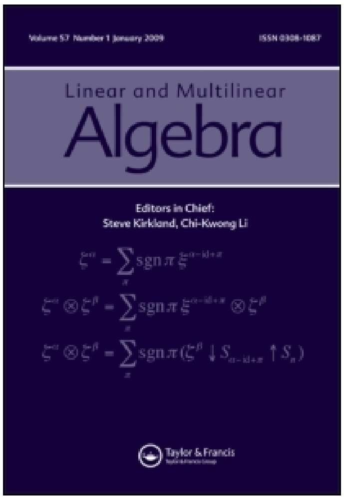

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A note on unimodular congruence of the Laplacian matrix of a graph

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Let G be a simple connected graph and $L(G)$ be its Laplacian matrix. In this note, we prove that $L(G)$ is congruent by a unimodular matrix to its Smith normal form if and only if G is a tree.

Keywords: graph; Laplacian matrix; Smith normal form; unimodular congruence

AMS Subject Classifications: 15A18; 05C50

1. Introduction

Let $G=(V,E)$ be a simple connected graph with vertex set $V=\{v_1,v_2,\dots,v_n\}$ and edge set $E=\{e_1,e_2,\dots,e_m\}$. For each edge $e_k=\{v_i,v_j\}$ choose one of v_i and v_j to be the positive end of e_k and the other to be the negative end. We call this an *orientation* of G. The vertex-edge incidence matrix $C=C(G)=(c_{ij})$ afforded by a fixed but arbitrary orientation of G is defined by

$$c_{ij}=\begin{cases}+1, & \text{if } v_i \text{ is the positive end of } e_j,\\ -1, & \text{if } v_i \text{ is the negative end of } e_j,\\ 0, & \text{otherwise.}\end{cases}$$

While C depends on the orientation of G, the Laplacian matrix $L(G)=CC^t$ does not, and $L(G)=D(G)-A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix of G.

If G is a graph with n vertices, denote by $\Delta_k(G)$ the greatest common divisor of the determinants of all the k-by-k submatrices of $L(G)$. It is convenient to define $\Delta_0(G)=1$. The invariant factors of $L(G)$ are $s_k(G)=\Delta_k(G)/\Delta_{k-1}(G)$, $1\le k\le n$. The *Smith normal form* of $L(G)$ is the n-square diagonal matrix $S(G)$ whose (k,k) entry is $s_k(G)$. For the Laplacian matrix of a graph G, it is easy to see that $s_1(G)=1$ and $s_n(G)=0$. Thus, the Smith normal form of $L(G)$ is

$$S(G)=\mathrm{diag}(1,s_2(G),\dots,s_{n-1}(G),0).$$

An integral matrix P is unimodular if P^{-1} is also integral, i.e. if $\det P=\pm 1$. Let A, B be two integral matrices. We say that A is unimodularly equivalent to B if there exist two

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unimodular matrices P and Q such that $B = PAQ$. It follows from [5, Theorem 2.9] that for every graph G, there must exist two unimodular matrices P and Q such that $PL(G)Q = S(G)$. The Smith normal form of $L(G)$ is complicated since it has a close connection with the critical group of G (for details, see [1]). But the Smith normal form of C^tC is of a simple form; in [3] Merris proved that, if a graph has n vertices and m edges, then the Smith normal form of C^tC is always $I_{n-2} \oplus (n) \oplus 0_{m-n+1}$.

Much more restricted than unimodular equivalence is unimodular congruence. Two integral matrices A and B are unimodularly congruent if there is a unimodular matrix P such that $B = PAP^t$, where P^t is the transpose of P. Let G_1 and G_2 be graphs with n vertices. In [6,7] Watkins proved that $L(G_1)$ is unimodularly congruent to $L(G_2)$ if and only if G_1 and G_2 are cycle isomophic. In [4], Merris proved that, for a graph G with n vertices and m edges, the matrix C^tC is unimodularly congruent to $(2I_{n-1} + X_{n-1} + X_{n-1}^t) \oplus 0_{m-n+1}$, where X_{n-1} is the $(0, 1)$ matrix with order $n-1$, whose (i, j)-entry is 1 if $j = i + 1$, and 0 otherwise.

In this note, we will prove that the Laplacian matrix of a simple connected graph G is unimodularly congruent to its Smith normal form if and only if G is a tree.

2. Lemmas and main results

In order to obtain the results, we need the following Lemmas.

LEMMA 2.1 [6, Corollary 4] *Let G be a graph with n vertices. Then G is a tree if and only if $L(G)$ is unimodularly congruent to $I_{n-1} \oplus [0]$.*

LEMMA 2.2 *Let G be a connected graph with n vertices. If $L(G)$ is unimodularly congruent to its Smith normal form, then G has a cut-edge.*

Proof By the definition of unimodular congruence, if $L(G)$ is unimodularly congruent to the Smith normal form $S(G)$, then there exists a unimodular matrix $P = (p_{ij})$ of order n, such that

$$PL(G)P^t = S(G). \tag{1}$$

Consider the $(1, 1)$-entry of both sides of (1), then we have

$$\sum_{v_i \sim v_j} (p_{1i} - p_{1j})^2 = 1, \tag{2}$$

where $v_i \sim v_j$ means that v_i and v_j in G are adjacent. Since P is an integral matrix, it follows that exactly one term on the left side of (2) is non-zero, and it is equal to 1. If we assume that the term $(p_{1s} - p_{1t})^2$ corresponding to edge $e_k = \{v_s, v_t\}$ is equal to 1, then we can say that e_k is a cut-edge of G. Otherwise, the edge e_k must belong to some cycle Q in G. Let $v_s - v_{i_1} - \cdots - v_{i_l} - v_t - v_s$ denote the cycle Q. It follows from the fact that every term on the left side of (2) corresponding to the edge within the chain $v_s - v_{i_1} - \cdots - v_{i_l} - v_t$ is equal to 0, and then $p_{1s} = p_{1i_1} = \cdots = p_{1i_l} = p_{1t}$. However, it contradicts that $(p_{1s} - p_{1t})^2 = 1$. So $e_k = \{v_s, v_t\}$ does not belong to any cycle in G. It follows that e_k is a cut-edge of G. ∎

Suppose $G = (V, E)$ is a simple graph. If $e = \{v_i, v_j\} \in E$ does not lie on a cycle of length 3, we may retract along e to obtain a new simple graph $G' = (V', E')$, where

$E' = E \backslash \{e\}$, and V' is derived from V by identifying vertices v_i and v_j. We call G' a retraction of G along e.

LEMMA 2.3 *Suppose that G is a graph and the Lapacian matrix L(G) is unimodularly congruent to its Smith normal form S(G). Then L(G) is unimodularly congruent to* $\begin{pmatrix} 1 & 0 \\ 0 & L(G') \end{pmatrix}$, *where G′ is the retraction of G along a cut-edge which is admitted by Lemma 2.2, and L(G′) is the Laplacian matrix of graph G′.*

Proof It follows from Lemma 2.2 that the graph G has at least one cut-edge. Suppose that e_k is one of the cut edges in G. Then, we denote the retraction of G along e_k by G' and the two components of $G - e_k$ by G_1 and G_2. Assume that the vertex set of G has been ordered as $V(G) = \{v_1, v_2, \ldots, v_n\}$, so that $V(G_1) = \{v_1, v_{k+1}, \ldots, v_n\}$, $V(G_2) = \{v_2, \ldots, v_k\}$, $e_k = \{v_1, v_2\}$, the set of neighbours of v_1 in G_1 is $\{v_{n-d_1+2}, \ldots, v_{n-1}, v_n\}$, and the set of neighbours of v_2 in G_2 is $\{v_{k-d_2+2}, \ldots, v_{k-1}, v_k\}$. Let $v_1{}'$ denote the new vertex of G' obtained from G by identifying v_1 and v_2. Then the vertex set of G' is $V(G') = \{v_1{}', v_3, \ldots, v_n\}$. If $L_0 = (0) \oplus L(G')$, then with respect to the ordering of vertices, we have $L(G) = L_0 + F$, where $F = (F_{ij})$ is a 3×3 block partitioned matrix: $F_{11} = \begin{pmatrix} d_1 & -1 \\ -1 & 2-d_1 \end{pmatrix}$, where d_1 is the degree (in G) of v_1; F_{13} is the 2-by-$(d_1 - 1)$ matrix whose first row consists entirely of -1's and whose second row is all $+1$'s; $F_{31} = F_{13}^t$, and the other blocks are appropriately sized zero matrices. (In particular, F_{22} and F_{33} are square blocks, whereas F_{23} is $(n - d_1 - 1)$-by-$(d_1 - 1)$.)

Let $T = (t_{ij})$ and $W = (w_{ij})$ be matrices of order n, where

$$t_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = 1,\ k+1 \le j \le n, \\ 1 & \text{if } i = 2,\ j = 1, \\ 0 & \text{otherwise;} \end{cases} \qquad w_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = 2,\ k+1 \le j \le n, \\ -1 & \text{if } i = 1,\ k+1 \le j \le n, \\ -1 & \text{if } i = 2,\ j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $TW = WT = I$, where I is the identity matrix of order n. Thus T is a unimodular matrix. A direct calculation will show that $TL(G)T^t = \begin{pmatrix} 1 & 0 \\ 0 & L(G') \end{pmatrix}$.

LEMMA 2.4 *Let G be a simple graph. Then the Laplacian matrix L(G) is unimodularly congruent to its Smith normal form S(G) if and only if L(G′) is unimodularly congruent to the Smith normal form S(G′).*

Proof If $L(G)$ is unimodularly congruent to its Smith normal form $S(G)$, then there exists a unimodular matrix $P = (p_{ij})$ such that $PL(G)P^t = S(G)$. By Lemma 2.2, there is a cut-edge in G. As in the proof of Lemma 2.3, we order $V(G)$ so that v_1 and v_2 are the two end-points of one of the cut edges of G, then by the proof of Lemma 2.2 we have $|p_{11} - p_{12}| = 1$. It follows from Lemma 2.3 that

$$L(G) = T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & L(G') \end{pmatrix} (T^{-1})^t.$$

Thus

$$(PT^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & L(G') \end{pmatrix} (PT^{-1})^t = S(G). \tag{3}$$

Suppose $PT^{-1}=R=(r_{ij})$, then R is unimodular and

$$|r_{11}| = |(p_{11}, p_{12}, \ldots, p_{1n})(1-1, 0, \ldots, 0)^t| = |p_{11} - p_{12}| = 1.$$

Write R with the form $R=\left(\begin{smallmatrix} r_{11} & R_{21} \\ R_{12} & R_{22}\end{smallmatrix}\right)$. Then

$$\begin{pmatrix} r_{11} & R_{12} \\ R_{21} & R_{22}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & L(G')\end{pmatrix}\begin{pmatrix} r_{11} & R_{12} \\ R_{21} & R_{22}\end{pmatrix}^t = S(G). \tag{4}$$

We now compare both sides of (4) and this gives

$$\begin{cases} r_{11}^2 + R_{12}L(G')R_{12}^t = 1, \\ r_{11}R_{21}^t + R_{12}L(G')R_{22}^t = 0, \\ R_{21}R_{21}^t + R_{22}L(G')R_{22}^t = \operatorname{diag}(s_2, s_3, \ldots, s_{n-1}, 0). \end{cases} \tag{5}$$

Since $r_{11}^2=1$, it follows from the first equation of (5) that $R_{12}L(G')R_{12}^t=0$. Thus $(R_{12}C(G'))(R_{12}C(G'))^t=0$, where $C(G')$ is the vertex-edge incidence matrix of G'. So $R_{12}C(G')=0$, and it results that $R_{12}L(G')R_{22}^t=0$. Then by the second equation of (5), we have $R_{21}^t=0$, this implies that R_{22} is a unimodular matrix. Now the third equation of (5) implies that

$$R_{22}L(G')R_{22}^t = \operatorname{diag}(s_2, s_3, \ldots, s_{n-1}, 0).$$

Since $s_2|s_3|\cdots|s_{n-1}$, this means that $\operatorname{diag}(s_2, s_3, \ldots, s_{n-1}, 0)$ is the Smith normal form of $L(G')$. So $L(G')$ is unimodularly congruent to its Smith normal form.

Now assume that $L(G')$ is unimodularly congruent to its Smith normal form $S(G')$, then there exists a unimodular matrix $H=(\mathrm{h}_{ij})$ of order $n-1$ such that

$$HL(G')H^t = S(G').$$

Then

$$\begin{aligned} L(G) &= T^{-1}\begin{pmatrix} 1 & 0 \\ 0 & L(G')\end{pmatrix}(T^{-1})^t \\ &= T^{-1}\begin{pmatrix} 1 & 0 \\ 0 & H^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & S(G')\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (H^{-1})^t\end{pmatrix}(T^{-1})^t. \end{aligned}$$

Let $P=\left(\begin{smallmatrix} 1 & 0 \\ 0 & H\end{smallmatrix}\right)T$, then $PL(G)P^t=\left(\begin{smallmatrix} 1 & 0 \\ 0 & S(G')\end{smallmatrix}\right)$. Clearly, the matrix P is unimodular and it implies that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & S(G')\end{smallmatrix}\right)$ is the Smith normal form of $L(G)$. So $L(G)$ is unimodularly congruent to its Smith normal form. ■

THEOREM 2.5 *Let $G=(V,E)$ be a simple connected graph with n vertices. Then its Laplacian matrix $L(G)$ is unimodularly congruent to its Smith normal form if and only if G is a tree.*

Proof By the well-known matrix-tree theorem [2, Theorem 2.5.3], the greatest common divisor of the determinants of all the $(n-1)\times(n-1)$ submatrices of $L(G)$ is the number of spanning trees in G. So $L(G)$ has Smith normal form $I_{n-1}\oplus[0]$ if and only if G is a tree. Now suppose G is a tree, then it follows from Lemma 2.1 that $L(G)$ is unimodularly congruent to its Smith normal form.

We use induction on n for the reverse implication. The result is certainly valid for both $n=1$ and $n=2$ so that we may assume that $n \geq 3$. Since the Laplacian matrix $L(G)$ is unimodularly congruent to its Smith normal form, then it follows from Lemma 2.3 that G has at least one cut-edge e. Let G' be the retraction of G along the cut-edge e. From Lemma 2.4, $L(G')$ is unimodularly congruent to its Smith normal form $S(G')$. The graph G' has $n-1$ vertices and, by the induction hypothesis, must be a tree. Therefore, the graph G is a tree with n vertices. ■

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