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The Critical Group of $K_m \times C_n$

Jian WANG Yong Liang PAN[1)] **Jun Ming XU**
Department of Mathematics, University of Science and Technology of China, Hefei 230026, *P. R. China*
E-mail: *wangj12@mail.ustc.edu.cn* *ylpan@ustc.edu.cn* *xujm@ustc.edu.cn*

Abstract In this paper, the structure of the critical group of the graph $K_m \times C_n$ is determined, where $m, n \geq 3$.

Keywords Graph, Laplacian matrix, critical group, invariant factor, Smith normal form, tree number

MR(2000) Subject Classification 05C50

1 Introduction and Statement of Results

The critical group of a connected graph is a finite abelian group whose structure is a subtle isomorphism invariant of the graph. It is closely connected with the graph Laplacian.

Let $G = (V, E)$ be a finite connected graph without self-loops, but with multiple edges allowed. Then the Laplacian matrix of G is the $|V| \times |V|$ matrix defined by

$$L(G)_{uv} = \begin{cases} d(u), & \text{if } u = v, \\ -a_{uv}, & \text{if } u \neq v, \end{cases} \tag{1.1}$$

where a_{uv} is the number of the edges joining u and v, and $d(u)$ is the degree of u.

Regarding $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$, its cokernel $\operatorname{coker}(L(G)) = \mathbb{Z}^{|V|}/\operatorname{im}(L(G))$ is an abelian group. For $1 \leq i \leq |V|$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^t \in \mathbb{Z}^{|V|}$ be the i-th standard basis, and x_i be its image in $\operatorname{coker}(L(G))$. We know that $\operatorname{coker}(L(G))$ is determined by the generators $x_1, \ldots, x_{|V|}$ and the relations $(x_1, \ldots, x_{|V|})L(G) = 0$. Since $L(G)$ is symmetric, we can rewrite the relations as follows:

$$\begin{cases} l_{11}x_1 + l_{12}x_2 + \cdots + l_{1|V|}x_{|V|} = 0, \\ l_{21}x_1 + l_{22}x_2 + \cdots + l_{2|V|}x_{|V|} = 0, \\ \quad \vdots \\ l_{|V|1}x_1 + l_{|V|2}x_2 + \cdots + l_{|V||V|}x_{|V|} = 0. \end{cases} \tag{1.2}$$

Two integral matrices A and B are equivalent (written $A \sim B$) if there are unimodular matrices P and Q such that $B = PAQ$ (an integral matrix P is unimodular if P^{-1} is also integral, i.e., if $\det P = \pm 1$). Equivalently, B is obtainable from A by a sequence of elementary

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1) Corresponding author

row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by -1, (3) the addition of any integer times of one row (resp. column) to another row (resp. column).

It is easy to see that $A \sim B$ implies that $\operatorname{coker}(A) \cong \operatorname{coker}(B)$. The Smith normal form is a diagonal canonical form for our equivalence relation: Every $n \times n$ integral matrix A is equivalent to a unique diagonal matrix $\operatorname{diag}(s_1(A), \ldots, s_n(A))$, where $s_i(A)$ divides $s_{i+1}(A)$ for $i = 1, 2, \ldots, n-1$. The i-th diagonal entry of the Smith normal form of A is usually called the i-th invariant factor of A. We will use the fact that the values $s_i(A)$ can also be interpreted as follows: For each i, the product $s_1(A)s_2(A)\cdots s_i(A)$ is the greatest common divisor of all $i \times i$ minors of A.

The classification theorem for finitely generated abelian groups asserts that $\operatorname{coker}(L(G))$ has a direct sum decomposition

$$\operatorname{coker}(L(G)) \cong (\mathbb{Z}/t_1\mathbb{Z}) \oplus (\mathbb{Z}/t_2\mathbb{Z}) \oplus \cdots \oplus \left(\mathbb{Z}/t_{|V|}\mathbb{Z}\right), \tag{1.3}$$

where the nonnegative integers t_i are the diagonal entries of the Smith normal form of the relation matrix $L(G)$, of course, they satisfy that t_i divides $t_{i+1}\,(1 \le i < |V|)$. Since G is connected, it is not hard to see that $L(G)$ has rank $|V| - 1$, and the kernel of $L(G)$ is spanned by the vectors in $\mathbb{R}^{|V|}$ which are constant on the vertices. It follows that $t_{|V|} = 0$ and $t_1 \cdots t_{|V|-1} \neq 0$.

Now we can write

$$\operatorname{coker}(L(G)) = \mathbb{Z}^{|V|}/\operatorname{im}(L(G)) \cong \mathbb{Z} \oplus K(G), \tag{1.4}$$

where

$$K(G) = (\mathbb{Z}/t_1\mathbb{Z}) \oplus (\mathbb{Z}/t_2\mathbb{Z}) \oplus \cdots \oplus \left(\mathbb{Z}/t_{|V|-1}\mathbb{Z}\right). \tag{1.5}$$

The finite abelian group $K(G)$ is defined to be the critical group of G. And we will call the positive integers $t_1, \ldots, t_{|V|-1}$ the invariant factors of $K(G)$. The critical group $K(G)$ is also known as the Picard group and the Jacobian group of G in [1–3], while in the physics literature it is known as the abelian sandpile group, and it has a close connection with the critical configuration in a certain dollar game on G, see [3, 4]. For the general theory of the critical group, we refer the reader to Biggs [2, 3], Godsil [4, Chapter 14], Cori, et al. [5, 6], Dartois et al. [7], and Bacher, et al. [1].

The well-known Kirchhoff's matrix-tree theorem [4, Theorem 13.2.1] shows that $t_1 \cdots t_{|V|-1}$ equals the number κ of spanning trees of G. It follows that the invariant factors of $K(G)$ can be used to distinguish pairs of non-isomorphic graphs which have the same κ, and so there is considerable interest in their properties. If G is a simple connected graph, then its Laplacian matrix $L(G)$ has some entry which is equal to -1. Since the invariant factor t_1 of $K(G)$ is equal to the greatest common divisor of all the entries of $L(G)$, it follows that t_1 must be equal to 1. But the other invariant factors of $K(G)$ are not easy to be determined.

Compared to the number of the results on the spanning tree number κ, there are relatively few results describing the critical group structure of $K(G)$ in terms of the structure of G. Recently, there are some families of graphs for which the critical group structure has been completely determined: wheel graphs [3]; cycles [8]; complete graphs [9]; complete multipartite graphs and cartesian products of complete graphs [10]; a subclass of the threshold graphs [11];

the Möbius ladder graphs [12]; the Cayley graph $\mathcal{D}_n$ of the dihedral group [7]; the square cycle graphs C_n^2 [13]; etc.

Given two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the Cartesian product of them is denoted by $G_1 \times G_2$. It has vertex set $V_1 \times V_2 = \{(u_i, v_j) | u_i \in V_1, v_j \in V_2\}$, where (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1 = u_2$ and $(v_1, v_2) \in E_2$, or $(u_1, u_2) \in E_1$ and $v_1 = v_2$. One may view $G_1 \times G_2$ as the graph obtained from G_2 by replacing each of its vertices by a copy of G_1, and each of its edges by $|V_1|$ edges joining the corresponding vertices of G_1 in the two copies.

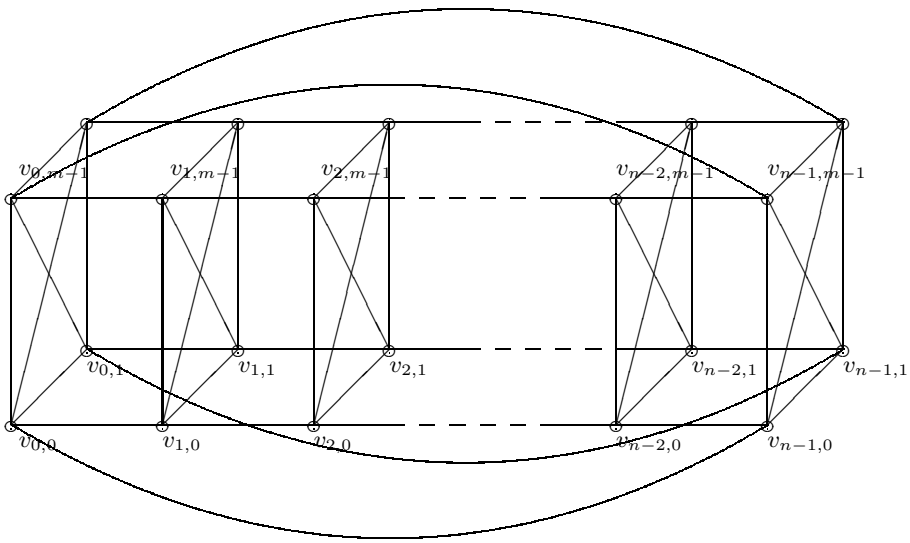

Figure 1 Graph $K_m \times C_n$

The structure of the critical group of $K_m \times P_n$ has been obtained in [14], where K_m is the complete graph on m vertices and P_n is the path on n vertices. In this paper we will describe the structure of the critical group on $K_m \times C_n$ with $n, m \geq 3$, where C_n is the cycle on n vertices. From the definition of the Cartesian product of two graphs, it is easy to see that there are n layers of $K_m \times C_n$, each of which is a copy of K_m. Let $\mathbb{Z}_n$ denote $\mathbb{Z}/n\mathbb{Z}$. Then for $i \in \mathbb{Z}_n,\ j \in \mathbb{Z}_m$, we may let $v_{i,j}$ denote the j-th vertex in the i-th layer of $K_m \times C_n$. The vertex $v_{i,j}$ is adjacent to vertices $v_{l,j}$ with $l = i \pm 1 \pmod n$, and to the vertices $v_{i,k}$, $k \in \mathbb{Z}_m, k \neq j$ (see Figure 1).

Before the main result can be stated, we need some technical definitions.

If m is a positive integer, let $\alpha = \frac{1}{2}(m + 2 + \sqrt{m^2 + 4m})$, $\beta = \frac{1}{2}(m + 2 - \sqrt{m^2 + 4m})$. Then for $p \in \mathbb{Z}$, we set $u_p := \frac{1}{\alpha - \beta}\left(\alpha^p - \beta^p\right)$, $v_p := \alpha^p + \beta^p$, $\tau_p := \frac{1}{m}(p - u_p)$, $h_p := u_p + u_{p+1}$, and $g_p := \tau_p + \tau_{p+1}$. For the integers $a_1, a_2, \ldots, a_k$, we will let $(a_1, a_2, \ldots, a_k)$ denote their greatest common divisor, and use $a_1 \mid a_2 \mid \cdots \mid a_k$ to mean that a_1 divides a_2, a_2 divides a_3, etc.

Now, we can state our main result in this article as follows:

Theorem 1.1 *If $n = 2s + 1$, the critical group of $K_m \times C_n$ $(m, n \geq 3)$ is*

$$\mathbb{Z}_{(n,g_s)} \oplus \mathbb{Z}_{h_s} \oplus \underbrace{\mathbb{Z}_{h_s} \oplus \cdots \oplus \mathbb{Z}_{h_s}}_{m-2} \oplus \mathbb{Z}_\gamma \oplus \underbrace{\mathbb{Z}_{mh_s} \oplus \cdots \oplus \mathbb{Z}_{mh_s}}_{m-3} \oplus \mathbb{Z}_\varphi,$$

where $\gamma = \frac{h_s}{(n,g_s)}(n,h_s)$ *and* $\varphi = \frac{nmh_s}{(n,h_s)}$.

If $n = 2s$, *the critical group of* $K_m \times C_n$ $(m,n \geq 3)$ *is*

$$\mathbb{Z}_{(u_s,2\tau_s)} \oplus \mathbb{Z}_\zeta \oplus \underbrace{\mathbb{Z}_{(m,2)u_s} \oplus \cdots \oplus \mathbb{Z}_{(m,2)u_s}}_{m-3} \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\rho \oplus \underbrace{\mathbb{Z}_\chi \oplus \cdots \oplus \mathbb{Z}_\chi}_{m-3} \oplus \mathbb{Z}_\xi,$$

where

$$\begin{cases} \zeta = \dfrac{u_s\,(n,u_s,4\tau_s)}{(u_s,2\tau_s)}, \\ \eta = \dfrac{u_s(m,2)\,(n,u_s-4\tau_s)}{(n,u_s,4\tau_s)}, \\ \rho = \dfrac{(m+4)u_s\,(mn,(m+4)u_s,2n)}{(n,u_s-4\tau_s)\,(m,2)}, \\ \chi = \dfrac{m(m+4)u_s}{(m,2)}, \\ \xi = \dfrac{nm(m+4)u_s}{(mn,\ (m+4)u_s,\ 2n)}. \end{cases}$$

An immediate consequence of Theorem 1.1 is the following corollary.

Corollary 1.2 *The spanning tree number of* $K_m \times C_n$ *is*

$$\frac{n}{m}\left(\left(\frac{m+2+\sqrt{m^2+4m}}{2}\right)^n + \left(\frac{m+2-\sqrt{m^2+4m}}{2}\right)^n - 2\right)^{m-1}.$$

2 Propositions and Lemmas

We first present some obvious and some less obvious propositions of the sequences u_p, v_p, τ_p, h_p and g_p.

Note that $\alpha^p \mp \beta^p = (\alpha+\beta)(\alpha^{p-1} \mp \beta^{p-1}) - \alpha\beta(\alpha^{p-2} \mp \beta^{p-2})$. With the above definitions, it is easy to see that $\alpha+\beta = m+2$ and $\alpha\beta = 1$. So we have the following Proposition 2.1.

Proposition 2.1 *If* p *is integral, then*

$$\begin{cases} u_p = (m+2)u_{p-1} - u_{p-2}, \\ u_0 = 0, \quad u_1 = 1, \\ v_p = (m+2)v_{p-1} - v_{p-2}, \\ v_0 = 2, \quad v_1 = m+2. \end{cases} \tag{2.1}$$

From (2.1), it is easy to see that for every integer p, u_p and v_p are integral. The following Proposition 2.2 can be proved by induction on p.

Proposition 2.2 *If* p *is integral, then*

$$u_p \equiv p \pmod{m}, \quad v_p \equiv 2 \pmod{m}. \tag{2.2}$$

By (2.2), we see that $m \mid (p-u_p)$, i.e., τ_p is integral for $p \in \mathbb{Z}$. In fact, we further have the following Proposition 2.3.

Proposition 2.3 *If* p *is integral, then*

$$\tau_p = (m+2)\tau_{p-1} - \tau_{p-2} - (p-1). \tag{2.3}$$

Proof Since $u_p = p - m\tau_p$, it follows from Proposition 2.1 that $p - m\tau_p = (m+2)(p - 1 - m\tau_{p-1}) - (p - 2 - m\tau_{p-2})$. So $m\tau_p = m(m+2)\tau_{p-1} - m\tau_{p-2} - m(p-1)$, and then (2.3) holds. □

Proposition 2.4 *If p is a nonnegative integer, then*

$$\begin{aligned} &u_{p-1}u_{p+1} - u_p^2 - 1 + (u_{p+1} - u_{p-1}) \\ &\quad = v_p - 2 = \begin{cases} mh_s^2, & \text{if } p = 2s+1, \\ m(m+4)u_s^2, & \text{if } p = 2s. \end{cases} \end{aligned} \tag{2.4}$$

Proof A direct calculation can show

$$\begin{aligned} &u_{p-1}u_{p+1} - u_p^2 - 1 + (u_{p+1} - u_{p-1}) \\ &= \frac{\alpha^{p-1} - \beta^{p-1}}{\alpha - \beta} \cdot \frac{\alpha^{p+1} - \beta^{p+1}}{\alpha - \beta} - \left(\frac{\alpha^p - \beta^p}{\alpha - \beta}\right)^2 - 1 + \left(\frac{\alpha^{p+1} - \beta^{p+1}}{\alpha - \beta} - \frac{\alpha^{p-1} - \beta^{p-1}}{\alpha - \beta}\right) \\ &= -1 - 1 + \alpha^p + \beta^p = v_p - 2. \end{aligned}$$

So the first equality of (2.4) holds. Now we verify the second equality.

If $p = 2s+1$, then

$$\begin{aligned} mh_s^2 &= m(u_{s+1} + u_s)^2 = \frac{m}{(\alpha - \beta)^2}(\alpha^{s+1} - \beta^{s+1} + \alpha^s - \beta^s)^2 \\ &= \frac{m}{m^2 + 4m}(\alpha^{2s+2} + \beta^{2s+2} + \alpha^{2s} + \beta^{2s} + 2\alpha^{2s+1} + 2\beta^{2s+1} - 2 - 2 - 2\alpha - 2\beta) \\ &= \frac{1}{m+4}((m+4)v_p - 2(m+4)) = v_p - 2. \end{aligned}$$

If $p = 2s$, then

$$(m+4)u_s^2 = \frac{m(m+4)}{(\alpha - \beta)^2}(\alpha^s - \beta^s)^2 = \alpha^{2s} + \beta^{2s} - 2 = v_p - 2.$$ □

Proposition 2.5 *If p is integral, then*

$$(u_{p+1} - 1,\ u_p) = (u_p,\ u_{p-1} + 1) = \begin{cases} h_s, & \text{if } p = 2s+1, \\ (m,\ 2)u_s, & \text{if } p = 2s. \end{cases} \tag{2.5}$$

Proof For $i \in \mathbb{Z}$, set $\theta_i := u_{p-i} + u_i$. Note that $\alpha\beta = 1$ implies that $u_{-i} = -u_i$. Then it follows from (2.1) that

$$\begin{aligned} \theta_{i+1} &= u_{p-i-1} + u_{i+1} = -u_{i+1-p} + u_{i+1} \\ &= -((m+2)u_{i-p} - u_{i-1-p}) + (m+2)u_i - u_{i-1} \\ &= (m+2)u_{p-i} + u_{i-1-p} + (m+2)u_i - u_{i-1} \\ &= (m+2)(u_{p-i} + u_i) - (u_{p-(i-1)} + u_{i-1}) \\ &= (m+2)\theta_i - \theta_{i-1}. \end{aligned} \tag{2.6}$$

Here we are using the fact that $(a, b) = (a, ax - b)$ for $a, b, x \in \mathbb{Z}$. Thus

$$\begin{aligned} (u_{p+1} - 1,\ u_p) &= (\theta_{-1}, \theta_0) = ((m+2)\theta_0 - \theta_{-1}, \theta_0) = (\theta_1, \theta_0) \\ &= (u_p,\ u_{p-1} + 1). \end{aligned}$$

Moreover, $(\theta_0, \theta_1) = (\theta_1, (m+2)\theta_1 - \theta_0) = (\theta_1, \theta_2) = \cdots = (\theta_{s-1}, \theta_s)$, where $s = \lfloor \frac{p}{2} \rfloor$.

If $p = 2s+1$, then $\theta_{s-1} = u_{s+2} + u_{s-1} = (m+1)h_s$ and $\theta_s = u_{s+1} + u_s = h_s$. Thus

$$(\theta_{s-1}, \theta_s) = h_s.$$

If $p = 2s$, then $\theta_{s-1} = u_{s+1} + u_{s-1} = (m+2)u_s$ and $\theta_s = 2u_s$. Therefore,

$$(\theta_{s-1}, \theta_s) = ((m+2)u_s, 2u_s) = (m,\ 2)u_s. \qquad \square$$

The following Lemmas 2.6 and 2.7 will be used in the proof of Theorem 1.1.

Lemma 2.6 *For $n \in \mathbb{N}$, let $B = \begin{pmatrix} n & \tau_{n-1} & \tau_n \\ 0 & \tau_n - \tau_{n-1} & \tau_{n+1} - \tau_n \\ 0 & u_n & u_{n+1} - 1 \end{pmatrix}$ and $\mathrm{diag}(s_1(B), s_2(B), s_3(B))$ be its Smith normal form.*

If $n = 2s+1$, then

$$\begin{cases} s_1(B) = (n, g_s), \\ s_2(B) = h_s, \\ s_3(B) = \dfrac{nh_s}{(n, g_s)}. \end{cases} \tag{2.7}$$

If $n = 2s$, then

$$\begin{cases} s_1(B) = (u_s, 2\tau_s), \\ s_2(B) = \dfrac{u_s\,(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}, \\ s_3(B) = \dfrac{n(m+4)u_s}{(n, u_s - 4\tau_s)}. \end{cases} \tag{2.8}$$

Proof Recall that $s_1(B)$ equals the greatest common divisor of all entries of B. So

$$\begin{aligned} s_1(B) &= (n, \tau_{n-1}, \tau_n, \tau_n - \tau_{n-1}, \tau_{n+1} - \tau_n, u_n, u_{n+1} - 1) \\ &= (n, \tau_{n-1}, \tau_n, \tau_{n+1}, u_n, u_{n+1} - 1). \end{aligned}$$

Since we have (2.3) and

$$\begin{cases} u_n = n - m\tau_n, \\ u_{n+1} - 1 = (m+1)n + m\tau_{n-1} - m(m+2)\tau_n, \end{cases}$$

it follows that

$$s_1(B) = (n, \tau_n, \tau_{n-1}) = \left(n, \frac{1}{m}(n - \theta_0),\ \frac{1}{m}(n - \theta_1)\right). \tag{2.9}$$

From (2.6), we have $\frac{1}{m}(n - \theta_2) = (m+2)\frac{1}{m}(n - \theta_1) - \frac{1}{m}(n - \theta_0) - n$. Therefore

$$\begin{aligned} s_1(B) &= \left(n, \frac{1}{m}(n - \theta_0),\ \frac{1}{m}(n - \theta_1)\right) \\ &= \left(n, \frac{1}{m}(n - \theta_1),\ \frac{1}{m}(n - \theta_2)\right) \\ &= \cdots \\ &= \left(n, \frac{1}{m}(n - \theta_{s-1}),\ \frac{1}{m}(n - \theta_s)\right), \end{aligned}$$

where $s = \lfloor \frac{n}{2} \rfloor$.

• If $n = 2s+1$, then $\theta_{s-1} = u_{s+2} + u_{s-1} = (m+1)h_s$ and $\theta_s = u_{s+1} + u_s = h_s$. There results that

$$s_1(B) = \left(n, \frac{1}{m}(n-(m+1)h_s), \frac{1}{m}(n-h_s)\right) = (n, g_s)\,. \tag{2.10}$$

• If $n = 2s$, then $\theta_{s-1} = u_{s+1} + u_{s-1} = (m+2)u_s$ and $\theta_s = 2u_s$. There results that

$$s_1(B) = \left(n, \frac{1}{m}(n-(m+2)u_s), \frac{1}{m}(n-2u_s)\right) = (2s, 2\tau_s, u_s) = (u_s, 2\tau_s). \tag{2.11}$$

Recall that $s_1(B)s_2(B)$ equals the greatest common divisor of all 2×2 minors of B. So

$$s_1(B)s_2(B) = (\Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{22}, \Delta_{23}, \Delta_{31}, \Delta_{32}, \Delta_{33}),$$

where Δ_{ij} is the determinant of the submatrix formed by deleting the i-th row and j-th column of B for $1 \le i,\ j \le 3$. It is straightforward to see that

$$\begin{aligned}
\Delta_{11} &= \det\begin{pmatrix} \tau_n - \tau_{n-1} & \tau_{n+1} - \tau_n \\ u_n & u_{n+1} - 1 \end{pmatrix} = \frac{1}{m}(u_{n-1}u_{n+1} - u_n^2 - 1 + (u_{n+1} - u_{n-1})) \\
&\overset{(2.4)}{=\!=} \frac{1}{m}(v_n - 2); \\
\Delta_{12} &= \det\begin{pmatrix} 0 & \tau_{n+1} - \tau_n \\ 0 & u_{n+1} - 1 \end{pmatrix} = 0; \\
\Delta_{13} &= \det\begin{pmatrix} 0 & \tau_n - \tau_{n-1} \\ 0 & u_n \end{pmatrix} = 0; \\
\Delta_{21} &= \det\begin{pmatrix} \tau_{n-1} & \tau_n \\ u_n & u_{n+1} - 1 \end{pmatrix} = \frac{1}{m}((n-1-u_{n-1})(u_{n+1}-1) - (n-u_n)u_n) \\
&= \frac{1}{m}(n(u_{n+1} - 1 - u_n)) - \frac{1}{m}(u_{n-1}u_{n+1} - u_n^2 - 1 + (u_{n+1} - u_{n-1})) \\
&\overset{(2.4)}{=\!=} n(\tau_n - \tau_{n+1}) - \frac{1}{m}(v_n - 2); \\
\Delta_{22} &= \det\begin{pmatrix} n & \tau_n \\ 0 & u_{n+1} - 1 \end{pmatrix} = n(u_{n+1} - 1); \\
\Delta_{23} &= \det\begin{pmatrix} n & \tau_{n-1} \\ 0 & u_n \end{pmatrix} = nu_n; \\
\Delta_{31} &= \det\begin{pmatrix} \tau_{n-1} & \tau_n \\ \tau_n - \tau_{n-1} & \tau_{n+1} - \tau_n \end{pmatrix} \\
&= \frac{1}{m^2}((n-1-u_{n-1})(u_n - u_{n+1} + 1) - (n-u_n)(u_{n-1} + 1 - u_n)) \\
&= \frac{1}{m^2}((u_{n-1}u_{n+1} - u_n^2 - 1 + (-u_{n-1} + u_{n+1})) + 2nu_n - n(u_{n+1} + u_{n-1})) \\
&\overset{(2.4)}{=\!=} \frac{1}{m}(\frac{v_n - 2}{m} - nu_n);
\end{aligned}$$

$$\Delta_{32} = \det \begin{pmatrix} n & \tau_n \\ 0 & \tau_{n+1} - \tau_n \end{pmatrix} = n(\tau_{n+1} - \tau_n);$$

$$\Delta_{33} = \det \begin{pmatrix} n & \tau_{n-1} \\ 0 & \tau_n - \tau_{n-1} \end{pmatrix} = n(\tau_n - \tau_{n-1}).$$

Note that $\Delta_{33} = \Delta_{23} + \Delta_{32}$, $\Delta_{21} = -\Delta_{32} - \Delta_{11}$ and $\Delta_{11} = m\Delta_{31} + \Delta_{23}$. Recall that $\theta_i = u_{n-i} + u_i$, so

$$\begin{aligned} s_1(B)s_2(B) &= (\Delta_{22}, \Delta_{23}, \Delta_{31}, \Delta_{32}) \\ &= \left(n(u_{n+1} - 1), nu_n, \frac{1}{m}\left(\frac{v_n - 2}{m} - nu_n\right), n(\tau_{n+1} - \tau_n) \right) \\ &= \left(n\theta_{-1}, n\theta_0, \frac{1}{m}\left(\frac{v_n - 2}{m} - n\theta_0\right), \frac{n}{m}(\theta_0 - \theta_{-1}) \right). \end{aligned} \tag{2.12}$$

With the aid of (2.6), it is easy to verify that $\frac{n}{m}(\theta_0 - \theta_{-1}) + n\theta_0 = \frac{n}{m}(\theta_1 - \theta_0)$. Moreover, we have $\frac{1}{m}(\frac{v_n-2}{m} - n\theta_0) - \frac{n}{m}(\theta_1 - \theta_0) = \frac{1}{m}(\frac{v_n-2}{m} - n\theta_1)$. From (2.12), it follows that

$$\begin{aligned} s_1(B)s_2(B) &= \left(n\theta_0, n\theta_1, \frac{n}{m}(\theta_1 - \theta_0), \frac{1}{m}\left(\frac{v_n - 2}{m} - n\theta_1\right) \right) \\ &= \left(n\theta_1, n\theta_2, \frac{n}{m}(\theta_2 - \theta_1), \frac{1}{m}\left(\frac{v_n - 2}{m} - n\theta_2\right) \right) \\ &= \cdots \\ &= \left(n\theta_{s-1}, n\theta_s, \frac{n}{m}(\theta_s - \theta_{s-1}), \frac{1}{m}\left(\frac{v_n - 2}{m} - n\theta_s\right) \right), \end{aligned} \tag{2.13}$$

where $s = \lfloor \frac{n}{2} \rfloor$.

• If $n = 2s+1$, then $\theta_{s-1} = u_{s+2} + u_{s-1} = (m+1)h_s$ and $\theta_s = h_s$. So from (2.4) and (2.13), we can see that

$$\begin{aligned} s_1(B)s_2(B) &= \left(n(m+1)h_s, nh_s, nh_s, \frac{1}{m}(h_s^2 - nh_s) \right) \\ &= \left(nh_s, \frac{1}{m}(h_s^2 - nh_s) \right) \\ &= h_s \left(n, \frac{1}{m}(h_s - n) \right) = h_s(n, g_s). \end{aligned} \tag{2.14}$$

• If $n = 2s$, then $\theta_{s-1} = u_{s+1} + u_{s-1} = (m+2)u_s$, and $\theta_s = 2u_s$. So from (2.4) and (2.13), we can see that

$$\begin{aligned} s_1(B)s_2(B) &= \left(n\theta_{s-1}, n\theta_s, \frac{n}{m}(\theta_s - \theta_{s-1}), \frac{1}{m}((m+4)u_s^2 - 2nu_s) \right) \\ &= \left(nu_s, \frac{1}{m}((m+4)u_s^2 - 2nu_s) \right) \\ &= u_s \left(n, \frac{1}{m}((m+4)u_s - 2n) \right) \\ &= u_s \left(n, u_s - 4\tau_s \right). \end{aligned} \tag{2.15}$$

Recall that $s_1(B)s_2(B)s_3(B)$ equals the determinant of B. So

$$s_1(B)s_2(B)s_3(B) = \det(B) = n\Delta_{11}(B) = \frac{n}{m}(v_n - 2).$$

Thus we have

$$\begin{cases} s_1(B)s_2(B)s_3(B) = nh_s^2, & \text{if } n = 2s+1, \\ s_1(B)s_2(B)s_3(B) = n(m+4)u_s^2, & \text{if } n = 2s. \end{cases} \tag{2.16}$$

Combining (2.10), (2.11), (2.14), (2.15) and (2.16), we obtain the formulas (2.7) and (2.8). □

Lemma 2.7 *For $n \in \mathbb{N}$, let $W = \begin{pmatrix} u_{n-1}+1 & u_n \\ u_n & u_{n+1}-1 \end{pmatrix}$, and $\mathrm{diag}(s_1(W), s_2(W))$ be its Smith normal form.*

If $n = 2s+1$, then

$$\begin{cases} s_1(W) = h_s, \\ s_2(W) = mh_s. \end{cases}$$

If $n = 2s$, then

$$\begin{cases} s_1(W) = (m,2)u_s, \\ s_2(W) = \dfrac{m(m+4)u_s}{(m,2)}. \end{cases}$$

Proof Recall that $s_1(W)$ equals the greatest common divisor of all the entries of W. So

$$\begin{aligned} s_1(W) &= (u_{n-1}+1, u_n, u_{n+1}-1) = (u_{n-1}+1, u_n) \\ &\overset{(2.5)}{=\!=} \begin{cases} s_1(W) = h_s, & \text{if } n = 2s+1, \\ s_1(W) = (m,2)u_s, & \text{if } n = 2s. \end{cases} \end{aligned} \tag{2.17}$$

Recall that $s_1(W)s_2(W)$ equals the greatest common divisor of all 2×2 minors of W. So

$$\begin{aligned} s_1(W)s_2(W) &= \det(W) = u_{n-1}u_{n+1} - u_n^2 - 1 + (u_{n+1} - u_{n-1}) \\ &\overset{(2.4)}{=\!=} \begin{cases} s_1(W)s_2(W) = mh_s^2, & \text{if } n = 2s+1, \\ s_1(W)s_2(W) = m(m+4)u_s^2, & \text{if } n = 2s. \end{cases} \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18), we can obtain

$$s_2(W) = \begin{cases} mh_s, & \text{if } n = 2s+1, \\ \dfrac{m(m+4)u_s}{(m,2)}, & \text{if } n = 2s. \end{cases} \tag{2.19}$$

3 Proofs of Theorem 1.1 and Corollary 1.2

Observe that the critical group of graph G is completely determined by the cokernel of $L(G)$. Thus, it is sufficient to compute the Smith normal form of the Laplacian matrix $L(G)$.

The proof of Theorem 1.1 contains the following steps:

(1) First, we prove that there is a matrix $A \in \mathbb{Z}^{2m\times 2m}$ such that $L(G) \sim I_{nm-2m} \oplus A$ (see (3.9) and (3.10)).

(2) Next, we prove there are two matrices $B \in \mathbb{Z}^{3\times 3}$ and $W \in \mathbb{Z}^{2\times 2}$ such that $A \sim 0_1 \oplus B \oplus \underbrace{W \oplus \cdots \oplus W}_{m-2}$; the Smith normal forms of B and W are given in Lemmas 2.6 and 2.7 respectively.

(3) Finally, we compute the Smith normal form of A from those of B and W.

After the three steps, the Smith normal form of $L(G)$ will be obtained.

Step 1 Now we work on the system of relations of the cokernel of the Laplacian of $K_m \times C_n$. Let $e_{i,j} = (0, \ldots, 0, 1, 0, \ldots, 0)^t \in \mathbb{Z}^{mn}$, whose unique nonzero 1 is in the position corresponding to vertex $v_{i,j}$, and let $x_{i,j}$ be its image in $\mathrm{coker}(L(K_m \times C_n))$. Then it follows from the relations (1.2) of $\mathrm{coker}(L(K_m \times C_n))$ that we can get the system of equations:

$$(m+1)x_{i,j} - \sum_{k \in \mathbb{Z}_m, k \neq j} x_{i,k} - x_{i-1,j} - x_{i+1,j} = 0, \quad i \in \mathbb{Z}_n,\ j \in \mathbb{Z}_m. \tag{3.1}$$

Let $M_i = \sum_{j \in \mathbb{Z}_m} x_{i,j}$, for $i \in \mathbb{Z}_n$. Then from (3.1) we have

$$(m+1)M_i - (m-1)M_i - M_{i+1} - M_{i-1} = 0. \tag{3.2}$$

This identity implies that

$$M_{i+1} = 2M_i - M_{i-1}. \tag{3.3}$$

Recursively using identity (3.3), we can rewrite all M_i's as integral linear combinations of M_0 and M_1.

$$M_i = iM_1 - (i-1)M_0, \quad 2 \leq i \leq n-1. \tag{3.4}$$

So from (3.1) and (3.4), we have

$$x_{i,j} = (m+2)x_{i-1,j} - x_{i-2,j} + (i-2)M_0 - (i-1)M_1, \tag{3.5}$$

where $2 \leq i \leq n-1$, $0 \leq j \leq m-1$.

Lemma 3.1 *For $0 \leq i \leq n-1$, $0 \leq j \leq m-1$, we have*

$$x_{i,j} = -u_{i-1}x_{0,j} + u_i x_{1,j} - \tau_{i-1}M_0 + \tau_i M_1. \tag{3.6}$$

Proof This lemma is valid in the cases $i = 0, 1, 2$. Suppose that $x_{l,j} = -u_{l-1}x_{0,j} + u_l x_{1,j} - \tau_{l-1}M_0 + \tau_l M_1$, for $l \leq h-1$, where $h \geq 3$. Then from the induction assumption and the equations (3.5), it follows that

$$\begin{aligned}
x_{h,j} &= (m+2)x_{h-1,j} - x_{h-2,j} + (h-2)M_0 - (h-1)M_1, \\
&= (m+2)\left(-u_{h-2}x_{0,j} + u_{h-1}x_{1,j} - \tau_{h-2}M_0 + \tau_{h-1}M_1\right) \\
&\quad - \left(-u_{h-3}x_{0,j} + u_{h-2}x_{1,j} - \tau_{h-3}M_0 + \tau_{h-2}M_1\right) + (h-2)M_0 - (h-1)M_1 \\
&= \left(-(m+2)u_{h-2} + u_{h-3}\right)x_{0,j} + \left((m+2)u_{h-1} - u_{h-2}\right)x_{1,j} \\
&\quad + \left(-(m+2)\tau_{h-2} + \tau_{h-3} + (h-2)\right)M_0 + \left((m+2)\tau_{h-1} - \tau_{h-2} - (h-1)\right)M_1 \\
&= -u_{h-1}x_{0,j} + u_h x_{1,j} - \tau_{h-1}M_0 + \tau_h M_1.
\end{aligned}$$

Recalling (2.1) and (2.3), we know that (3.6) holds by induction. □

In view of Lemma 3.1, we only need at most $2m$ generators for the system of equations (3.1). Indeed each $x_{i,j}$ can be expressed in terms of $x_{0,0}, x_{1,0}, x_{0,1}, x_{1,1}, \ldots, x_{0,m-1}, x_{1,m-1}$. So we know that there are at least $nm - 2m$ diagonal entries of the Smith normal form of $L(G)$ equal to 1 and the remaining invariant factors of the abelian group $\mathrm{coker}L(K_m \times C_n)$ are the diagonal entries of the Smith normal form of the relations matrix induced by $x_{0,0}, x_{1,0}, x_{0,1}, x_{1,1}, \ldots, x_{0,m-1}, x_{1,m-1}$.

From (3.6) and the cyclic structure of $K_m \times C_n$, it follows that, for $0 \leq j \leq m-1$,

$$\begin{cases} x_{0,j} = x_{n,j} = -u_{n-1}x_{0,j} + u_n x_{1,j} - \tau_{n-1}M_0 + \tau_n M_1, \\ x_{1,j} = x_{n+1,j} = -u_n x_{0,j} + u_{n+1}x_{1,j} - \tau_n M_0 + \tau_{n+1}M_1. \end{cases} \tag{3.7}$$

Therefore, for $0 \le j \le m-1$,

$$\begin{cases} (-u_{n-1}-\tau_{n-1}-1)x_{0,j}+(u_n+\tau_n)x_{1,j}-\tau_{n-1}\sum\limits_{k\neq j}x_{0,k}+\tau_n\sum\limits_{k\neq j}x_{1,k}=0, \\ (-u_n-\tau_n)x_{0,j}+(u_{n+1}+\tau_{n+1}-1)x_{1,j}-\tau_n\sum\limits_{k\neq j}x_{0,k}+\tau_{n+1}\sum\limits_{k\neq j}x_{1,k}=0. \end{cases} \tag{3.8}$$

Let

$$E=\begin{pmatrix} -u_{n-1}-1-\tau_{n-1} & u_n+\tau_n \\ -u_n-\tau_n & u_{n+1}-1+\tau_{n+1} \end{pmatrix}, \quad F=\begin{pmatrix} -\tau_{n-1} & \tau_n \\ -\tau_n & \tau_{n+1} \end{pmatrix},$$

and

$$Y=(x_{0,0},x_{1,0},x_{0,1},x_{1,1},\ldots,x_{0,m-1},x_{1,m-1})^T.$$

Then from the equalities in (3.8), we have that

$$AY=0, \tag{3.9}$$

where

$$A=\begin{pmatrix} E & F & F & \cdots & F \\ F & E & F & \cdots & F \\ F & F & E & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F & F & F & \cdots & E \end{pmatrix} \in \mathbb{Z}^{2m\times 2m}. \tag{3.10}$$

Step 2 The matrix A in equation (3.9) is the relation matrix induced by the generators $x_{0,0},x_{1,0},x_{0,1},x_{1,1},\ldots,x_{0,m-1},x_{1,m-1}$. We now discuss the Smith normal form of the relation matrix A.

Let

$$H=\begin{pmatrix} I_2 & 0 & 0 & \cdots & 0 \\ -(m-1)I_2 & I_2 & I_2 & \cdots & I_2 \\ -I_2 & 0 & I_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -I_2 & 0 & 0 & \cdots & I_2 \end{pmatrix} \in \mathbb{Z}^{2m\times 2m},$$

where I_2 is the 2×2 identity matrix. Then it is not difficult to verify that

$$H^{-1}=\begin{pmatrix} I_2 & 0 & 0 & \cdots & 0 \\ I_2 & I_2 & -I_2 & \cdots & -I_2 \\ I_2 & 0 & I_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_2 & 0 & 0 & \cdots & I_2 \end{pmatrix} \in \mathbb{Z}^{2m\times 2m}.$$

By a direct calculation, we have

$$HAH^{-1} = \begin{pmatrix} E+(m-1)F & F & 0 & 0 & 0 \\ 0 & E-F & 0 & 0 & 0 \\ 0 & 0 & E-F & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & E-F \end{pmatrix}. \tag{3.11}$$

Note that

$$\begin{pmatrix} E+(m-1)F & F \\ 0 & E-F \end{pmatrix} = \begin{pmatrix} -n & n & -\tau_{n-1} & \tau_n \\ -n & n & -\tau_n & \tau_{n+1} \\ 0 & 0 & -u_{n-1}-1 & u_n \\ 0 & 0 & -u_n & u_{n+1}-1 \end{pmatrix}.$$

Let $Q_1 = \begin{pmatrix} m & -m & 1 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then it is easy to see that Q_1 and Q_2 are unimodular matrices, and a careful calculation can show

$$Q_1 \begin{pmatrix} E+(m-1)F & F \\ 0 & E-F \end{pmatrix} Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \tag{3.12}$$

where the matrix B is just the one defined in Lemma 2.6.

Note that $(E-F)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_{n-1}+1 & u_n \\ u_n & u_{n+1}-1 \end{pmatrix} = W$, which is just the one considered in Lemma 2.7. Therefore, from (3.11) and (3.12) we have

$$\begin{aligned} A &\sim HAH^{-1} \sim 0_1 \oplus B \oplus \underbrace{W \oplus \cdots \oplus W}_{m-2}. \\ &\sim \operatorname{diag}(s_1(B), s_2(B), s_3(B), s_1(W), \ldots, s_1(W), s_2(W), \ldots, s_2(W), 0) \\ &\sim \operatorname{diag}(s_1(B), s_2(B), s_1(W), \ldots, s_1(W), s_2(W), \ldots, s_2(W), s_3(B), 0). \end{aligned} \tag{3.13}$$

Step 3 Now we distinguish two cases to compute the Smith normal form of A.

Case 1 $n = 2s+1$.

Then by Lemmas 2.6 and 2.7 we have $s_1(B) = (n, g_s)$, $s_2(B) = h_s$, $s_3(B) = \frac{nh_s}{(n,g_s)}$, $s_1(W) = h_s$, $s_2(W) = mh_s$. Since $s_1(B)|s_2(B)$, $s_1(W)|s_2(W)$ and $s_2(B) = s_1(W)$, it follows that $s_1(B)|s_2(W)$.

Write $\gamma = (s_2(W), s_3(B))$ and $\varphi = \frac{s_2(W)s_3(B)}{\gamma}$. Then

$$\operatorname{diag}(s_2(W), s_3(B)) \sim \operatorname{diag}(\gamma, \varphi), \tag{3.14}$$

where

$$\begin{aligned} \gamma &= \left(mh_s, \frac{nh_s}{(n,g_s)}\right) = \frac{h_s}{(n,g_s)}(m(n,g_s), n) \\ &= \frac{h_s}{(n,g_s)}(mn, n-h_s, n) \quad (\text{here } mg_s = n-h_s.) \\ &= \frac{h_s}{(n,g_s)}(n, h_s), \end{aligned} \tag{3.15}$$

and

$$\varphi = \frac{nmh_s}{(n, h_s)}. \tag{3.16}$$

Note that $s_1(W) = s_2(B)$, it implies that $s_1(W)|s_3(B)$. So $s_1(W)|\gamma|s_2(W)$. Moreover, it is easy to see that $s_2(W) \mid \varphi$.

Therefore, (3.13) and (1.14) implies that

$$\operatorname{diag}(s_1(B), s_2(B), \underbrace{s_1(W), \ldots, s_1(W)}_{m-2}, \gamma, \underbrace{s_2(W), \ldots, s_2(W)}_{m-3}, \varphi, 0) \tag{3.17}$$

is the Smith normal form of A.

Case 2 $n = 2s$.

Then by Lemmas 2.6 and 2.7, we know that in this case

$$s_1(B) = (u_s, 2\tau_s) \quad s_2(B) = \frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}, \quad s_3(B) = \frac{n(m+4)u_s}{(n, u_s - 4\tau_s)},$$
$$s_1(W) = (m, 2)u_s, \quad s_2(W) = \frac{m(m+4)u_s}{(m, 2)}.$$

It is obvious that

$$s_1(B)|s_2(B)|s_3(B), \quad s_1(B)|s_1(W)|s_2(W), \quad s_2(B)|s_2(W). \tag{3.18}$$

Clearly, we have

$$\operatorname{diag}(s_2(W), s_3(B)) \sim \operatorname{diag}(\rho, \xi), \tag{3.19}$$

where

$$\begin{aligned}\rho = (s_2(W), s_3(B)) &= \left(\frac{m(m+4)u_s}{(m, 2)}, \frac{n(m+4)u_s}{(n, u_s - 4\tau_s)}\right) \\ &= \frac{(m+4)u_s}{(n, u_s - 4\tau_s)(m, 2)}(mn, (m+4)u_s - 2n, mn, 2n) \\ &= \frac{(m+4)u_s(mn, (m+4)u_s, 2n)}{(n, u_s - 4\tau_s)(m, 2)},\end{aligned} \tag{3.20}$$

and

$$\xi = \frac{s_2(W)s_3(B)}{\rho} = \frac{mn(m+4)u_s}{(mn,\ (m+4)u_s,\ 2n)}. \tag{3.21}$$

Therefore, It follows from (3.13) that

$$A \sim \operatorname{diag}(s_1(B), s_2(B), \overbrace{s_1(W), \ldots, s_1(W)}^{m-2}, \rho, \overbrace{s_2(W), \ldots, s_2(W)}^{m-3}, \xi, 0). \tag{3.22}$$

We also have

$$\operatorname{diag}(s_2(B), s_1(W)) \sim \operatorname{diag}(\zeta, \eta), \tag{3.23}$$

where

$$\begin{aligned}\zeta = (s_2(B), s_1(W)) &= \left(\frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}, (m, 2)u_s\right) \\ &= \frac{u_s\,(n, u_s - 4\tau_s, mu_s, 2m\tau_s, 2u_s, 4\tau_s)}{(u_s, 2\tau_s)} \\ &= \frac{u_s\,(n, u_s, 4\tau_s)}{(u_s, 2\tau_s)},\end{aligned} \tag{3.24}$$

and

$$\eta = \frac{s_2(B)s_1(W)}{\zeta} = \frac{u_s(m,2)\,(n, u_s - 4\tau_s)}{(n, u_s, 4\tau_s)}. \tag{3.25}$$

Combining (3.22) and (3.23), we have

$$A \sim \operatorname{diag}(s_1(B), \zeta, \underbrace{s_1(W), \ldots, s_1(W)}_{m-3}, \eta, \rho, \underbrace{s_2(W), \ldots, s_2(W)}_{m-3}, \xi, 0).$$

Since $(n, u_s - 4\tau_s)\,|n$, and $(m,2)\,u_s|(m+4)u_s$, it follows that $s_1(W)|s_3(B)$, and hence $s_1(W)|(s_2(W), s_3(B))$. Furthermore, since $s_2(B)|s_3(B)$ and $s_2(B)|s_2(W)$, it implies that $s_2(B)$ divides $(s_2(W), s_3(B))$. So, $(s_2(W), s_3(B))$, i.e., ρ is a common multiple of $s_1(W)$ and $s_2(B)$. Note that η is the least common multiple of $s_1(W)$ and $s_2(B)$, it divides ρ. According to (3.18), it is easy to see $s_1(B)|\zeta$. And it is clear that we have $s_1(W)|\eta$ and $s_2(W)|\xi$. Thus

$$\operatorname{diag}(s_1(B), \zeta, \underbrace{s_1(W), \ldots, s_1(W)}_{m-3}, \eta, \rho, \underbrace{s_2(W), \ldots, s_2(W)}_{m-3}, \xi, 0) \tag{3.26}$$

is the Smith normal form of A.

Now, the proof of Theorem 1.1 is completed. □

Proof of Corollary 1.2 If $n = 2s+1$, then we have $s_1(W)s_2(W) = mh_s^2 \overset{(2.4)}{=} v_n - 2$ and $s_1(B)s_2(B)s_1(W)\gamma\varphi = (n, g_s)h_s h_s \frac{h_s}{(n,g_s)}(n, h_s)\frac{nmh_s}{(n,h_s)} = nmh_s^4 \overset{(2.4)}{=} \frac{n}{m}(v_n-2)^2$. It follows that the spanning tree number of $K_m \times C_n$ is $\frac{n}{m}(v_n-2)^2 \times (v_n-2)^{m-3} = \frac{n}{m}(v_n-2)^{m-1}$.

If $n = 2s$, then $s_1(W)s_2(W) = (m,2)\,u_s\frac{m(m+4)u_s}{(m,2)} = m(m+4)u_s^2 \overset{(2.4)}{=} v_n - 2$ and $s_1(B)\zeta\eta\rho\xi = nm(m+4)^2u_s^4 \overset{(2.4)}{=} \frac{n}{m}(v_n-2)^2$. It follows that the spanning tree number of $K_m \times C_n$ is $\frac{n}{m}(v_n-2)^2 \times (v_n-2)^{m-3} = \frac{n}{m}(v_n-2)^{m-1}$. (In fact, by (3.13) we know that the spanning tree number of graph $K_m \times C_n$ is $|\det(\operatorname{diag}(B \oplus \underbrace{W \oplus \cdots \oplus W}_{m-2}))| = \det(B) \times (\det(W))^{m-2}$. In the proofs of Lemmas 2.6 and 2.7, we have seen that $\det(B) = \frac{n}{m}(v_n-2)$ and $\det(W) = v_n - 2$. So the spanning tree number is $\frac{n}{m}(v_n-2)^{m-1}$.) □

4 Remarks

(I) If $n = 1$, then $K_m \times C_n$ is the complete graph K_m; from [9] we know its critical group is $(\mathbb{Z}_m)^{m-2}$.

(II) If $n = 2$, then $\operatorname{coker}(K_m \times C_2)$ is determined by the generators $x_{0,j}$, $x_{1,j}$, and the relations

$$\begin{cases} (m+1)x_{0,j} - \sum\limits_{\substack{k\in\mathbb{Z}_m\\ k\neq j}} x_{0,k} - 2x_{1,j} = 0, & (4.1)\\[2ex] (m+1)x_{1,j} - \sum\limits_{\substack{k\in\mathbb{Z}_m\\ k\neq j}} x_{1,k} - 2x_{0,j} = 0, & (4.2)\end{cases}$$

where $j \in \mathbb{Z}_m$. From (4.1), we get

$$2x_{1,j} = (m+1)x_{0,j} - \sum_{\substack{k\in\mathbb{Z}_m\\ k\neq j}} x_{0,k}, \quad j \in \mathbb{Z}_m. \tag{4.3}$$

Substituting (4.3) into (4.2) $\times$ 2 gives the following

$$(m+4)(m-1)x_{0,j} - \sum_{\substack{k\in\mathbb{Z}_m\\ k\neq j}} (m+4)x_{0,k} = 0, \quad j \in \mathbb{Z}_m.$$

So we can simplify (up to equivalence) the Laplacian matrix of $K_m \times C_2$ into

$$(m+4)\begin{pmatrix} m-1 & -1 & \cdots & -1 \\ -1 & m-1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & -1 \\ -1 & \cdots & -1 & m-1 \end{pmatrix}_{m\times m} = (m+4)L(K_m).$$

Therefore, the critical group of $K_m \times C_2$ is $\mathbb{Z}_{m+4} \oplus (\mathbb{Z}_{m(m+4)})^{m-2}$.

(III) If $m = 1$, then $K_m \times C_n$ is just the cycle C_n. So from [8], its critical group is $\mathbb{Z}_n$. In fact, when $m = 1$, it is easy to see the matrix A in (3.10) has the following property:

$$A = E = \begin{pmatrix} -n & n \\ -n & n \end{pmatrix} \sim (0) \oplus n.$$

The known result is obtained immediately.

(IV) If $m = 2$, the graph $K_2 \times C_n$ is just the Cayley graph $\mathcal{D}_n$ of dihedral group. The result of this case was obtained in [7]. In the following we will try to get the result again.

From (3.10) and (3.13), we know

$$A = \begin{pmatrix} E & F \\ F & E \end{pmatrix} \sim (0) \oplus B \sim \mathrm{diag}(s_1(B), s_2(B), s_3(B), 0).$$

• If $n = 2s+1$, then from (2.7) we have

$$\begin{cases} s_1(B) = (n, g_s), \\ s_2(B) = h_s, \\ s_3(B) = \dfrac{nh_s}{(n, g_s)}. \end{cases}$$

Note that $g_s = \frac{n-h_s}{2}$ and n is odd, then $(n, g_s) = (n, 2g_s) = (n, n-h_s) = (n, h_s)$. Therefore,

$$\begin{cases} s_1(B) = (n, h_s), \\ s_2(B) = h_s, \\ s_3(B) = \dfrac{nh_s}{(n, h_s)}. \end{cases}$$

• If $n = 2s$, then from (2.8) we have

$$\begin{cases} s_1(B) = (u_s, 2\tau_s), \\ s_2(B) = \dfrac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}, \\ s_3(B) = \dfrac{6nu_s}{(n, u_s - 4\tau_s)}. \end{cases}$$

Note that $s - u_s$ is even by (2.2), and $2^{t+1}|u_n$ if $2^t|n$ (see [7, Corollary 3.4]). Hence

$$\begin{aligned} s_1(B) &= (u_s, 2\tau_s) = (u_s, s-u_s) = (u_s, s) \\ &= \begin{cases} (u_s, 2s) = (u_s, n), & \text{if } s \text{ is odd}, \\ \dfrac{(u_s, 2s)}{2} = \dfrac{(u_s, n)}{2}, & \text{if } s \text{ is even}. \end{cases} \end{aligned}$$

Also we have $3^t|u_n$ if $3^t|n$ (see [7, Corollary 3.4]), then

$$(n, u_s - 2(s - u_s)) = (2s, 3u_s) = (n, u_s).$$

So

$$s_2(B) = \frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)} = \frac{u_s(n, u_s)}{(u_s, 2\tau_s)} = \begin{cases} u_s, & \text{if } s \text{ is odd,} \\ 2u_s, & \text{if } s \text{ is even.} \end{cases}$$

Then $s_3(B) = \frac{6nu_s}{(n, u_s - 4\tau_s)} = \frac{6nu_s}{(n, u_s)}$.

It is easy to see the result of the case $m = 2$ here is the same as the result obtained in [7].

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