Connectivity and super-connectivity of Cartesian product graphs*

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Abstract

This paper determines that the connectivity of the Cartesian product $G_1 \square G_2$ of two graphs G_1 and G_2 is equal to $\min\{\kappa_1 v_2, \kappa_2 v_1, \delta_1 + \delta_2\}$, where v_i, κ_i, δ_i is the order, the connectivity and the minimum degree of G_i, respectively, for $i = 1, 2$, and gives some necessary and sufficient conditions for $G_1 \square G_2$ to be maximally connected and super-connected.

1 Introduction

All graphs in this paper are finite and simple. For graph theoretical terminology and notation not defined here, we refer the reader to [5]. Let G_1 and G_2 be two graphs, v_i, δ_i, κ_i and V_i denote the number of vertices, the minimum degree, the connectivity and the vertex-set of G_i, respectively, for $i = 1, 2$. The Cartesian product graph $G_1 \square G_2$ has the vertex-set $V = V_1 \times V_2 = \{xy |\ x \in V_1, y \in V_2\}$, and two vertices x_1x_2 and y_1y_2 are adjacent if and only if either $x_1 = y_1$, x_2 and y_2 are adjacent in G_2, or $x_2 = y_2$, x_1 and y_1 are adjacent in G_1. A graph is said to be *maximally connected* if $\kappa = \delta$. A connected graph is said to be *super-connected* if every minimum cut-set is the neighbor-set of some vertex. It is clear that any super-connected graph is certainly maximally connected.

The recent study on connectivity of the Cartesian product can be found in [1, 2, 3, 4], where the lower bounds of the connectivity of $G_1 \square G_2$ and some sufficient conditions for it to be maximally or super-connected are given. In the present paper, we determine that $\kappa(G_1 \square G_2) = \min\{\kappa_1 v_2, \kappa_2 v_1, \delta_1 + \delta_2\}$ and give some necessary and sufficient conditions for $G_1 \square G_2$ to be maximally connected and super-connected.

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2 Connectivity

Lemma 1 *Let p, q, a, b be integers with $1 \leqslant a \leqslant p-1$ and $1 \leqslant b \leqslant q-1$. Then $a(q-b)+b(p-a) \geqslant p+q-2$ and the equality holds if and only if one of the following conditions holds*

i) $q = 2, b = 1$,
ii) $p = 2, a = 1$,
iii) $a = 1, b = 1$,
vi) $a = p-1, b = q-1$.

Proof. If $q \geqslant 2b$, then

$$\begin{aligned} a(q-b)+b(p-a) &= (q-2b)a + p\,b \\ &\geqslant (q-2b) + p\,b \\ &= p+q-2+(p-2)(b-1) \\ &\geqslant p+q-2. \end{aligned}$$

If $q < 2b$, then

$$\begin{aligned} a(q-b)+b(p-a) &= (q-2b)a + p\,b \\ &\geqslant (q-2b)(p-1) + p\,b \\ &= p+q-2+(p-2)(q-b-1) \\ &\geqslant p+q-2. \end{aligned}$$

And it is easy to check the conditions for the equality to hold. □

Lemma 2 *Let G be a graph and $A \subseteq V(G)$. Then $|A \cup N(A)| \geqslant \delta(G)+1$.*

Proof. Arbitrarily take a vertex x in A. Its neighbors must be in $A \cup N(A) - \{x\}$. Thus $|A \cup N(A)| = |\{x\}| + |A \cup N(A) - \{x\}| \geqslant 1 + d_G(x) \geqslant 1 + \delta(G)$. □

Two vertices x_1x_2 and y_1y_2 in $G_1 \Box G_2$ are said to be *parallel with* G_1 (resp. G_2) if $x_2 = y_2$ (resp. $x_1 = y_1$). Two vertices are said to be *parallel* if they are parallel with either G_1 or G_2.

Theorem 1 *For every two connected graphs $G_1 \neq K_1$ and $G_2 \neq K_1$,*

$$\kappa(G_1 \Box G_2) = \min\{\kappa_1 v_2, \kappa_2 v_1, \delta_1 + \delta_2\}$$

Proof. Let $G = G_1 \Box G_2$. Clearly, $\kappa(G) \leqslant \delta(G) = \delta_1 + \delta_2$. If G_2 is not a complete graph, let S_0 be a minimum cut-set of G_2, then $V_1 \times S_0$ is a cut-set of G, which implies $\kappa(G) \leqslant \kappa_2 v_1$; if G_2 is a complete graph, then $\kappa_2 = \delta_2$, therefore $\kappa(G) \leqslant \delta_1 + \delta_2 \leqslant \delta_2(\delta_1+1) \leqslant \kappa_2 v_1$. By symmetry, we have $\kappa(G) \leqslant \kappa_1 v_2$. So it remains to prove that $\kappa(G_1 \Box G_2) \geqslant \min\{\kappa_1 v_2, \kappa_2 v_1, \delta_1 + \delta_2\}$. Let S be a minimum cut-set in G.

Case 1: There exist no pair of parallel vertices in distinct components of $G-S$. Take a component C of $G-S$, let $A = \{x \in V_1 | xy \in V(C) \text{ for some } y\} \subseteq V_1$ and $B = \{y \in V_2 | xy \in V(C) \text{ for some } x\} \subseteq V_2$. Obviously, $|A| \geqslant 1$. Because vertices in other components of $G - S$ must not be parallel with any vertex in C, we have $|A| \leqslant v_1 - 1$. Similarly, $1 \leqslant |B| \leqslant v_2 - 1$. Thus, $(V_1 - A) \times B$ and $A \times (V_2 - B)$ must be in S because vertices in them are parallel with some vertex in C and not in C. Let $a = |A|, b = |B|$, by Lemma 1, we have

$$\begin{aligned} \kappa(G) = |S| &\geqslant |(V_1 - A) \times B| + |A \times (V_2 - B)| \\ &= (v_1 - a)b + a(v_2 - b) \\ &\geqslant v_1 + v_2 - 2 \\ &\geqslant \delta_1 + \delta_2. \end{aligned} \tag{1}$$

Case 2: There exist a pair of parallel vertices in distinct components of $G-S$. Without loss of generality, suppose that u and w are parallel vertices with G_2 and are in components C_1 and C_2 of $G - S$, respectively. Let $V_1 = \{x_1, x_2, \cdots, x_{v_1}\}$ and $S_i = S \cap (\{x_i\} \times V_2)$. Without loss of generality, assume $u, w \in \{x_1\} \times V_2$. Note that if $\{x_i\} \times V_2$ contains vertices of distinct components of $G-S$, then $|S_i| \geqslant \kappa_2$. If for each $x_i \in V_1$, $\{x_i\} \times V_2$ contains vertices in both C_1 and C_2, then

$$\kappa(G) = |S| = \sum_{i=1}^{v_1} |S_i| \geqslant v_1 \kappa_2. \tag{2}$$

So we may suppose that there exist $x \in V(G_1)$ such that $\{x\} \Box G_2$ does not contain vertices of C_1. Split the vertex-set of G_1 into two subsets X_1 and X_2, X_1 containing the vertices x such that $xy \notin C_1$ for all $y \in V(G_2)$ and X_2 all the other vertices of G_1. Since G_1 is connected there is an edge e with one end-vertex in X_1 and the other in X_2. We may assume the two end-vertices of e are x_k and x_1. Let $H = \{x_1\} \Box G_2$. Let $D = C_1 \cap V(H)$ and D' be the neighbors of D in $\{x_k\} \Box G_2$. It is clear that both D' and $N_H(D)$ must be in S. By Lemma 2, $|D'| + |N_H(D)| = |D| + |N_H(D)| \geqslant \delta_2 + 1$. Besides x_k, the vertex x_1 has at least $\delta_1 - 1$ neighbors in G_1. For each $x_i \in N_{G_1}(x_1) - \{x_k\}$, $S_i \neq \emptyset$, otherwise u and w will be connected through $\{x_i\} \Box G_2$, a contraction. Therefore,

$$\begin{aligned} \kappa(G) = |S| &\geqslant (|D'| + |N_H(D)|) + \sum_{x_i \in N_{G_1}(x_1) - \{x_k\}} |S_i| \\ &\geqslant (\delta_2 + 1) + (\delta_1 - 1) \\ &= \delta_1 + \delta_2. \end{aligned} \tag{3}$$

In all cases, we prove $\kappa(G) \geqslant \min\{\kappa_1 v_2, \kappa_2 v_1, \delta_1 + \delta_2\}$. The proof of the theorem is complete. $\Box$

From Theorem 1, we obtain the following corollary, a necessary and sufficient condition for the Cartesian product graph to be maximally connected, immediately.

Corollary 1 *Let G_1 and G_2 be two connected graphs, then $G_1 \Box G_2$ is maximally connected if and only if* $\min\{\kappa_1 v_2, \kappa_2 v_1\} \geqslant \delta_1 + \delta_2$.

3 Super-connectivity

We say a connected graph G to have the *property* $\mathscr{P}$ if there is a subset $A \subset V(G)$ with $|A| \geqslant 2$ and $|A \cup N(A)| = \delta(G) + 1$ such that $G - N(A)$ is disconnected. It follows from the definition that A is a complete subgraph of G and that any vertex from A is adjacent to every vertex from $N(A)$. So $|A| \geqslant 2$ can be replaced by $|A| = 2$ in the definition without changing the meaning.

Lemma 3 *Any maximally connected graph has no property $\mathscr{P}$.*

Proof. Suppose to the contrary that there is a maximally connected graph G with the property $\mathscr{P}$. Then there is a subset $A \subset V(G)$ with $|A| \geqslant 2$ and $|A \cup N(A)| = \delta(G) + 1$ such that $G - N(A)$ is disconnected. Thus, $1 + \delta(G) = |A \cup N(A)| \geqslant 2 + \kappa(G) = 2 + \delta(G)$, a contradiction. □

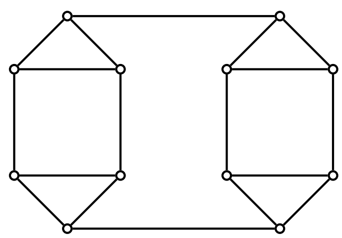

Figure 1: A non-maximally connected graph without the property $\mathscr{P}$

The graph shown in Figure 1 shows that the reverse of Lemma 3 is not always true. The importance of the property $\mathscr{P}$ in the study of super-connectivity of Cartesian graphs is indicated in the following lemma.

Lemma 4 *Let G_1 and G_2 be two connected graphs, G_1 has the property $\mathscr{P}$ and $\delta_2 = 1$. Then $G_1 \Box G_2$ is not super-connected.*

Proof. Suppose to the contrary that $G_1 \Box G_2$ is super-connected. Then $G_1 \Box G_2$ is maximally connected, i.e., $\kappa(G_1 \Box G_2) = \delta_1 + \delta_2$. Since G_1 has the property $\mathscr{P}$, there is a subset $A \subset V_1$ with $|A| \geqslant 2$ and $|A \cup N(A)| = \delta_1 + 1$ such that $G_1 - N(A)$ is disconnected. Let x be a vertex of degree one in G_2 and y be the only neighbor of x. Then $S = (N(A) \times \{x\}) \cup (A \times \{y\})$ is a cut-set of $G = G_1 \Box G_2$ and $|S| = |N(A) \cup A| = \delta_1 + 1 = \delta_1 + \delta_2 = \kappa(G_1 \Box G_2)$, which implies that S is a minimum cut-set. If A and $N(A)$ both have at least two vertices then the set S is not a neighborhood of any vertex. $|A| \geqslant 2$ by definition. If $|N(A)| = 1$, then S is a neighborhood of a vertex if and only if $N(N(A)) = A$, that is, G_1 is a complete graph. Since complete graphs do not have property $\mathscr{P}$, $|N(A)| \geqslant 2$. So there is no isolated vertex in $G_1 \Box G_2 - S$, a contradiction. This completes the proof. $\Box$

Another class of graphs, which will be called the locally complete graphs, also gives rise to non-super-connected Cartesian product graphs. A connected non-complete graph with $\delta \geqslant 2$ is said to be *locally complete* if it has a block isomorphic to $K_{\delta+1}$. By the definition, a connected locally complete graph has connectivity $\kappa = 1$ and has the property $\mathscr{P}$. For a connected graph, the relations among the property $\mathscr{P}$, locally complete and maximally connected are shown on Figure 2.

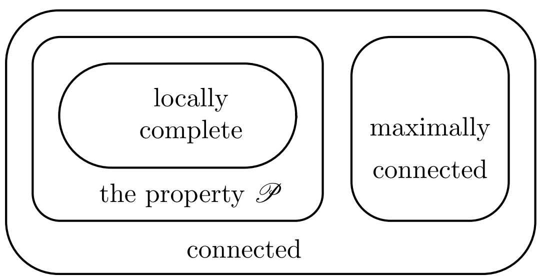

Figure 2: Relations among the property $\mathscr{P}$, locally complete and maximally connected

Lemma 5 *Let G_1 and G_2 be two connected locally complete graphs, then $G_1 \Box G_2$ is not super-connected.*

Proof. Suppose to the contrary that $G_1 \Box G_2$ is super-connected. Then $G_1 \Box G_2$ is maximally connected, i.e., $\kappa(G_1 \Box G_2) = \delta_1 + \delta_2$. By the hypothesis, let $\{x_0, x_1, \cdots, x_{\delta_1}\}$ and $\{y_0, y_1, \cdots, y_{\delta_2}\}$ be the vertex-set of a complete block of G_1 and G_2, respectively. And assume that x_0 is a cut-vertex of G_1 and that y_0 is a cut-vertex of G_2. Then $S = \{x_1 y_0, x_2 y_0, \cdots, x_{\delta_1} y_0\} \cup$

$\{x_0y_1, x_0y_2, \cdots, x_0y_{\delta_2}\}$ is a cut-set of $G_1\Box G_2$ and $|S| = \delta_1 + \delta_2$. But there are no isolated vertices in $G_1\Box G_2 - S$, a contradiction. □

Lemma 6 *Let G be a connected graph with $\kappa = 1$ and $\delta \geqslant 2$, $D \subset V(G)$ with $|D \cup N(D)| = \delta + 1$ and $|D| \geqslant 2$. Then any element of D and at least one element of $V(G) - D - N(D)$ are not cut-vertices of G.*

Proof. We first note that $N(x) = D \cup N(D) - \{x\}$ for each vertex $x \in D$ since $|D \cup N(D) - \{x\}| = |D \cup N(D)| - 1 = \delta$. This fact means that each vertex in D is adjacent to all vertices in $N(D)$. As $|D| \geqslant 2$, the neighbors of x are still connected in $G - x$ for any $x \in D$, which implies any vertex in D is not a cut-vertex of G.

It is clear that $N(D) \neq \emptyset$ and $V(G) - D - N(D) \neq \emptyset$ since $\kappa = 1$ and $\delta \geqslant 2$. If $y \in V(G) - D - N(D)$ is a cut-vertex of G, then at least one of connected components of $G - y$ contains no vertices in $D \cup N(D)$ since any two vertices of $D \cup N(D)$ is connected in $G - y$. Choose such a cut-vertex $y \in V(G) - D - N(D)$ such that the number of vertices of the smallest component C of $G - y$ which contains no vertices in $D \cup N(D)$ is as small as possible. Let y' be a neighbor of y in C. If y' is a cut-vertex, then $G - y'$ has a component $C' \subset C$ as $y' \notin C'$, which contradicts to our choice of y. So y' is not a cut-vertex. □

Lemma 7 *Let G_1 and G_2 be two connected graphs, $\kappa_2 = 1$, $\delta_2 \geqslant 2$. Let $S \subset V_1 \times V_2$, S has no vertices parallel with G_2 and $|S| < v_1$. Then $G_1\Box G_2 - S$ is connected.*

Proof. Let $V_1 = \{x_1, x_2, \cdots, x_n\}$ and $S_i = S \cap (\{x_i\} \times V_2)$, by the hypothesis, $|S_i| \leqslant 1$. Without loss of generality, assume that $|S_i| = 1$ for $1 \leqslant i \leqslant t = |S|$. We need the following simple fact:

Fact 1 *If x_j and x_h are adjacent, then for each vertex v in $\{x_j\}\Box G_2 - S_j$ there exist a vertex w in $\{x_h\}\Box G_2 - S_h$ such that v and w are connected in $G[x_j, x_h]\Box G_2 - S_j - S_h$.*

Proof of Fact 1. Because $\kappa_2 = 1$ and $\delta_2 \geqslant 2$, $v_2 \geqslant 5$, $\{x_i\}\Box G_2 - S_i$ is either connected with at least 4 vertices, or disconnected with each component having at least two vertices. If the neighbor v' of v in $\{x_h\}\Box G_2$ does not belong to S_h, $P = vv'$ is the desired path and $w = v'$. If $v' \in S_h$, because v is always in a component of at least two vertices in $\{x_j\}\Box G_2 - S_j$, let w' be a neighbor of v in the component, and w be the neighbor of w' in $\{x_h\}\Box G_2$. So $P = vw'w$ is a vw-path. □

Come back to the proof of the lemma. Because $t = |S| < v_1$, there exist $x_k (k > t)$ such that $S_k = \emptyset$, namely $\{x_k\}\Box G_2 - S_k$ is connected. For each vertex u in $\{x_i\}\Box G_2 - S_i$ for $i \neq k$, there is a path from x_i to x_k, following that path, u can be connected to some vertex in $\{x_k\}\Box G_2 - S_k$ in $G_1\Box G_2 - S$ by Fact 1. $\Box$

It is ready to present our second major result.

Theorem 2 *Let $G_1 \neq K_1$ and $G_2 \neq K_1$ be two connected graphs, then $G_1\Box G_2$ is super-connected if and only if one of the following conditions is satisfied:*

i) $G_1\Box G_2$ is isomorphic to $K_2\Box K_2$ or $K_2\Box K_3$,

ii) $\min\{v_1\kappa_2, v_2\kappa_1\} > \delta_1 + \delta_2$ but none of following three situation: $\delta_1 = 1$, G_2 has the property $\mathscr{P}$; $\delta_2 = 1$, G_1 has the property $\mathscr{P}$; both G_1 and G_2 are locally complete.

Proof. Let $G = G_1\Box G_2$. We prove the necessity first. Assume G is super-connected, then it is maximally connected, by Corollary 1, $\kappa_1 v_2 \geqslant \delta_1 + \delta_2$ and $\kappa_2 v_1 \geqslant \delta_1 + \delta_2$. If $\kappa_1 v_2 = \delta_1 + \delta_2$, then G_1 must be a complete graph. Otherwise, let S_1 be a minimum cut-set of G_1, then $S_1 \times V_2$ is a minimum cut-set of G without isolated vertices, a contradiction. So G_1 is a complete graph, we have $\delta_1 + \delta_2 = \kappa_1 v_2 = \delta_1 v_2 \geqslant \delta_1(\delta_2 + 1)$. From this inequality, we have $\delta_1 = 1$ and $v_2 = \delta_2 + 1$, which means $G_1 = K_2$ and G_2 is also a complete graph. If $G_2 = K_n$ with $n \geqslant 4$, let R be a set of two adjacent vertices of $\{x_1\}\Box G_2$, where $x_1 \in V_1$. Then $N_G(R)$ is a minimum cut-set without leaving isolated vertices, a contradiction. So G_2 must be K_2 or K_3. Thus the condition i) is satisfied. If $\kappa_2 v_a = \delta_1 + \delta_2$, the same argument gives that G_1 and G_2 satisfy the condition i).

Now assume $\min\{v_1\kappa_2, v_2\kappa_1\} > \delta_1 + \delta_2$. If $\delta_1 = 1$ and G_2 has the property $\mathscr{P}$, or $\delta_2 = 1$ and G_1 has the property $\mathscr{P}$, then $G_1\Box G_2$ is not super-connected by Lemma 4. If both G_1 and G_2 are locally complete then $G_1\Box G_2$ is not super-connected by Lemma 5. Thus, the condition ii) is satisfied.

Next, we will show either of the two conditions is sufficient for G to be super-connected. Clearly, the condition i) is sufficient since both $K_2\Box K_2$ and $K_2\Box K_3$ are super-connected. If the condition ii) holds, then G is maximally connected by Corollary 1. Let S be a minimum cut-set, then $|S| = \delta_1 + \delta_2$. We only need to prove that $G - S$ contains isolated vertices. Following the notations and the argument of Theorem 1, we consider two cases.

Case 1: There exist no pair of parallel vertices in distinct components of $G - S$. In this case, all the equalities in the inequality (1) in the proof of Theorem 1 hold since $|S| = \delta_1 + \delta_2$. So $|S| = |(V_1 - A) \times B| + |A \times (V_2 - B)|$. And both G_1 and G_2 are complete graphs by $v_1 + v_2 - 2 = \delta_1 + \delta_2$. But neither of them is K_2, otherwise if, for example, $G_1 = K_2$, then $v_2\kappa_1 = v_2 \cdot 1 = 1 + \delta_2 = \delta_1 + \delta_2$, which contradicts the hypothesis. So $v_1 \neq 2$ and $v_2 \neq 2$. Therefore, by $(v_1 - a)b + a(v_2 - b) = v_1 + v_2 - 2$ and Lemma 1, either $a = b = 1$ or $a = v_1 - 1$ and $b = v_2 - 1$, in both situations, there is an isolated vertex in $G - S$.

Case 2: There exist some pair of parallel vertices in distinct components of $G - S$. Assume that u and w in $\{x_1\} \times V_2$ are parallel with G_2 and belong to components C_1 and C_2, respectively. If for each $x_i \in V_1$, $\{x_i\} \times V_2$ contains vertices of both C_1 and C_2, then $|S| \geqslant v_1\kappa_2 > \delta_1 + \delta_2$ by the inequality (2), a contradiction.

Thus, there is some $x \neq x_1$ such that $\{x\} \times V_2$ contains no vertices of C_1. Since $|S| = \delta_1 + \delta_2$, all the equalities in the inequality (3) hold. So

$$|S| = (|D'| + |N_H(D)|) + \sum_{x_i \in N_{G_1}(x_1) - \{x_k\}} |S_i|.$$

Furthermore, $d_{G_1}(x_1) = \delta_1$ and $|D'| + |N_H(D)| = \delta_2 + 1$.

If $\delta_1 = 1$, by the hypothesis, G_2 does not have the property $\mathscr{P}$, so $H = \{x_1\}\Box G_2$ does not have the property $\mathscr{P}$. Note that $|D| + |N_H(D)| = |D'| + |N_H(D)| = |S| = \delta_2 + 1$, therefore $|D| = 1$, so D is an isolated vertex in $G - S$.

Now assume $\delta_1 \geqslant 2$. We proceed by considering three subcases. The outline of each subcase is as follows. We first prove $|D| = 1$, then prove that $(G_1 - x_1)\Box G_2 - S$ is connected. If so, let $D = \{u\}$, and one of its neighbors belongs to D' and hence to S. So each vertex of $\{x_1\}\Box G_2 - S - D$ has at least one neighbor in $(G_1 - x_1)\Box G_2 - S$ and this makes $G - S - D$ connected. Therefore $D = \{u\}$ must be the other component of $G - S$, which will complete the proof.

It remains for us to show that $|D| = 1$ and $(G_1 - x_1)\Box G_2 - S$ is connected. We mention some more facts which are obvious but used often in the rest of the proof.

Fact 2 *Let G_1 and G_2 be two connected graphs with $\min\{v_1\kappa_2, v_2\kappa_1\} > \delta_1 + \delta_2$. If $\kappa_1 = 1$, then $v_2 > \delta_1 + \delta_2$ and G_2 is not a complete graph. If $\kappa_2 = 1$, then $v_1 > \delta_1 + \delta_2$.*

Subcase A: $\delta_2 = 1$. So $|D| = |N_H(D)| = 1$. Let $K \subseteq V_1$ such that if $x_i \in K$, then $\{x_i\}\Box G_2$ contains vertices of distinct components of $G - S$. Obvious, $x_1 \in K$ and $K \subseteq \{x_1\} \cup N_{G_1}(x_1)$. Because $\delta_2 = 1$, $V_1 - \{x_1\} - N_{G_1}(x_1) \neq \emptyset$ by Fact 2. Note that each vertex in K is not adjacent with those in $V_1 - \{x_1\} - N_{G_1}(x_1)$. Thus $N_{G_1}(K) = \{x_1\} \cup N_{G_1}(x_1) - K$ is a cut-set of G_1 and $|K \cup N_{G_1}(K)| = |\{x_1\} \cup N_{G_1}(x_1)| = \delta_1 + 1$. Because G_1 does not have the property $\mathscr{P}$, $|K| = 1$, namely $K = \{x_1\}$. So for each $x_i \neq x_1$, the vertices of $\{x_i\}\Box G_2 - S$ are in the same component of $G - S$. If $\kappa_1 \geqslant 2$, then $G_1 - x_1$ is connected, hence $(G_1 - x_1)\Box G_2 - S$ is connected. If $\kappa_1 = 1$, then $v_2 > \delta_1 + \delta_2$ by Fact 2, so there exists $y \in V_2$ such that

$G_1\Box\{y\}$ contains no vertices in S, which implies that $(G_1 - x_1)\Box G_2 - S$ connected. In either case, $(G_1 - x_1)\Box G_2 - S$ is connected.

Subcase B: $\kappa_2 \geqslant 2$. First, we deduce $|D| = 1$. Suppose to the contrary that $|D| \geqslant 2$. Then $|N_H(D)| < \delta_2$ and so there is no isolated vertex in $H - S_1$. Because $\kappa_2 \geqslant 2$, but for any $x_i \in N_{G_1}(x_1) - \{x_k\}$, $|S_i| = 1$, we have $\{x_i\}\Box G_2 - S$ is connected. Thus all distinct components of $H - S$ will be connected through $\{x_i\}\Box G_2 - S$, a contradiction. So $|D| = 1$, $|S_{t_1}| = |D'| = |D| = 1$, and $\{x_k\}\Box G_2 - S$ is also connected. Therefore, for any $x_i \in V_1$ except x_1, $\{x_i\}\Box G_2 - S$ is connected. As in **Subcase A**, if $\kappa_1 \geqslant 2$, then $G_1 - x_1$ is connected. If $\kappa_1 = 1$, there exists $y \in V_2$ such that $G_1\Box\{y\}$ contains no vertices in S. So $(G_1 - x_1)\Box G_2 - S$ is connected.

Subcase C: $\kappa_2 = 1$ and $\delta_2 \geqslant 2$. As before, first prove $|D| = 1$. Suppose to the contrary that $|D| \geqslant 2$. Let $D_0 = \{y \in V_2 | x_1 y \in D\}$. By applying G_2 to Lemma 6, any vertex of D_0 is not a cut-vertex of G_2 and $V_2 - D_0 - N_{G_2}(D_0)$ contains at least one non-cut-vertex. Consider each $x_i \in N_{G_1}(x_1) - \{x_k\}$. Because $|S_i| = 1$, the element of S_i must be a cut-vertex of $\{x_i\}\Box G_2$, otherwise $H - S$ would be connected through $\{x_i\}\Box G_2 - S_i$. So S consists of $N(D)$, D' and $\delta_1 - 1$ cut-vertices (of $\{x_i\}\Box G_2$). Let $u = x_1 y_1$, then $G_1\Box\{y_1\}$ contains exactly one vertex of S, that is $x_k y_1$. If $G_1 - x_k$ is connected, because $\kappa_2 = 1$, let x_j be a vertex besides x_1 and its neighbors in V_1(x_j exists by Fact 2). If $G_1 - x_k$ is not connected but x_1 lies in a component that there exist a vertex besides itself and its neighbors, let x_j denote that vertex. In either case, there is an (x_1, x_j)-path in $G_1 - x_k$ and $\{x_j\}\Box G_2$ contains no vertices of S. Furthermore there exist a non-cut-vertex z in $V_2 - D_0 - N(D_0)$, thus $G_1\Box\{z\}$ contains no vertices of S. Then $u = x_1 y_1$ is connected with $x_1 z$ through $(G_1 - x_k)\Box\{y_1\}$, $\{x_j\}\Box G_2$ and $G_1\Box\{z\}$, as illustrated in Figure 3, a contradiction.

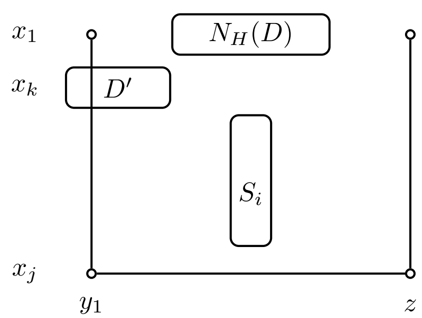

Figure 3: $x_1 y_1 \overset{(G_1 - x_k)\Box\{y_1\}}{\longrightarrow} x_j y_1 \overset{\{x_j\}\Box G_2}{\longrightarrow} x_j z \overset{G_1\Box\{z\}}{\longrightarrow} x_1 z$

Now there is one condition we have not yet considered: $G_1 - x_k$ is not connected and x_1 lies in a component that consist of only itself and its neighbors, which means that G_1 is locally complete. Then by hypothesis G_2 must not be locally complete, which imply $|N(D)| \geqslant 2$. Let $x_2 \in N_{G_1}(x_1) - \{x_k\}$, $x_j \in N_{G_1}(x_k) - \{x_1\} - N_{G_1}(x_1)$, $y_1 \in D_0$, $z \in V_2 - D_0 - N_{G_2}(D_0)$. And choose $y_2 \in N_{G_2}(D_0)$ such that $x_2y_2 \notin S_2$ (y_2 exists because $|S_2| = 1$ and $|N(D)| \geqslant 2$). Then x_1y_1 and x_1z is connected in $G - S$ as follows (see Figure 4), a contradiction.

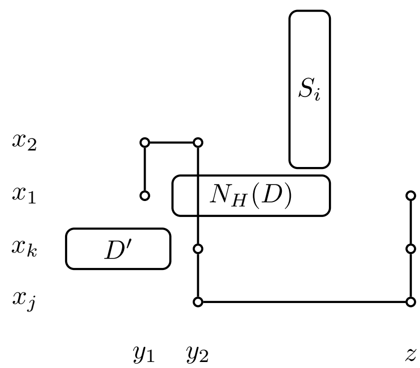

Figure 4: $x_1y_1 \to x_2y_1 \to x_2y_2 \to x_ky_2 \to x_jy_2 \overset{\{x_j\}\Box G_2}{\longrightarrow} x_jz \to x_kz \to x_1z$

So $|D| = 1$, next we will show $(G_1 - x_1)\Box G_2 - S$ is connected. If $G_1 - x_1$ is connected, just apply $G_1 - x_1$ and G_2 to Lemma 7. If $G_1 - x_1$ is disconnected, $\kappa_1 = 1$ and $\delta_1 \geqslant 2$, then the number of neighbors of x_1 in each component F is strictly less than δ_1, thus each component contains vertices besides those neighbors of x_1. By applying F and G_2 to Lemma 7, we know that $F\Box G_2 - S$ is connected. And as $\kappa_1 = 1$, $v_2 > \delta_1 + \delta_2$, there exists a $y \in V_2$ such that $G_1\Box\{y\}$ contains no vertices of S, and connects each $F\Box G_2 - S$.

Thus in all cases, $G - S$ isolates a vertex, this completes the proof. □

The following result proved in [1] will be a direct consequence of Theorem 2.

Corollary 2 *[1] Assume $G_1\Box G_2 \ncong K_2\Box K_n$ for $n \geqslant 4$. If G_i is regular and maximally connected for $i = 1, 2$, then $G_1\Box G_2$ is super-connected.*

Proof. Because both G_1 and G_2 are maximally connected, $v_1\kappa_2 = v_1\delta_2 \geqslant (\delta_1 + 1)\delta_2 \geqslant \delta_1 + \delta_2$. By the same reason, $v_2\kappa_1 \geqslant \delta_1 + \delta_2$. If $v_1\kappa_2 = \delta_1 + \delta_2$, because G_2 is maximally connected, $\delta_1 + \delta_2 = v_1\kappa_2 = v_1\delta_2 \geqslant (\delta_1 + 1)\delta_2 = \delta_1\delta_2 + \delta_2$. So $\delta_2 = 1$ and $v_1 = \delta_1 + 1$, which means that $G_2 = K_2$ (because G_2

is regular) and G_1 is a complete graph, hence $G_1 \Box G_2$ must be isomorphic to $K_2 \Box K_n$. By the hypothesis, $n = 2, 3$. Thus the condition i) of Theorem 2 is satisfied. If $v_2\kappa_1 = \delta_1 + \delta_2$, the same argument shows the condition i) of Theorem 2 is also satisfied. Now assume that $\min\{v_1\kappa_2, v_2\kappa_1\} > \delta_1 + \delta_2$. By Lemma 3, a maximally connected graph is neither locally complete nor have the the property $\mathscr{P}$ (see Figure 2). Thus the condition ii) of Theorem 2 is always satisfied. This completes the proof. □

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