

The p -Bondage Number of Trees

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Abstract Let p be a positive integer and $G = (V, E)$ be a simple graph. A p -dominating set of G is a subset $D \subseteq V$ such that every vertex not in D has at least p neighbors in D . The p -domination number of G is the minimum cardinality of a p -dominating set of G . The p -bondage number of a graph G with $\Delta(G) \geq p$ is the minimum cardinality among all sets of edges $B \subseteq E$ for which $\gamma_p(G - B) > \gamma_p(G)$. For any integer $p \geq 2$ and tree T with $\Delta(T) \geq p$, this paper shows that $1 \leq b_p(T) \leq \Delta(T) - p + 1$, and characterizes all trees achieving the equalities.

Keywords Domination · Bondage number · p -Domination · p -Bondage number · Trees

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1 Introduction

For notation and graph theory terminology we follow [5, 11, 12]. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood*, the *closed neighborhood* and the *degree* of a vertex $v \in V(G)$ are denoted by

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$N_G(v) = \{u \in V(G) | uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$ and $\deg_G(v) = |N_G(v)|$, respectively. The *maximum degree* $\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}$. For a pair of vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ between u and v is the length of the shortest uv -paths in G .

For any $S \subseteq V(G)$, the subgraph induced by S (resp., $V(G) \setminus S$) is denoted by $G[S]$ (resp., $G - S$). For $B \subseteq E(G)$, we use $G - B$ to denote the subgraph with vertex set $V(G)$ and edge set $E(G) \setminus B$. For convenience, for $u \in V(G)$ and $xy \in E(G)$, we denote $G - \{u\}$ and $G - \{xy\}$ by $G - u$ and $G - xy$, respectively.

Let T be a tree. A vertex u is called a *leaf* of T if $\deg_T(u) = 1$ and the unique neighbor of u is said to be the *support vertex* of u . The symbol $S(a, b)$ will denote the double star obtained by adding an edge between the central vertices of two stars $K_{1,a-1}$ and $K_{1,b-1}$. And the vertex with degree a (resp., b) in $S(a, b)$ is called the *L-central vertex* (resp., *R-central vertex*) of $S(a, b)$. If T is a rooted tree, then, for every $v \in V(T)$, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v , and define $D[v] = D(v) \cup \{v\}$. A set $S \subseteq V(G)$ is a *dominating set* if for each vertex $v \in V(G) - S$, $S \cap N_G(v) \neq \emptyset$. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets in G .

In Ref. [8], Fink and Jacobson introduced the concept of p -domination. Let p be a positive integer. A subset S of $V(G)$ is a *p -dominating set* of G if, for every $v \in V(G) - S$, $|S \cap N_G(v)| \geq p$. The *p -domination number* $\gamma_p(G)$ is the minimum cardinality among all p -dominating sets of G . Any p -dominating set of G with cardinality $\gamma_p(G)$ will be called a γ_p -set of G . Note that the γ_1 -set is the classic minimum dominating set. For any $S, T \subseteq V(G)$, S p -dominates T in G if, for every $v \in T - S$, $|S \cap N_G(v)| \geq p$. Notice that every graph has a p -dominating set since the vertex set $V(G)$ is such a set. We also note that the 1-dominating set is a dominating set, and so $\gamma(G) = \gamma_1(G)$. Thus, the p -domination number has received much research attention (see, for example [1, 2, 4, 7, 8]). From the definition of a p -dominating set, the follow result is straightforward and useful.

Lemma 1 ([2]) *Every p -dominating set of a graph contains any vertex of degree at most $p - 1$.*

By this lemma, to avoid happening the trivial case, we always assume that the maximum degree of a graph G considered in this paper is no less than p , that is, $\Delta(G) \geq p$.

In 1990, Fink et al. [9] introduced the bondage number as a parameter measuring the vulnerability of the interconnection network under link failure. The *bondage number* $b(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $B \subseteq E(G)$ such that $\gamma(G - B) > \gamma(G)$. This topic was further investigated in many papers (see, for example [3, 6, 10, 13–23]). We generalize this concept to the *p -bondage number* of G , denoted by $b_p(G)$, as the minimum cardinality among all sets of edges $B \subseteq E(G)$ such that $\gamma_p(G - B) > \gamma_p(G)$. Clearly, $b_1(G) = b(G)$.

Recently, Blidia et al. [2] have given some bounds on the p -domination number in trees. It is easy to prove that $1 \leq b(T) \leq 2$ for any nontrivial tree T . Teschner [24], Hartnell and Rall [13] characterized all trees bondage number one and two, respectively. In this paper, for any integer $p \geq 2$ and tree T with $\Delta(T) \geq p$, we show that $1 \leq b_p(T) \leq \Delta(T) - p + 1$, and characterize all trees achieving the equalities.

In the remainder of this paper, we assume that $p \geq 2$ is an integer. The rest of this article is organized as follows. Section 2 characterizes all trees with p -bondage number one. Section 3 shows that $b_p(T) \leq \Delta(T) - p + 1$ for any tree T with $\Delta(T) \geq p$, and Sect. 4 characterizes all trees attaining the upper bound.

2 Characterization of Trees with $b_p(T) = 1$

From the definition of p -bondage, it is obvious that $b_p(T) \geq 1$ for any tree T with $\Delta(T) \geq p$. In this section, we will characterize all trees with $b_p(T) = 1$.

In order to state our result, we need some notations. Let G be a graph and D be a subset of $V(G)$. For any $x \in D$, a vertex y not in D is said to be a p -private neighbor of x with respect to D if y is a neighbor of x and $|D \cap N_G(y)| = p$. The p -private neighborhood of x with respect to D , denoted by $PN_p(x, D, G)$, is the set of all p -private neighbors of x with respect to D in G .

Theorem 1 *Let T be a tree with $\Delta(T) \geq p$. Then $b_p(T) = 1$ if and only if T has an edge xy satisfying either $x \in D$ and $y \in PN_p(x, D, T)$ or $y \in D$ and $x \in PN_p(y, D, T)$ for any γ_p -set D of T .*

Proof Assume that $b_p(T) = 1$. Then there is an edge, denoted by xy , in T such that $\gamma_p(T - xy) > \gamma_p(T)$. Let T_x (resp., T_y) denote the component of $T - xy$ containing x (resp., y). Clearly, $\gamma_p(T - xy) = \gamma_p(T_x) + \gamma_p(T_y)$. Let D be a γ_p -set of T .

Suppose that D satisfies one of the following four conditions:

- (1) $D \cap \{x, y\} = \{x, y\}$;
- (2) $D \cap \{x, y\} = \emptyset$;
- (3) $D \cap \{x, y\} = \{x\}$ and $|D \cap N_T(y)| \geq p + 1$;
- (4) $D \cap \{x, y\} = \{y\}$ and $|D \cap N_T(x)| \geq p + 1$.

Then, since D is a γ_p -set of T , we can check easily that $D \cap V(T_x)$ and $D \cap V(T_y)$ p -dominate T_x and T_y , respectively. So

$$\gamma_p(T - xy) = \gamma_p(T_x) + \gamma_p(T_y) \leq |D \cap V(T_x)| + |D \cap V(T_y)| = |D| = \gamma_p(T),$$

a contradiction with $\gamma_p(T - xy) > \gamma_p(T)$. Therefore, by the definition of γ_p -sets, we have either $D \cap \{x, y\} = \{x\}$ and $|D \cap N_T(y)| = p$ or $D \cap \{x, y\} = \{y\}$ and $|D \cap N_T(x)| = p$. That is, edge xy satisfies either $x \in D$ and $y \in PN_p(x, D, T)$ or $y \in D$ and $x \in PN_p(y, D, T)$.

Conversely, we assume that the edge xy of T satisfies either $x \in D$ and $y \in PN_p(x, D, T)$ or $y \in D$ and $x \in PN_p(y, D, T)$ for any γ_p -set D of T . Then we only need to prove $\gamma_p(T - xy) > \gamma_p(T)$. Let D be a γ_p -set of T . Without loss of generality, say $x \in D$ and $y \in PN_p(x, D, T)$.

To the contrary that $\gamma_p(T - xy) = \gamma_p(T)$. Let T_x (resp., T_y) denote the component of $T - xy$ which contains x (resp., y), and let $D_x = D \cap V(T_x)$ and $D_y = D \cap V(T_y)$. Then

$$|D_x| + |D_y| = |D| = \gamma_p(T) = \gamma_p(T - xy) = \gamma_p(T_x) + \gamma_p(T_y).$$

Since D is a γ_p -set of T and $x \in D$, D_x is a p -dominating set of T_x . Thus $|D_x| \geq \gamma_p(T_x)$, and so $|D_y| = \gamma_p(T_x) + \gamma_p(T_y) - |D_x| \leq \gamma_p(T_y)$.

If $|D_y| \leq \gamma_p(T_y) - 1$, then $|D_x| = \gamma_p(T_x) + \gamma_p(T_y) - |D_y| \geq \gamma_p(T_x) + 1$. Let D'_x be a γ_p -set of T_x , then set $S = D'_x \cup \{y\} \cup D_y$ is a p -dominating set of T and $|S| = |D'_x| + 1 + |D_y| \leq \gamma_p(T_x) + \gamma_p(T_y) = \gamma_p(T)$. That is, S is a γ_p -set of T . Since D'_x is a γ_p -set of T_x , by the definition of γ_p -sets, $x \in D'_x$ or $x \notin D'_x$ and $|D'_x \cap N_{T_x}(x)| \geq p$. Thus we have either $S \cap \{x, y\} = \{x, y\}$ or $S \cap \{x, y\} = \{y\}$ and $|S \cap N_T(x)| = |D'_x \cap N_{T_x}(x)| + |\{y\}| \geq p + 1$, a contradiction with the assumption of xy .

If $|D_y| = \gamma_p(T_y)$, then $|D_x| = \gamma_p(T_x) + \gamma_p(T_y) - |D_y| = \gamma_p(T_x)$, and so D_x is a γ_p -set of T_x . For any γ_p -set D'_y of T_y , it is obvious that set $S' = D_x \cup D'_y$ is a p -dominating set of T and $|S'| = |D_x| + |D'_y| = \gamma_p(T_x) + \gamma_p(T_y) = \gamma_p(T)$. So S' is a γ_p -set of T . By $x \in D_x$, we have $x \in S'$. By the condition of xy , we have $y \in PN_p(x, S', T)$, that is, $y \notin S'$ and $|S' \cap N_T(y)| = p$. Hence, $y \notin D'_y$ and $|D'_y \cap N_{T_y}(y)| = |S' \cap N_T(y)| - |\{x\}| = p - 1$, which contradicts that D'_y is a γ_p -set of T_y .

The proof of the theorem is complete. \square

3 The Upper Bound for $b_p(T)$

In this section, we will present an upper bound for $b_p(T)$.

Theorem 2 *Let T be a tree with $\Delta(T) = \Delta \geq p$. Then $b_p(T) \leq \Delta - p + 1$.*

Proof Let x be a leaf of T and we root T at x . Let

$$I = \{y \mid deg_T(y) \geq p \text{ and all vertices of } D(y) \text{ have degree at most } p - 1\}.$$

Since $\Delta(T) = \Delta \geq p$, $I \neq \emptyset$, and so we can choose a vertex, denoted by u , from I satisfying $d_T(u, r) = \max_{y \in I} \{d_T(y, x)\}$. Denote the father of u by v . Then, by the choice of u , all vertices of $D(v) \setminus C(v)$ have degree at most $p - 1$. By Lemma 1, every p -dominating set of T contains $D(v) \setminus C(v)$.

Case 1 $C(v)$ has a vertex of degree at least $p + 1$ in T . Without loss of generality, say $deg_T(u) \geq p + 1$.

Since $deg_T(u) \geq p + 1$, for any γ_p -set D' of $T - D[u]$, $D' \cup D(u)$ is a p -dominating set of T . So

$$\gamma_p(T - D[u]) + |D(u)| \geq \gamma_p(T).$$

Let $C(u) = \{a_1, \dots, a_{p-1}, a_p, \dots, a_{deg_T(u)-1}\}$ and $B = \{uv, ua_p, \dots, ua_{deg_T(u)-1}\}$, then $deg_{T-B}(u) = p - 1$. Thus, by Lemma 1, every γ_p -set of $T - B$ contains $D[u]$. So

$$\begin{aligned}\gamma_p(T - B) &= \gamma_p(T - D[u]) + |D[u]| \\ &= \gamma_p(T - D[u]) + |D(u)| + 1 \\ &\geq \gamma_p(T) + 1,\end{aligned}$$

which implies that $b_p(T) \leq |B| = \deg_T(v) - p + 1 \leq \Delta - p + 1$.

Case 2 Each vertex of $C(v)$ has degree at most p in T . By the choice of u , we have $\deg_T(u) = p$. By the definition, every γ_p -set of T contains exactly one of u and v .

Let $C(v) = \{u, u_1, \dots, u_{\deg_T(v)-2}\}$ and let

$$B = \begin{cases} \{vu\} & \text{if } \deg_T(v) \leq p; \\ \{vu, vu_{p-1}, \dots, vu_{\deg_T(v)-2}\} & \text{if } \deg_T(v) > p. \end{cases}$$

Hence both u and v have degree at most $p - 1$ in $T - B$. Let D' be a γ_p -set of $T - B$. Clearly, D' is a p -dominating set of T and, by Lemma 1, $\{u, v\} \subseteq D'$. Note that every γ_p -set of T contains exactly one of u and v . We have $\gamma_p(T - B) = |D'| > \gamma_p(T)$, which implies that

$$\begin{aligned}b_p(T) \leq |B| &= \begin{cases} 1 & \text{if } \deg_T(v) \leq p \\ \deg_T(v) - p + 1 & \text{if } \deg_T(v) > p \end{cases} \\ &\leq \Delta - p + 1.\end{aligned}$$

The proof of the theorem is complete. \square

Corollary 1 If T is a tree with $\Delta(T) = p$, then $b_p(T) = 1$.

4 Trees with Maximum p -Bondage Number

To characterize all trees attaining the upper bound given in Theorem 2, we define three types of operation on a tree T with $\Delta(T) = \Delta \geq p + 1$.

Type-1: Attach a vertex u to a vertex v of T by joining u to v , where $\deg_T(v) \leq p - 2$.

Type-2: Attach a star $K_{1, \Delta-1}$ to a vertex v of T by joining its central vertex to v , where v is in a γ_p -set of T and $\deg_T(v) \leq \Delta - 1$.

Type-3: Attach a double star $S(p, \Delta - 1)$ to a leaf v of T by coinciding its R -central vertex with v , where the support vertex of v is in a γ_p -set of T .

Let \mathcal{A} be a family of trees with maximum degree $\Delta \geq p + 1$ and p -bondage number $\Delta - p + 1$, that is

$$\mathcal{A} = \{T : T \text{ is a tree satisfying } \Delta(T) = \Delta \geq p + 1 \text{ and } b_p(T) = \Delta - p + 1\}.$$

We also define a family \mathcal{B} as follows:

$\mathcal{B} = \{T : T \text{ is obtained from } K_{1,\Delta} \text{ or } S(p, \Delta) \text{ by a finite sequence of operations of Type-}i, \text{ where } 1 \leq i \leq 3\}.$

We will show $\mathcal{A} = \mathcal{B}$.

Let t be an integer and G be a graph. We use $s_t(G)$ to denote the cardinality of the set of vertices with degree at least t in G .

From the definitions of γ_p -set and three types of operations, we obtain the following lemma immediately.

Lemma 2 *Let T be a tree obtained from T' by the operation Type- i , $1 \leq i \leq 3$. Then*

- (1) *If $i = 1$, then $\gamma_p(T) = \gamma_p(T') + 1$;*
- (2) *If $i = 2$, then $\gamma_p(T) = \gamma_p(T') + (\Delta - 1)$;*
- (3) *If $i = 3$, then $\gamma_p(T) = \gamma_p(T') + (\Delta + p - 3)$.*

Lemma 3 *If T is obtained from T' by the operation of Type-1, then*

$$s_p(T) = s_p(T') \quad \text{and} \quad b_p(T) = b_p(T').$$

Proof It is trivial for $s_p(T) = s_p(T')$. So we merely prove $b_p(T) = b_p(T')$.

Assume that T is obtained from T' by adding a vertex u to a vertex v of T' . Since $\deg_{T'}(v) \leq p - 2$, we have $\deg_{T-B}(v) \leq \deg_T(v) \leq p - 1$ for any $B \subseteq E(T)$. By Lemma 1, v is in every p -dominating set of T , T' , $T - B$, or $T' - (E(T') \cap B)$. Hence

$$\gamma_p(T') + 1 = \gamma_p(T) \quad \text{and} \quad \gamma_p(T' - (E(T') \cap B)) + 1 = \gamma_p(T - B).$$

Let B' be a subset of $E(T')$ satisfying $|B'| = b_p(T')$ and $\gamma_p(T' - B') > \gamma_p(T')$, then

$$\gamma_p(T - B') = \gamma_p(T' - B') + 1 > \gamma_p(T') + 1 = \gamma_p(T),$$

which implies that $b_p(T) \leq |B'| = b_p(T')$. Let B be a subset of $E(T)$ satisfying $|B| = b_p(T)$ and $\gamma_p(T - B) > \gamma_p(T)$, then

$$\gamma_p(T' - (E(T') \cap B)) = \gamma_p(T - B) - 1 > \gamma_p(T) - 1 = \gamma_p(T'),$$

and we can see that $b_p(T') \leq |E(T') \cap B| \leq |B| = b_p(T)$. So $b_p(T) = b_p(T')$, and the lemma follows. \square

Lemma 4 *If $T \in \mathcal{A}$ is a tree with $s_p(T) = 1$ or 2, then $T \in \mathcal{B}$.*

Proof Since $T \in \mathcal{A}$, we have $\Delta(T) = \Delta \geq p + 1$ and $b_p(T) = \Delta - p + 1$.

Case 1 $s_p(T) = 1$.

Assume that $v \in V(T)$ satisfies $\deg_T(v) \geq p$. Then, we have $\deg_T(v) = \Delta$ and the degree of every vertex of $V(T) \setminus \{v\}$ is at most $p - 1$ in T . Note that a star $K_{1,\Delta} \in \mathcal{B}$ and T can be obtained from $K_{1,\Delta}$ by $|V(T)| - (\Delta + 1)$ operations of Type-1. So $T \in \mathcal{B}$.

Case 2 $s_p(T) = 2$.

Let u and v be two vertices with degree at least p in T . Then every vertex of $V(T) \setminus \{u, v\}$ has degree at most $p - 1$. By Lemma 1, $V(T) \setminus \{u, v\}$ is contained by any p -dominating set of T . Let $a = \deg_T(u)$ and $b = \deg_T(v)$.

If $d_T(u, v) = 1$, then T can be obtained from a double star $S(a, b)$ by $|V(T)| - (a+b)$ operations of Type-1. By Lemma 3 and $b_p(T) = \Delta - p + 1$, $s_p(S(a, b)) = s_p(T) = 2$ and $b_p(S(a, b)) = b_p(T) = \Delta - p + 1$. It is easy to check that $S(a, b) = S(p, \Delta) \in \mathcal{B}$. So $T \in \mathcal{B}$.

If $d_T(u, v) \geq 2$, then $p \geq 3$ and $b_p(T) = \min\{a - p + 1, b - p + 1\}$ since $a \geq p$ and $b \geq p$. So $a = b = \Delta$. Let T' be a tree obtained by placing a path of length $\deg_T(u, v)$ between the central vertices of two stars $K_{1,\Delta-1}$. It is easy to see that T can be obtained from T' by $|V(T)| - |V(T')|$ operations of Type-1, and T' can be obtained from a star $K_{1,\Delta}$ by $|d_T(u, v)| - 2$ operations of Type-1 and one operation of Type-2. Hence $T \in \mathcal{B}$. \square

Lemma 5 *If $T' \in \mathcal{A}$ and T is obtained from T' by the operation Type- i , $i \in \{1, 2, 3\}$, then $T \in \mathcal{A}$.*

Proof Since $T' \in \mathcal{A}$, we have $\Delta(T') = \Delta \geq p + 1$ and $b_p(T') = \Delta - p + 1$. So $\Delta(T) = \Delta$. By Theorem 2 and Lemma 3, we only need to prove that $b_p(T) \geq \Delta - p + 1$ for $i = 2$ and $i = 3$, respectively. Suppose to the contrary that $b_p(T) \leq \Delta - p$. Then there is a subset $B \subseteq E(T)$ with $|B| = \Delta - p$ such that $\gamma_p(T - B) > \gamma_p(T)$. Let $B' = E(T') \cap B$.

Case 1 $i = 2$.

Assume that T is obtained from T' by attaching a star $K_{1,\Delta-1}$ with the central vertex u to a vertex v of T' such that $uv \in E(T)$, where v is in a γ_p -set D' of T' . By Lemma 2 (2), we have $\gamma_p(T) = \gamma_p(T') + (\Delta - 1)$.

If $B' = \emptyset$, since $\deg_{T-B}(u) = \deg_T(u) - |B| = \Delta - (\Delta - p) = p$, then $D' \cup N_T(u)$ is a p -dominating set of $T - B$. It follows that

$$\begin{aligned}\gamma_p(T - B) &\leq |D' \cup N_T(u)| \\ &= |D'| + |N_T(u)| - |D' \cap N_T(u)| \\ &= \gamma_p(T') + \Delta - 1 \\ &= \gamma_p(T),\end{aligned}$$

which contradicts with $\gamma_p(T - B) > \gamma_p(T)$.

If $B' \neq \emptyset$, then $|B - B'| \leq \Delta - p - 1$ and

$$|N_{T-B}(u) \setminus \{v\}| \geq \deg_{T-B}(u) - 1 = \Delta - |B - B'| - 1 \geq p.$$

Thus, for any γ_p -set S' of $T' - B'$, $S' \cup (N_T(u) \setminus \{v\})$ is a p -dominating set of $T - B$. Since $b_p(T') = \Delta - p + 1$ and $|B'| \leq |B| = \Delta - p$, we have $\gamma_p(T' - B') = \gamma_p(T')$. Therefore,

$$\begin{aligned}\gamma_p(T - B) &\leq |S' \cup (N_T(u) \setminus \{v\})| \\ &= \gamma_p(T' - B') + \Delta - 1 \\ &= \gamma_p(T') + \Delta - 1 \\ &= \gamma_p(T),\end{aligned}$$

a contradiction with $\gamma_p(T - B) > \gamma_p(T)$.

Case 2 $i = 3$.

Assume that T is obtained from T' by attaching a double star $T_0 = S(p, \Delta - 1)$ to a leaf v of T' by coinciding the R-central vertex of T_0 with v , where the L-central vertex of T_0 and the support vertex of v in T' are denoted as u and w , respectively. By Lemma 2(3), we have $\gamma_p(T) = \gamma_p(T') + (\Delta + p - 3)$. Let $T_1 = T[V(T_0) \cup \{w\}]$ and $B_1 = E(T_1) \cap B$. Clearly, T_1 is a double star $S(p, \Delta)$ and $\gamma_p(T_1 - B_1) = \gamma_p(T_1) = \Delta + p - 1$.

Subcase 2.1 $B' = \emptyset$ or $B' = \{vw\}$.

In this case, we have $B_1 = B$. By $\deg_{T_1-B}(v) \geq \Delta - |B| \geq p$ and Lemma 1, $V(T_1) \setminus \{v\}$ p -dominates $T_1 - B$. Let S be a γ_p -set of T' containing w . Since v is a leaf of T' , we have $v \in S$. Thus, $(S \setminus \{v\}) \cup (V(T_1) \setminus \{v\})$ is a p -dominating set of $T - B$ and

$$\begin{aligned}|(S \setminus \{v\}) \cup (V(T_1) \setminus \{v\})| \\ &= |S \setminus \{v\}| + |V(T_1) \setminus \{v\}| - |(S \setminus \{v\}) \cap (V(T_1) \setminus \{v\})| \\ &= (\gamma_p(T') - 1) + (\Delta + p - 1) - 1 \\ &= \gamma_p(T).\end{aligned}$$

Therefore, $\gamma_p(T - B) \leq |(S \setminus \{v\}) \cup (V(T_1) \setminus \{v\})| = \gamma_p(T)$, a contradiction with $\gamma_p(T - B) > \gamma_p(T)$.

Subcase 2.2 $B' \neq \emptyset$ and $B' \neq \{vw\}$.

In this case, we have $|B_1| \leq |B| - 1 = \Delta - p - 1$. Let E_u be a set of edges incident with u in T .

If $E_u \cap B = \emptyset$, then it holds that $V(T_1) \setminus \{u, w\}$ p -dominates $(T_1 - B_1) - w$. Since $b_p(T') = \Delta - p + 1$ and $|B'| \leq |B| = \Delta - p$, we have $\gamma_p(T' - B') = \gamma_p(T')$. Thus, there is a γ_p -set S' of $T' - B'$ with $|S'| = \gamma_p(T')$. By $\deg_{T'-B'}(v) \leq 1$, we have $v \in S'$. Therefore, $(S' \setminus \{v\}) \cup (V(T_1) \setminus \{u, w\})$ is a p -dominating set of $T - B$ and

$$\begin{aligned}|(S' \setminus \{v\}) \cup (V(T_1) \setminus \{u, w\})| &= (|S'| - 1) + (\Delta + p - 2) \\ &= \gamma_p(T') + (\Delta + p - 3) \\ &= \gamma_p(T).\end{aligned}$$

This implies that $\gamma_p(T - B) \leq |(S' \setminus \{v\}) \cup (V(T_1) \setminus \{u, w\})| = \gamma_p(T)$, a contradiction with $\gamma_p(T - B) > \gamma_p(T)$.

If $E_u \cap B \neq \emptyset$, then $|B' \cup \{vw\}| \leq |B'| + 1 \leq |B| = \Delta - p$, and so $\gamma_p(T' - (B' \cup \{vw\})) = \gamma_p(T')$. Let S'' be a γ_p -set of $T' - (B' \cup \{vw\})$. Clearly, $v \in S''$ and $S'' \setminus \{v\}$ p -dominates $(T' - B') - \{v\}$. Since $|B_1| \leq \Delta - p - 1$, $\deg_{T_1 - B_1}(v) \geq \Delta - |B_1| \geq p + 1$, and so $V(T_1) \setminus \{v, w\}$ p -dominates $(T_1 - B_1) - \{w\}$. Thus, $(S'' \setminus \{v\}) \cup (V(T_1) \setminus \{v, w\})$ is a p -dominating set of $T - B$, and

$$\begin{aligned}\gamma_p(T - B) &\leq |(S'' \setminus \{v\}) \cup (V(T_1) \setminus \{v, w\})| \\ &= (|S''| - 1) + (\Delta + p - 2) \\ &= \gamma_p(T') + (\Delta + p - 3) \\ &= \gamma_p(T),\end{aligned}$$

which contradicts with $\gamma_p(T - B) > \gamma_p(T)$.

The proof of the lemma is complete. \square

Lemma 6 *If $T \in \mathcal{A}$, then $T \in \mathcal{B}$.*

Proof Since $T \in \mathcal{A}$, we have $\Delta(T) = \Delta \geq p + 1$ and $b_p(T) = \Delta - p + 1$. We will prove $T \in \mathcal{B}$ by induction on $s_p(T)$. Let $s_p(T) = s$.

If $s \in \{1, 2\}$, then, by Lemma 4, $T \in \mathcal{B}$. This establishes the base case. Assume that, for any tree T' with $\Delta(T') = \Delta$ and $1 \leq s_p(T') < s$, if $b_p(T') = \Delta - p + 1$ then $T' \in \mathcal{B}$. We assume that $s \geq 3$.

Let x be a leaf of T and we root T at x . We follow the symbol I and u which are defined in the proof of Theorem 2. Let v be the father of u and denote the father of v by w in T . From the definition of I and u , we can see that every vertex of $D(v) \setminus C(v)$ has degree at most $p - 1$. We distinguish the follow two cases according to the degree of u in T .

Case 1 $\deg_T(u) \geq p + 1$.

Let $T' = T - (D(u) \setminus C(u))$ and $T'' = T - D[u]$. Clearly, T can be obtained from T' by $|D(u) \setminus C(u)|$ operations of Type-1. So we only need to prove that $T' \in \mathcal{B}$. By Lemma 3, $b_p(T') = b_p(T) = \Delta - p + 1$ and $s_p(T') = s_p(T) = s$. Since $\deg_T(u) \geq p + 1$, from the definition of γ_p -set, we can easily derive that

$$\begin{aligned}\gamma_p(T'') + (\deg_T(u) - 1) &= \gamma_p(T'), \text{ and} \\ \gamma_p(T'' - B) + (\deg_T(u) - 1) &= \gamma_p(T' - B) \text{ for any } B \subseteq E(T'').\end{aligned}$$

Let $C(u) = \{a_1, \dots, a_{\deg_T(u)-1}\}$.

We first prove that $\deg_T(u) = \Delta$. Let $F = T' - \{uv, ua_p, \dots, ua_{\deg_T(u)-1}\}$. Then $\deg_F(u) = p - 1$. By Lemma 1, we have

$$\gamma_p(F) = \gamma_p(T'') + \deg_T(u) = \gamma_p(T') + 1.$$

It follows that

$$\begin{aligned}\Delta - p + 1 &= b_p(T') \leq |\{uv, ua_p, \dots, ua_{\deg_T(u)-1}\}| \\ &= \deg_T(u) - p + 1 \\ &\leq \Delta - p + 1,\end{aligned}$$

which implies that $\deg_T(u) = \Delta$.

Secondly, we show that v belongs to some γ_p -set of T'' . Let $G = T' - \{ua_p, \dots, ua_{\Delta-1}\}$. Since $\deg_G(u) = p$, we can choose S from $V(G)$ as a γ_p -set of G such that $u \notin S$ and $v \in S$. Obviously, $C(u) \subseteq S$ and $S \cap V(T'')$ is a p -dominating set of T'' containing v . Because $|\{ua_p, \dots, ua_{\Delta-1}\}| = \Delta - p < b_p(T')$, we have

$$|S \cap V(T'')| = |S| - |C(u)| = \gamma_p(G) - (\Delta - 1) = \gamma_p(T') - (\Delta - 1) = \gamma_p(T'').$$

Hence $S \cap V(T'')$ is a γ_p -set of T'' containing v . Furthermore, T' is obtained from T'' by the operation of Type-2.

Now we prove the $T' \in \mathcal{B}$. For any $B \subseteq E(T'')$ with $|B| \leq \Delta - p$, by $b_p(T') = \Delta - p + 1$, we have

$$\gamma_p(T'' - B) = \gamma_p(T' - B) - (\Delta - 1) = \gamma_p(T') - (\Delta - 1) = \gamma_p(T'').$$

It follows from Theorem 2 that

$$\Delta - p + 1 \leq b_p(T'') \leq \Delta(T'') - p + 1 \leq \Delta - p + 1,$$

which implies that $\Delta(T'') = \Delta$ and $b_p(T'') = \Delta - p + 1$. Note that $s_p(T'') = s_p(T') - 1 = s - 1$. Applying the induction hypothesis, we have $T'' \in \mathcal{B}$, and so $T' \in \mathcal{B}$.

Case 2 $\deg_T(u) = p$.

Note that every vertex of $D(v) \setminus C(v)$ has degree at most $p - 1$. By Case 1, we can assume that the degree of every vertex of $C(v)$ in T is at most p . Suppose that $C(v)$ contains at least two vertices with degree p in T . By the definition, for any γ_p -set D of T , we have $v \in D$ and $u \in PN_p(v, D, T)$. By Theorem 1, $b_p(T) = 1 < \Delta - p + 1 = b_p(T)$, a contradiction. Therefore, all vertices of $C(v)$ have degree at most $p - 1$ in T except u .

Let T' , T'' and T_w be the subtrees $T - [D(v) \setminus (C(v) \cup C(u))]$, $T - D(v)$ and $T - D[v]$, respectively. Let $T'_v = T' - V(T_w)$. Clearly, T can be obtained from T' by $|D(v) \setminus (C(v) \cup C(u))|$ operations of Type-1. Thus, we only need to prove $T' \in \mathcal{B}$. By Lemma 3, $s_p(T') = s_p(T) = s$ and $b_p(T') = b_p(T) = \Delta - p + 1$.

We will prove that $\deg_T(v) = \Delta$. Suppose that $\deg_T(v) \leq p$. Then, by Lemma 1 and $\deg_T(u) = p$, it holds that either $v \in D$ and $u \in PN_p(v, D, T)$ or $u \in D$ and $v \in PN_p(u, D, T)$ for any γ_p -set D of T . By Theorem 1, $b_p(T) = 1 < \Delta - p + 1$,

a contradiction with $b_p(T) = \Delta - p + 1$. So $\deg_T(v) \geq p + 1$. Let $C(v) = \{u, u_1, \dots, u_{\deg_T(v)-2}\}$ and

$$B = \begin{cases} \{vu, vw\} & \text{if } \deg_T(v) = p + 1; \\ \{vu, vw, vu_p, \dots, vu_{\deg_T(v)-2}\} & \text{if } \deg_T(v) \geq p + 2. \end{cases}$$

Then $\deg_{T'-B}(v) = p - 1$, $|B| = \deg_T(v) - p + 1$ and $\gamma_p(T' - B) = \gamma_p(T_w) + |V(T'_v)|$. On the other hand, for any γ_p -set S_w of T_w , it is easy to see that $S_w \cup (V(T'_v) \setminus \{u\})$ p -dominates T' . It follows that $\gamma_p(T') \leq \gamma_p(T_w) + |V(T'_v)| - 1$, which implies that $\gamma_p(T' - B) > \gamma_p(T')$. Thus

$$\Delta - p + 1 = b_p(T') \leq |B| = \deg_T(v) - p + 1 \leq \Delta - p + 1.$$

Therefore, $\deg_T(v) = \Delta$ and T'_v is a double star $S(p, \Delta - 1)$.

Next we show that w belongs to some γ_p -set of T'' . Let

$$B' = \begin{cases} \{vu\} & \text{if } \Delta = p + 1; \\ \{vu, vu_p, \dots, vu_{\Delta-2}\} & \text{if } \Delta \geq p + 2, \end{cases}$$

and $H = T' - B'$. Then $\deg_H(u) = p - 1$, $\deg_H(v) = p$ and $|B'| = \Delta - p$. Since $b_p(T') = \Delta - p + 1$, we have $\gamma_p(H) = \gamma_p(T')$. Because v is a leaf of T'' , v belongs to any γ_p -set of T'' . It follows that

$$\gamma_p(T'') + |V(T'_v)| - 2 = \gamma_p(T'') + |V(T'_v) \setminus \{u, v\}| \geq \gamma_p(T').$$

Let S be a γ_p -set of H . Then $(S \cap V(T'')) \cup \{v\}$ is a p -dominating set of T'' and

$$\begin{aligned} |(S \cap V(T'')) \cup \{v\}| &\leq |S \cap V(T'')| + 1 \\ &= |S| - |V(T'_v) \setminus \{v\}| + 1 \\ &= \gamma_p(H) - (|V(T'_v)| - 2) \\ &= \gamma_p(T') - (|V(T'_v)| - 2) \\ &\leq \gamma_p(T''). \end{aligned}$$

Therefore, $(S \cap V(T'')) \cup \{v\}$ is a γ_p -set of T'' , $v \notin S$ and $\gamma_p(T') = \gamma_p(T'') + |V(T'_v)| - 2$. Since $v \notin S$ and $\deg_H(v) = p$, to p -dominate w , $w \in S$, we have $w \in (S \cap V(T'') \cup \{v\})$. Furthermore, T' can be obtained from T'' by the operation of Type-3.

To the end, we only need to prove $T'' \in \mathcal{B}$. Suppose that there is a subset $B'' \subseteq E(T'')$ with $|B''| = \Delta - p$ such that $\gamma_p(T'' - B'') > \gamma_p(T'')$. Since $\deg_{T'-B''}(u) = p$, there is a γ_p -set S' of $T' - B''$ containing v , and so $S' \cap V(T'')$ p -dominates $T'' - B''$. Hence, by $\gamma_p(T') = \gamma_p(T'') + |V(T'_v)| - 2$, we have

$$\begin{aligned}
\gamma_p(T' - B'') &= |S'| = |S' \cap V(T'')| + |V(T'_v) \setminus \{u, v\}| \\
&\geq \gamma_p(T'' - B'') + |V(T'_v)| - 2 \\
&> \gamma_p(T'') + |V(T'_v)| - 2 \\
&= \gamma_p(T'),
\end{aligned}$$

and we can see that $b_p(T') \leq |B''| = \Delta - p$, a contradiction with $b_p(T') = \Delta - p + 1$. Thus, by Theorem 2 and $\Delta(T'') \leq \Delta(T') = \Delta$, we have $\Delta(T'') = \Delta$ and $b_p(T'') = \Delta - p + 1$. Note that $s_p(T'') = s_p(T') - 2 < s$. Applying the induction hypothesis, we have $T'' \in \mathcal{B}$.

The proof of the lemma is complete. \square

Theorem 3 $\mathcal{A} = \mathcal{B}$.

Proof Note that $\{K_{1,\Delta}, S(p, \Delta)\} \subseteq \mathcal{A}$. Lemma 5 implies that $\mathcal{B} \subseteq \mathcal{A}$ and Lemma 6 implies that $\mathcal{A} \subseteq \mathcal{B}$. The theorem follows. \square

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