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# The Menger number of the Cartesian product of graphs ${ }^{*}$ 

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#### Abstract

In a real-time system, the Menger number $\zeta_{l}(G)$ is an important measure of the communication efficiency and fault tolerance of the system G. In this paper, we obtain a lower bound for the Cartesian product graph. We show that $\zeta_{l_{1}+l_{2}}\left(G_{1} \times G_{2}\right) \geq \zeta_{l_{1}}\left(G_{1}\right)+\zeta_{l_{2}}\left(G_{2}\right)$ for $l_{1} \geq 2$ and $l_{2} \geq 2$. As an application of the result, we determine the exact values $\zeta_{1}(G)$ of the grid network $G=G\left(m_{1}, m_{2}, \ldots, m_{n}\right)$ for $m_{i} \geq 2(1 \leq i \leq n)$ and $l \geq \sum_{i=1}^{n} m_{i}$. This example shows that the lower bound of $\zeta_{l_{1}+l_{2}}\left(G_{1} \times G_{2}\right)$ obtained is tight.


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## 1. Introduction

It is well known that underlying topology of an interconnection network can be represented by a graph $G=(V, E)$, where $V$ is the set of processors and $E$ is the set of communication links in the network. Throughout this paper, a graph $G=(V, E)$ always means a connected and simple graph (without loops and multiple edges), where $V=V(G)$ and $E=E(G)$ are the vertex set and the edge set of $G$, respectively. For graph terminology and notation not defined here, we follow [1].

Let $x$ and $y$ be two distinct vertices in a graph $G=(V, E)$. A path between $x$ and $y$ is denoted by the term $x y$-path. The distance $d_{G}(x, y)$ between $x$ and $y$ is the number of edges in a shortest $x y$-path, and the diameter of $G$ is $d(G)=\max \left\{d_{G}(x, y)\right.$ : $x, y \in V(G)\}$. For a vertex $x \in V(G)$, the set of neighbors of $x$ is denoted by $N_{G}(x)$ in $G$ and the degree of $x$ is $d_{G}(x)=\left|N_{G}(x)\right|$. The minimum degree of $G$ is $\delta(G)=\min \left\{d_{G}(v): v \in V(G)\right\}$.

When we use a graph to model a parallel computing or processing system, we can use internally disjoint paths to transmit messages simultaneously from a vertex $x$ to another vertex $y$. However, in a real-time system, the message delay must be limited within a given period since any message obtained beyond the bound may be worthless. A natural question is how many internally disjoint paths exist in the network to ensure message delay within the effective bounds. In the language of graph theory, this problem can be stated as follows.

Let $x$ and $y$ be two distinct vertices in a graph $G$. The $x y$-Menger number with respect to $l$, denoted by $\zeta_{l}(x, y)$, is the maximum number of internally disjoint $x y$-paths whose lengths are at most $l$ in $G$. The Menger number of $G$ with respect to $l$ is defined as $\zeta_{l}(G)=\min \left\{\zeta_{l}(x, y): x, y \in V(G)\right\}$. If $l<d(G)$, then $\zeta_{l}(G)=0$. To avoid the relatively trivial case in which $l<d(G)$ or $G$ is a complete graph, we assume that $l \geq d(G) \geq 2$ in this paper. Clearly, $\zeta_{l}(G) \leq \delta(G)$. For a graph $G$ with $d(G) \geq 2$ and $|V(G)|=n$, it is clear that $\zeta_{l}(G)$ is well defined for an integer $l$ with $d(G) \leq l \leq n-1$ and $\zeta_{d(G)}(G) \leq \zeta_{d(G)+1}(G) \leq \cdots \leq \zeta_{n-1}(G)$. There are many papers that have studied Menger-type parameters, such as [2-8].

We consider the Cartesian product $G_{1} \times G_{2}$ of graphs $G_{1}$ and $G_{2}$. For graphs $G_{1}$ and $G_{2}$, the Cartesian product $G_{1} \times G_{2}$ is the graph with vertex set $V\left(G_{1} \times G_{2}\right)=V\left(G_{1}\right) \times V\left(G_{2}\right)=\left\{x y \mid x \in V\left(G_{1}\right), y \in V\left(G_{2}\right)\right\}$ and edge set $E\left(G_{1} \times G_{2}\right)=\left\{\left(x_{1} x_{2}, y_{1} y_{2}\right) \mid\right.$ $x_{1}=y_{1}$ and $\left(x_{2}, y_{2}\right) \in E\left(G_{2}\right)$ or $x_{2}=y_{2}$ and $\left.\left(x_{1}, y_{1}\right) \in E\left(G_{1}\right)\right\}$. It is well known that the Cartesian product is an important

[^0]research topic in graph theory (see, e.g., [9-13]). It is also well known that, for designing large-scale interconnection networks, the Cartesian product is an important method to obtain large graphs from smaller ones, with a number of parameters that can be easily calculated from the corresponding parameters for those small initial graphs. The Cartesian product preserves many nice properties such as regularity, existence of Hamilton cycles and Euler circuits, and transitivity of the initial graphs (see, e.g., [1]). In fact, many well-known networks can be constructed by the Cartesian products of some simple graphs. For example, a torus is the Cartesian product of two cycles, a mesh is the Cartesian product of two paths, and a grid is the Cartesian product of several paths. What we are interested in is the Menger number of the Cartesian product of graphs.

## 2. Main results

For a vertex $x \in V\left(G_{1}\right)$ and a subgraph $H \subseteq G_{2}$, we use $x H$ to denote the subgraph of $G_{1} \times G_{2}$ induced by $\{x\} \times V(H)$. Similarly, for a vertex $y \in V\left(G_{2}\right)$, and a subgraph $H \subseteq G_{2}$, Hy denotes the subgraph of $G_{1} \times G_{2}$ induced by $V(H) \times\{y\}$. The symbol $l(P)$ denotes the length of a path $P$, which is the number of edges in $P$.

Now, we state our main result of this paper.
Theorem 1. For any two connected graphs $G_{1}$ and $G_{2}$, if $l_{i} \geq 2$ for $i=1$, 2, then $\zeta_{l_{1}+l_{2}}\left(G_{1} \times G_{2}\right) \geq \zeta_{l_{1}}\left(G_{1}\right)+\zeta_{l_{2}}\left(G_{2}\right)$.
Proof. Assume that $x=x_{1} x_{2}$ and $y=y_{1} y_{2}$ are two distinct vertices in $G_{1} \times G_{2}$, where $x_{1}, y_{1} \in V\left(G_{1}\right)$ and $x_{2}, y_{2} \in V\left(G_{2}\right)$.
If $x_{1} \neq y_{1}$, there must exist $\zeta_{l_{1}}\left(G_{1}\right)$ internally disjoint $x_{1} y_{1}$-paths $P_{1}, P_{2}, \ldots, P_{\zeta_{1}\left(G_{1}\right)}$ in $G_{1}$ such that $l\left(P_{i}\right) \leq l_{1}$ for any $i \in\left\{1,2, \ldots, \zeta_{l_{1}}\left(G_{1}\right)\right\}$. Without loss of generality, we may assume that $l\left(P_{1}\right) \leq l\left(P_{2}\right) \leq \cdots \leq l\left(P_{\zeta_{1}\left(G_{1}\right)}\right)$. Then $l\left(P_{i}\right) \geq 2$ for any $i \in\left\{2, \ldots, \zeta_{l_{1}}\left(G_{1}\right)\right\}$. Let $v_{i}$ be the first internal vertex in $P_{i}\left(2 \leq i \leq \zeta_{l_{1}}\left(G_{1}\right)\right)$. It is clear that $v_{i} \in N_{G_{1}}\left(x_{1}\right)$. Then $v_{i}$ splits the path $P_{i}$ into two subpaths $a_{i}$ and $P_{i}^{\prime}$, where $a_{i}$ is the first edge ( $x_{1}, v_{i}$ ) in $P_{i}$ and $P_{i}^{\prime}$ is the subpath of $P_{i}$ from $v_{i}$ to $y_{1}$. Hence the path $P_{i}$ can be expressed as

$$
P_{i}=x_{1} \xrightarrow{a_{i}} v_{i} \xrightarrow{P_{i}^{\prime}} y_{1}, \quad i=2,3, \ldots, \zeta_{1}\left(G_{1}\right) .
$$

Similarly, if $x_{2} \neq y_{2}$, there must exist $\zeta_{l_{2}}\left(G_{2}\right)$ internally disjoint $x_{2} y_{2}$-paths $W_{1}, W_{2}, \ldots, W_{\zeta_{2}\left(G_{2}\right)}$ in $G_{2}$ such that $l\left(P_{j}\right) \leq l_{2}$ for any $j \in\left\{1,2, \ldots, \zeta_{l_{2}}\left(G_{2}\right)\right\}$. Without loss of generality, we may assume that $l\left(W_{1}\right) \leq l\left(W_{2}\right) \leq \cdots \leq l\left(W_{\zeta_{1}\left(G_{2}\right)}\right)$. Then $l\left(W_{j}\right) \geq 2$ for any $j \in\left\{2, \ldots, \zeta_{l_{2}}\left(G_{2}\right)\right\}$. Let $u_{j}$ be the first internal vertex in $P_{j}\left(2 \leq j \leq \zeta_{l_{2}}\left(G_{2}\right)\right)$. Then the path $W_{j}$ can be expressed as $W_{j}=x_{2} \xrightarrow{b_{j}} u_{j} \xrightarrow{W_{j}^{\prime}} y_{2}, j=2,3, \ldots, \zeta_{l_{2}}\left(G_{2}\right)$, where $b_{j}$ is the first edge $\left(x_{2}, u_{j}\right)$ in $W_{j}$ and $W_{j}^{\prime}$ is the subpath of $W_{j}$ from $u_{j}$ to $y_{2}$. It is clear that $u_{j} \in N_{G_{2}}\left(x_{2}\right)$.

Using the above notations, we can construct $\zeta_{l_{1}}\left(G_{1}\right)+\zeta_{l_{2}}\left(G_{2}\right)$ internally disjoint $x y$-paths $R_{1}, R_{2}, \ldots R_{\zeta l_{1}\left(G_{1}\right)+\zeta_{l_{2}}\left(G_{2}\right)}$ with $l\left(R_{i}\right) \leq l_{1}+l_{2}$ for each $i$. Consider the following three cases.
Case 1. $x_{1} \neq y_{1}, x_{2} \neq y_{2}$.
Let $R_{1}=x_{1} x_{2} \xrightarrow{P_{1} x_{2}} y_{1} x_{2} \xrightarrow{y_{1} W_{1}} y_{1} y_{2}$; then $l\left(R_{1}\right)=l\left(P_{1}\right)+l\left(W_{1}\right) \leq l_{1}+l_{2}$.
For $i=2,3, \ldots, \zeta_{l_{1}}\left(G_{1}\right)$, let $R_{i}=x_{1} x_{2} \xrightarrow{a_{i} x_{2}} v_{i} x_{2} \xrightarrow{v_{i} W_{1}} v_{i} y_{2} \xrightarrow{P_{i}^{\prime} y_{2}} y_{1} y_{2}$; then $l\left(R_{i}\right)=1+l\left(W_{1}\right)+l\left(P_{i}^{\prime}\right) \leq l_{1}+l_{2}$.
Let $R_{\zeta l_{1}\left(G_{1}\right)+1}=x_{1} x_{2} \xrightarrow{x_{1} W_{1}} x_{1} y_{2} \xrightarrow{P_{1} y_{2}} y_{1} y_{2}$; then $l\left(R_{1}\right)=l\left(W_{1}\right)+l\left(P_{1}\right) \leq l_{1}+l_{2}$.
For $j=2,3, \ldots, \zeta_{l_{2}}\left(G_{2}\right)$, let $R_{\zeta l_{1}\left(G_{1}\right)+j}=x_{1} x_{2} \xrightarrow{x_{1} b_{j}} x_{1} u_{j} \xrightarrow{P_{1} u_{j}} y_{1} u_{j} \xrightarrow{y_{1} W_{j}^{\prime}} y_{1} y_{2}$; then $l\left(R_{\zeta_{1}\left(G_{1}\right)+j}\right)=1+l\left(P_{1}\right)+l\left(W_{j}^{\prime}\right) \leq l_{1}+l_{2}$.
Case 2. $x_{1}=y_{1}, x_{2} \neq y_{2}$.
Since $\left|N_{G_{1}}\left(x_{1}\right)\right|=d_{G_{1}}\left(x_{1}\right) \geq \delta\left(G_{1}\right) \geq \zeta_{l_{1}}\left(G_{1}\right), N_{G_{1}}\left(x_{1}\right) \backslash\left\{v_{2}, v_{3}, \ldots, v_{\zeta_{1}\left(G_{1}\right)}\right\} \neq \emptyset$. Let $v_{1} \in N_{G_{1}}\left(x_{1}\right) \backslash\left\{v_{2}, v_{3}, \ldots, v_{\zeta l_{1}\left(G_{1}\right)}\right\}$ and $a_{1}=\left(x_{1}, v_{1}\right)$.

For $i=1,2, \ldots, \zeta_{1}\left(G_{1}\right)$, let $R_{i}=x_{1} x_{2} \xrightarrow{a_{i} x_{2}} v_{i} x_{2} \xrightarrow{v_{i} W_{1}} v_{i} y_{2} \xrightarrow{a_{i} y_{2}} y_{1} y_{2}$; then $l\left(R_{i}\right)=1+l\left(W_{1}\right)+1 \leq l_{1}+l_{2}$.
For $j=1,2, \ldots, \zeta_{l_{2}}\left(G_{2}\right)$, let $R_{\zeta l_{1}\left(G_{1}\right)+j}=x_{1} x_{2} \xrightarrow{x_{1} W_{j}} x_{1} y_{2}=y_{1} y_{2}$; then $l\left(R_{\zeta l_{1}\left(G_{1}\right)+j}\right)=l\left(W_{j}\right)<l_{1}+l_{2}$.
Case 3. $x_{1} \neq y_{1}, x_{2}=y_{2}$.
For $i=1,2, \ldots, \zeta_{l_{1}}\left(G_{1}\right)$, let $R_{i}=x_{1} x_{2} \xrightarrow{P_{i} x_{2}} y_{1} x_{2}=y_{1} y_{2}$; then $l\left(R_{i}\right)=l\left(P_{i}\right)<l_{1}+l_{2}$.
Since $\left|N_{G_{2}}\left(x_{2}\right)\right|=d_{G_{2}}\left(x_{2}\right) \geq \delta\left(G_{2}\right) \geq \zeta_{l_{2}}\left(G_{2}\right), N_{G_{2}}\left(x_{2}\right) \backslash\left\{u_{2}, u_{3}, \ldots, u_{\zeta l_{2}\left(G_{2}\right)}\right\} \neq \emptyset$. Let $u_{1} \in N_{G_{2}}\left(x_{2}\right) \backslash\left\{u_{2}, u_{3}, \ldots, u_{\zeta l_{1}\left(G_{1}\right)}\right\}$ and $b_{1}=\left(x_{2}, u_{1}\right)$.

For $j=1,2, \ldots, \zeta_{l_{2}}\left(G_{2}\right)$, let $R_{\zeta l_{1}\left(G_{1}\right)+j}=x_{1} x_{2} \xrightarrow{x_{1} b_{j}} x_{1} u_{j} \xrightarrow{P_{1} u_{j}} y_{1} u_{j} \xrightarrow{y_{1} b_{j}} y_{1} x_{2}=y_{1} y_{2}$; then $l\left(R_{\zeta_{1}\left(G_{1}\right)+j}\right)=1+l\left(W_{j}\right)+1 \leq$ $l_{1}+l_{2}$.

It is easy to check that the $x y$-paths $R_{1}, R_{2}, \ldots, R_{\zeta l_{1}\left(G_{1}\right)+\zeta_{2}\left(G_{2}\right)}$ constructed above are internally disjoint in $G_{1} \times G_{2}$ whichever case occurs.

Since $l\left(R_{i}\right) \leq l_{1}+l_{2}$ for $1 \leq i \leq \zeta_{l_{1}}\left(G_{1}\right)+\zeta_{l_{2}}\left(G_{2}\right)$, the theorem follows.

The connectivity of a graph $G$, denoted by $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's theorem that $\kappa(G) \geq k$ if any two distinct vertices of $G$ are connected by at least $k$ internal vertex-disjoint paths. Generally, we have $\zeta_{l}(G) \leq \kappa(G)$. If $P$ is a path of length $m$, then $\zeta_{l}(P)=1=\kappa(P)$ for any $l \geq m$.

The grid network is defined as $G\left(m_{1}, m_{2}, \ldots, m_{n}\right)=P_{m_{1}} \times P_{m_{2}} \times \cdots \times P_{m_{n}}$, where $P_{m_{i}}$ is a path of length $m_{i}$ for each $i=1,2, \ldots, n$. As an application of Theorem 1, we obtain the Menger number of the grid network. The following lemma is useful in the proof of our conclusion.

Lemma 2 (Theorems 2.3.3 and 2.3.4 in [1]). Let $G_{1}, G_{2}, \ldots, G_{n}$ be $n$ simple graphs. Then $d\left(G_{1} \times G_{2} \times \cdots \times G_{n}\right)=d\left(G_{1}\right)+$ $d\left(G_{2}\right)+\cdots+d\left(G_{n}\right)$. If $\kappa\left(G_{i}\right)=\delta\left(G_{i}\right)>0$ for each $i=1,2, \ldots, n$, then $\kappa\left(G_{1} \times G_{2} \times \cdots \times G_{n}\right)=\kappa\left(G_{1}\right)+\kappa\left(G_{2}\right)+\cdots+\kappa\left(G_{n}\right)$.

Corollary 3. Let $G=G\left(m_{1}, m_{2}, \ldots, m_{n}\right)$ be a grid network. If $l \geq \sum_{i=1}^{n} m_{i}$ and $m_{i} \geq 2$ for each $i=1,2, \ldots$, $n$, then $\zeta_{l}(G)=\zeta_{m_{1}}\left(P_{m_{1}}\right)+\zeta_{m_{2}}\left(P_{m_{2}}\right)+\cdots+\zeta_{m_{n}}\left(P_{m_{n}}\right)=n$.
Proof. Since $d\left(P_{m_{i}}\right)=m_{i}$, by Lemma 2, we have $d(G)=\sum_{i=1}^{n} m_{i}$. For $l \geq d(G)$, we have $\zeta_{l}(G) \geq \zeta_{d(G)}(G)$. By Theorem 1 , using the associative law, we have $\zeta_{d(G)}(G) \geq \zeta_{m_{1}}\left(P_{m_{1}}\right)+\zeta_{m_{2}}\left(P_{m_{2}}\right)+\cdots+\zeta_{m_{n}}\left(P_{m_{n}}\right)=n$. By Lemma 2, we have $\kappa(G)=n$. By $\zeta_{l}(G) \leq \kappa(G)=n$ and $\zeta_{l}(G) \geq \zeta_{d(G)}(G)=n$, we have $\zeta_{l}(G)=n=\zeta_{m_{1}}\left(P_{m_{1}}\right)+\zeta_{m_{2}}\left(P_{m_{2}}\right)+\cdots+\zeta_{m_{n}}\left(P_{m_{n}}\right)$.

The corollary is proved.

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