots, k_n)$ is hamiltonian. Without loss of generality, assume $\{k_1, \ldots, k_t\}$ consists of all odd numbers in $\{k_1, \ldots, k_n\}$. Then $0 \leq t \leq n$. By the commutative law of Cartesian product,

$$C_n(k_1, \ldots, k_n) = (C_{k_1} \times \cdots \times C_{k_t}) \times (C_{k_{t+1}} \times \cdots \times C_{k_n}).$$

No matter what the value of t, applying Theorem 5, Lemma 7 and Lemma 8 as required yields the results. □

By Theorem 9, we immediately obtain the following two corollaries.

Corollary 10 *([17]) Let $n \geq 2$ and $k \geq 3$ be two integers. Then Q_n^k is bipanconnected.*

Corollary 11 *([10]) Let $n \geq 2$ be an integer and let $k \geq 4$ be an even integer. Then Q_n^k is bipanconnected.*

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