Fault-tolerant edge-pancyclicity of locally twisted cubes

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A B S T R A C T
The n-dimensional locally twisted cube LTQ n is a new variant of the hypercube, which possesses some properties superior to the hypercube. This paper investigates the fault-tolerant edge-pancyclicity of LTQ n, and shows that if LTQ n (n ≥ 3) contains at most \( \frac{n}{C_0} \) faulty vertices and/or edges then, for any fault-free edge \( e \) and any integer \( \ell \) with \( 6 \leq \ell \leq 2^n - f_v \), there is a fault-free cycle of length \( \ell \) containing the edge \( e \), where \( f_v \) is the number of faulty vertices. The result is optimal in some senses. The proof is based on the recursive structure of LTQ n.

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1. Introduction

As topological structures, interconnection networks play an important role in parallel processing and computing systems, particularly in multicomputer systems, which provide an effective mechanism of exchanging data between processors. The \( n \)-dimensional hypercube \( Q_n \), suggested first by Sullivan and Bashkow [32], is one of the most popular, versatile and efficient interconnection networks, which possesses many excellent features such as logarithmic number of links per node, logarithmic diameter, high symmetry, recursive structure, linear bisection width (see [12]) and, thus, becomes the first choice for the topological structure of parallel processing and computing systems (see [22,3]). The machines based on the hypercube have been implemented commercially such as the Cosmic Cube from Caltech [31], the IPSC/2 from Intel [28] and Connection Machines [14]. Parallel algorithms, and simple but efficient routing and broadcasting algorithms based on the hypercube have been developed [22,29].

It is well known that the diameter of the hypercube \( Q_n \) is \( n \). Hillis [14] showed that the hypercube \( Q_n \) does not have the smallest possible diameter relative to its number of vertices \( 2^n \). To achieve smaller diameter with the same number of vertices and links as the hypercube, a great number of variants of the hypercube were proposed, such as the Möbius cube [6], the crossed cube [7], the twisted cube [13], the augmented cube [5], the locally twisted cube [40], the bubble-sort graph [2], the star graph [23], and so on (see [37]). All these variants of the hypercube have only about half of the diameter of the hypercube.

It is well known that an interconnection network can be modeled by a connected graph \( G = (V,E) \), where \( V = V(G) \) is the vertex-set and \( E = E(G) \) is the edge-set of \( G \), in which vertices represent processors and edges represent communication links between processors.

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There are a lot of mutually conflicting requirements in designing an interconnection network. It is almost impossible to design an interconnection network which is optimum from all aspects. One has to design a suitable network depending on the requirements and its properties. One of the central issues in designing and evaluating an interconnection network is to study how well other existing networks can be embedded into this network. This problem can be modeled by the following design an interconnection network which is optimum from all aspects. One has to design a suitable network depending on the dilution, and a guest graph

Definition 1

Faulty elements occur, Chang et al. [4] and Park et al. [30], independently, showed that the others are all faulty. This condition is natural since, in practical applications, the probability is small for a vertex $x$ with at least two fault-free edges. This condition is optimal in some senses.

We, in this paper, improve the related result by proving that for any edge in $LTQ_n$, there are cycles of all lengths containing it. Ma and Xu [26] further improved this result by showing that for any two different vertices $x$ and $y$ with distance $d$ in $LTQ_n$, there exist $xy$-paths of all lengths from $d$ to $2^n - 1$ except for $d + 1$. Even when faulty elements occur, Chang et al. [4] and Park et al. [30], independently, showed that $LTQ_n$ still contains fault-free cycles of all lengths provided that faulty elements do not exceed $n - 2$. Very recently, Han et al. [10] have showed that $LTQ_n$ with at most $n - 3$ faulty elements contains paths of all lengths from $2^{n-1} - 1$ to $2^n - f$, where $f$ is the number of faulty vertices. Hsieh and Wu [19] have considered more faulty edges and showed that $LTQ_n$ contains a fault-free Hamiltonian cycle provided that faulty edges do not exceed $2n - 5$ and each vertex is incident with at least two fault-free edges. This condition is natural since, in practical applications, the probability is small for a vertex $x$ being isolated (all links incident with $x$ are faulty) or pendant (only one link incident with $x$ is fault-free and the others are all faulty).

We, in this paper, improve the related result by proving that if $LTQ_n$ contains at most $n - 3$ faulty elements then, for any fault-free edge $e$ and any integer $\ell$ with $6 \leq \ell \leq 2^n - f$, there is a fault-free cycle of length $\ell$ containing the edge $e$. We also demonstrate that this result is optimal in some senses.

The remainder of this paper is organized as follows. In Section 2, we recall the structure of $LTQ_n$, and some definitions and notations. In Section 3, we introduce some properties of $LTQ_n$ to used in our proofs. In Section 4, we give the proof of our result. Finally, we give some concluding remarks in Section 5.

2. Preliminaries

We follow Xu [36] for graph-theoretical terminology and notation. A graph $G = (V, E)$ consists of a vertex-set $V$ and an edge-set $E$, where $V = V(G)$ is a finite set and $E = E(G)$ is a subset of $\{xy | xy \text{ is an unordered pair of } V\}$. Two vertices $x$ and $y$ are adjacent if $xy$ is an edge of $G$, and are also the end-vertices of $xy$. For a vertex $x$, we call the vertices adjacent to it the neighbors of $x$. The degree of a vertex $x$ is the number of edges incident with it. A graph is called $k$-regular if each vertex has degree $k$. For two distinct vertices $x$ and $y$, an $xy$-path between $x$ and $y$ is a sequence of distinct vertices in which any two consecutive vertices are adjacent. The length of a path is the number of edges on the path. An $xy$-path of length at least three is called a cycle if $x = y$. A connected subgraph of $G$ is called a spanning tree if it contains all vertices of $G$ and no cycles, in which a distinguished vertex is called the root of the spanning tree.

The distance between two distinct vertices $x$ and $y$ in $G$ is the length of a shortest $xy$-path in $G$, and the diameter of $G$ is the maximum distance between any two vertices. The connectivity of $G$ is the minimum $\kappa$ for which there are $\kappa$ vertices whose removal results in a disconnected graph.

We now recall the definition of the $n$-dimensional locally twisted cube, proposed by Yang, Evans and Megson [41], which has $2^n$ vertices, and each vertex is an $n$-string on $\{0, 1\}$.

Definition 1 [41]. The $n$-dimensional locally twisted cube, denoted by $LTQ_n$ ($n \geq 2$), is recursively defined as follow.

1. $LTQ_2$ is a graph isomorphic to $Q_2$.
(2) For \( n \geq 3 \), \( LTQ_n \) is built from two disjoint copies of \( LTQ_{n-1} \) according to the following steps. Let \( LTQ_{n-1}^0 \) and \( LTQ_{n-1}^1 \) denote graphs obtained by prefixing labels of each vertex of one copy of \( LTQ_{n-1} \) with 0 and with 1, respectively, and connect a vertex \( x = x_0x_1\ldots x_n \) of \( LTQ_{n-1}^0 \) with another vertex \( y = 1(x_2 + x_0)x_3\ldots x_n \) of \( LTQ_{n-1}^1 \) by an edge \( xy \), where ‘+’ represents the modulo 2 addition.

The graphs shown in Fig. 1 are \( LTQ_3 \) and \( LTQ_4 \).

The locally twisted cube \( LTQ_n \) can be equivalently defined with the following non-recursive fashion.

**Definition 2** [41]. For \( n \geq 2 \), the \( n \)-dimensional locally twisted cube \( LTQ_n \) is a graph with all \( n \)-strings on \( \{0, 1\} \) as the vertex-set. Two vertices \( x = x_0x_1\ldots x_n \) and \( y = y_0y_1\ldots y_n \) of \( LTQ_n \) are adjacent if and only if either

(a) \( x_i = y_i \) and \( x_{i+1} = y_{i+1} + x_n \) for some \( 1 \leq i \leq n - 2 \), and \( x_j = y_j \) for all the remaining bits, where ‘+’ represents the modulo 2 addition, or

(b) \( x_i = y_i \) for some \( i \in \{n - 1, n\} \), and \( x_j = y_j \) for all the remaining bits.

According to the above definitions, it is not difficult to see that \( LTQ_n \) is \( n \)-regular. Furthermore, Yang, Evans and Megson [40,41] proved that the connectivity of \( LTQ_n \) is \( n \), the diameter \( n - 1 \) if \( n \geq 3 \) or 4, and \( \lceil \frac{1}{2}(n + 3) \rceil \) if \( n \geq 5 \), and that \( LTQ_n \) contains cycles of all lengths from 4 to \( 2^n \). Very recently, Hsieh and Tu [18] have showed that \( LTQ_n \) contains \( n \) edge-disjoint spanning trees. Lin et al. [24] further proved that any two spanning trees with the same root \( x \) and for any other vertex \( y \neq x \), the two different paths from \( x \) to \( y \), one path in each tree, are internally vertex-disjoint. For more properties of \( LTQ_n \), the reader can refer to [4,10,11,18,19,24–27,30,38–41].

We now make some remarks on the \( n \)-dimensional locally twisted cube. First, like to many variants of the hypercube such as the Möbius cube, the crossed cube, the twisted cube, the augmented cube and otherwise, the locally twisted cube not only keeps many nice properties of the hypercube such as regularity, high connectivity and high recursive constructability, but also has diameter of about half of that of the hypercube of the same size. Secondly, the locally twisted cube also keeps a nice property of the hypercube, that is, the labels of some neighboring vertices may differ in a large number of bits. As a result, a portion of good properties of hypercube are lost in these variants. For example, the design of efficient parallel algorithms on these variants turns out to be a difficult task [41].

Thirdly, the locally twisted cube \( LTQ_n \) contains cycles of all lengths from 4 to \( 2^n \) [40], but the hypercube \( Q_n \) contains only even cycles since it is a bipartite graph. Thus, \( LTQ_n \) is superior to \( Q_n \) in cycle embedding property.

Fourthly, the construction of the locally twisted cube \( LTQ_n \) is quite different from that of the twisted cube \( TQ_n \). The former is defined for any positive integer \( n \), while the latter only for odd integer.

Lastly, it should be noted that, like to many variants of the hypercube, the locally twisted cube \( LTQ_n \) is not vertex-transitive for \( n \geq 4 \) proved by Liu et al. [25].

**3. Properties**

In this section, we introduce some properties of \( LTQ_n \) to be used in our proofs in Section 4.

Let \( G = (V, E) \) be a graph. A non-empty subset of \( E(G) \) is called a matching of \( G \) if no two of its elements have a common end-vertices in \( G \). A matching \( M \) is perfect if every vertex of \( G \) is an end-vertex of some edge in \( M \).

From **Definition 1**, \( LTQ_n \) can be expressed as the union of two disjoint copies of \( LTQ_{n-1} \) by adding a perfect matching between them according to the specified rule. For short, we often write \( LTQ_n = L \sqcup R \), where \( L \cong LTQ_{n-1}^0 \) and \( R \cong LTQ_{n-1}^1 \). Let \( E_c \) be the perfect matching, a set of edges connecting \( L \) with \( R \). Obviously, \( |E_c| = 2^{n-1} \).

![Fig. 1. The locally twisted cubes \( LTQ_3 \) and \( LTQ_4 \).](image)
Yang, Evans and Megson [41] found an isomorphic expression of $LTQ_n$. For example, two graphs shown in Fig. 2(a) and (b) are other expressions of $LTQ_3$ and $LTQ_4$, respectively.

In general, they proved the following result.

**Lemma 1** [41]. Let $U'$ be the graph obtained from $Q_{n-1}$ by suffixing the labels of all vertices with 0. $R'$ be the graph obtained from a graph isomorphic to $Q_{n-1}$ by suffixing the labels of all vertices with 1. Then $LTQ_n$ is isomorphic to the graph obtained from $U'$ and $R'$ by adding a perfect matching $M$ between them, denoted by $LTQ_n = Q_{n-1} \oplus Q_{n-1}$, where $M$ is the set of edges by linking two vertices with difference only suffixes.

Let $LTQ_n = L \otimes R$ defined in Lemma 1. For convenience, for a vertex $u$ in $LTQ_n$, if $u$ is in $L$, we write $u_L$ for $u$, and use $u_R$ to denote its neighbor in $R$. Let $u_L$ and $v_L$ be two adjacent vertices in $L$. We say $v_L$ is a strong neighbor of $u_L$ in $L$ if their neighbors $u_R$ and $v_R$ are adjacent in $R$, and a weak neighbor of $u_L$ in $L$ if their neighbors $u_R$ and $v_R$ are not adjacent in $R$. Similarly, we can define a strong neighbor or a weak neighbor of two adjacent vertices $u_R$ and $v_R$ in $R$.

**Lemma 2.** Let $LTQ_n = L \otimes R$. If $n \geq 4$ then, for any vertex $u_L$ in $L$, there are $n - 2$ strong neighbors and one weak neighbor in $L$. Moreover, if $w_R$ is the weak neighbor of $u_L$, then the distance between $w_R$ and $w_L$ is two in $R$. The same conclusion holds for any vertex $u_R$ in $R$.

**Proof.** Let $u_L = 0x_2x_3x_4 \ldots x_{n-1}x_n$ be a vertex in $L$. Then its neighbor in $R$ is $u_R = 1(x_2 + x_n)x_3x_4 \ldots x_{n-1}x_n$. Let $s^2_L, s^3_L, \ldots, s^{n-1}_L, w_L$ be $n - 1$ neighbors of $u_L$ in $L$, where

$$s^2_L = 0x_2(x_3 + x_n)x_4 \ldots x_{n-1}x_n,$$
$$s^3_L = 0x_2x_3(x_4 + x_n)x_5 \ldots x_{n-1}x_n,$$
$$\ldots,$$
$$s^{n-1}_L = 0x_2x_3x_4 \ldots x_{n-3}x_{n-2}x_{n-1}x_n,$$
$$w_L = 0x_2x_3x_4 \ldots x_{n-1}x_n.$$ 

Let $s^2_R, s^3_R, \ldots, s^{n-1}_R, w_R$ be neighbors of $s^2_L, s^3_L, \ldots, s^{n-1}_L$, $w_L$ in $R$, respectively, where

$$s^2_R = 1(x_3 + x_n)(x_4 + x_n)x_5 \ldots x_{n-1}x_n,$$
$$s^3_R = 1(x_2 + x_n)x_3(x_4 + x_n) \ldots x_{n-1}x_n,$$
$$\ldots,$$
$$s^{n-1}_R = 1(x_2 + x_n)x_3x_4 \ldots x_{n-3}x_{n-2}x_{n-1}x_n,$$
$$w_R = 1(x_2 + x_n)x_3x_4 \ldots x_{n-1}x_n.$$ 

Since $x_2 + x_n = x_2 + x_n$, by definition of $LTQ_n$, $u_0s^i_R \in E(R)$ for $i = 2, 3, \ldots, n - 1$, but $u_R$ and $w_R$ are not adjacent in $R$. So $s^2_L, s^3_L, \ldots, s^{n-1}_L$ are strong neighbors of $u_L$ and $w_L$ is a weak neighbor of $u_L$ in $L$. In order to complete the proof of the lemma, we need to find a vertex $o_L$ such that $u_0o_Rw_R \in E(R)$. We can find this vertex $w_R$ as follow.

$$o_R = 1(x_2 + x_n)x_3x_4 \ldots x_{n-1}x_n \text{ when } x_n = 1;$$
$$o_R = 1(x_2 + x_n)x_3x_4 \ldots x_{n-1}x_n \text{ when } x_n = 0.$$ 

It is easy to see that $u_0o_Rw_R \in E(R)$, and so the distance between $u_R$ and $w_R$ is two in $R$. □

![Fig. 2. Another painting of LTQ3 and LTQ4.](image-url)
Lemma 3 ([27,20]). Let $e$ be any edge in $LTQ_n$ and $\ell$ be any integer with $4 \leq \ell \leq 2^n$. Then there is a cycle of length $\ell$ containing the edge $e$ in $LTQ_n$ for $n \geq 2$.

Lemma 4 [26]. Let $x$ and $y$ be any two different vertices in $LTQ_n$ and the distance between them be $d$. Then for any integer $\ell$ with $d \leq \ell \leq 2^n - 1$ except for $d + 1$, there exists an $xy$-path of length $\ell$ in $LTQ_n$ for $n \geq 3$.

Let $F$ be a set of faulty elements in a graph $G$. An edge or a vertex in $G$ is said to be fault-free if it is not in $F$, and a subgraph $H$ of $G$ to be fault-free if $H$ contains no elements in $F$. Throughout this paper, we use $f_e$ and $f_v$ to denote the numbers of vertices and edges in $F$, respectively.

Lemma 5 [10]. If $f_e + f_v < n - 3$ and $n \geq 3$, then for any integer $\ell$ with $2^{n-1} - 1 \leq \ell \leq 2^n - f_e - 1$, there is a fault-free path of length $\ell$ between any two distinct vertices in $LTQ_n$.

Lemma 6 [33]. Every fault-free edge (resp. vertex) of $Q_n$ lies on a fault-free cycle of every even length from $4$ to $2^n - 2f_v$ if $f_e + f_v < n - 2$ and $n \geq 3$.

4. Fault-tolerant edge-pancyclicity

To state and prove our main result, we need some notions and terminologies. Let $G$ be a connected graph, $x$ and $y$ be two distinct vertices in $G$. We use a sequence of distinct vertices $P = (x_0, x_1, \ldots, x_k)$, where $x_0 = x$ and $x_k = y$, to denote an $xy$-path. In addition, a path can be expressed as the union of several subpaths. For example, we can write $P = P(x_0, x_1) + P(x_1, x_2)$, where $P(x_0, x_1) = (x_0, x_1, \ldots, x_k)$ and $P(x_1, x_2) = (x_1, x_2, \ldots, x_k)$. Since the subpath $P(x_1, x_2)$ is a single edge $e_{x_1x_2}$, we write $x_1x_2$ instead of $P(x_1, x_2)$, that is, $P = P(x_0, x_1) + x_1x_2 + P(x_1, x_2)$.

A graph $G$ of order $n$ is $k$-pancyclic if it contains cycles of all lengths from $k$ to $n$, and edge-$k$-pancyclic if each of its edges lies on a cycle of every length from $k$ to $n$. Clearly, an edge-$k$-pancyclic graph is certainly $k$-pancyclic. A graph $G$ is $f$-fault-tolerant edge-$k$-pancyclic if $G - F$ is still edge-$k$-pancyclic for any $F \subseteq E(G) \cup V(G)$ with $|F| \leq f$.

In this section, we investigate the fault-tolerant edge-pancyclicity of $LTQ_n$ and show that $LTQ_n$ is $(n - 3)$-fault-tolerant edge-6-pancyclic. We state this result as the following theorem.

Theorem 1. If $f_e + f_v < n - 3$ and $n \geq 3$ then, for any fault-free edge $e$ in $LTQ_n$ and any integer $\ell$ with $6 \leq \ell \leq 2^n - f_v$, there is a fault-free cycle of length $\ell$ containing the edge $e$ in $LTQ_n$.

Proof. We use the expression $LTQ_n = L \cup R$, where $L \equiv LTQ_{n-1}$ and $R \equiv LTQ_{n-1}$. Let $F$ be a set of faulty elements in $LTQ_n$ with $|F| = f_e + f_v$, $L = F \cap L$, $R = F \cap R$, $L = F \cap E_c$, $F = F \cap V(L)$, $F^C = F \cap V(R)$, $F^C = F \cap V(L)$, and $F^C = F \cap V(R)$. Without loss of generality, we may assume $|F^C| \leq |F|_c$.

Let $e$ be an arbitrary fault-free edge in $LTQ_n$ and $\ell$ be any integer with $6 \leq \ell \leq 2^n - f_v$. We need to prove that there exists a fault-free cycle of length $\ell$ containing the edge $e$ in $LTQ_n$ if $|F| < n - 3$ and $n \geq 3$.

We proceed by induction on $n \geq 3$. For $n = 3$, there are no fault-free vertices or edges in $LTQ_3$, and so the theorem holds by Lemma 3. Assume that the theorem holds for $LTQ_n$ for $n < 4$.

Case 1. $|F^C| \leq n - 4$.

Subcase 1.1. The fault-free edge $e$ is in $L$ or $R$.

Since $|F^C| < n - 4$, without loss of generality, assume that the edge $e$ is in $L$ and let $e = uLvL$.

If $6 \leq \ell \leq 2^{n-1} - |F^C|$, then, by the induction hypothesis, there is a fault-free cycle of length $\ell$ containing the edge $e$ in $L$, so in $LTQ_n$. Thus, we only need to consider such an $\ell$ that satisfies $2^{n-1} - |F^C| + 1 \leq \ell \leq 2^n - f_v$.

If $n = 4$, since $0 \leq |F^C| \leq |F^C| \leq n - 4 = 0$, then $2^{4-1} + 1 = 9 \leq \ell \leq 2^4 - f_v$. Let $\ell = \ell' + 1$. Then $8 \leq \ell' \leq 2^4 - f_v - 1$. By Lemma 5, there is a fault-free $uLvL$-path $P$ of length $\ell'$ in $LTQ_n$ and so $P + uLvL$ is a fault-free cycle of length $\ell$ containing the edge $e$.

Now, assume $n \geq 5$ and write $\ell = \ell_1 + \ell_2$, where $2^{n-2} - |F^C| \leq \ell \leq 2^{n-1} - |F^C|$ and $2^n - 2 \leq \ell \leq 2^{n-1} - |F^C| - 1$. Since $2^{n-2} - |F^C| \geq 2^{n-2} - n + 6 \geq 6$ for $n \geq 5$, by the induction hypothesis, there is a fault-free cycle of length $\ell_1$ containing the edge $e$ in $L$. Note that a cycle of length $\ell_1$ contains a matching $M$ with $|M| \geq \ell_1 / 2$. Consider the following inequality.

$$\frac{\ell_1}{2} - |F^C| - |F| - |\{e\}| \geq \frac{2^{n-2} - |F^C|}{2} - |F^C| - |F| - 1 \geq 2^{n-3} - |F| - 1 \geq 2^{n-3} - n + 2.$$

Let $f(x) = 2x^2 - x + 2$. Since $f(x) = 2x^3 ln 2 - 2^x \geq 2^x$ for $x \geq 5$, $f(x)$ is an increasing function, which implies that $\frac{\ell_1}{2} - |F^C| - |F| \geq f(5)$, which implies that $\frac{\ell_1}{2} - |F^C| - |F| \geq 2^{n-3} - 5 = 2 + 1$. In other words, there is such an edge, say $x_{2}y_{2}$, in $M$ that $x_{2}y_{2} \neq e$ and two edges $x_{2}x_{3}$ and $y_{2}y_{3}$ are fault-free (see Fig. 3). Since $|F^C| < n - 4$, by Lemma 5, there is a fault-free $x_{2}y_{3}e'$-path $P$ of length $\ell_2$ in $R$. So $C = x_{2}y_{2} + y_{2}y_{3} + P + x_{2}x_{3}$ is a fault-free cycle of length $\ell = \ell_1 + \ell_2$ containing $e$ (see Fig. 3).

Subcase 1.2. The fault-free edge $e$ is in $E_c$.

Let $e = uLvR$ and $uL = 0x_2, x_3, \ldots, x_n$. Then $uR = 1(x_2 + x_n)x_3, \ldots, x_n$. 

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Assume $\ell = 6$. We use the second expression $LTQ_n = Q_n \oplus Q_{n-1}$ by Lemma 1. Since $u_R$ and $u_L$ have the same suffix $x_n$, $u_R$ and $u_L$ are in the same $Q_{n-1}$. Since $|F| \leq n - 3 = (n - 1) - 2$, by Lemma 6, there is a fault-free cycle of length 6 containing the edge $e$ in $Q_{n-1}$, and so in $LTQ_n$.

We now consider $\ell = 7$. We can construct $n - 2$ cycles of length 7 that are disjoint each other except a common edge $e = u_i u_k$ by considering two cases depending on $x_n = 0$ or $x_n = 1$.

Subcase 1.2.1. $x_n = 0$.

The required $n - 2$ cycles $C_3, \ldots, C_{n-1}, C_n$ of length 7 are constructed as follows. For each $i = 3, 4, \ldots, n - 1$, let

$$C_i = \begin{cases} u_i = 0x_2x_3 \ldots x_{n-1}0, \\
s_i^L = 0x_2x_3 \ldots x_{n-1}x_{n-1} \ldots x_{n-1}1, \\
w_i^L = 0x_2x_3 \ldots x_{n-1}x_{n-1} \ldots x_{n-1}1, \\
\alpha_i = 0(x_2 + 1)x_3 \ldots x_{n-1}0, \\
w_L = 0(x_2 + 1)x_3 \ldots x_{n-1}1, \\
\sigma_k = 1(x_2 + 1)x_3 \ldots x_{n-1}x_{n-1} \ldots x_{n-1}0, \\
s_k^R = 1x_2 \ldots x_{n-2}x_{n-1}1, \\
w_R = 1x_2 \ldots x_{n-2}x_{n-1}1, \\
u_R = 1x_2 \ldots x_{n-2}x_{n-1}0 \\
u_k = 1x_2 \ldots x_{n-2}x_{n-1}0, \\
\end{cases}$$

where $s_i^L$ is a strong neighbor of $u_i$ in $L$, $w_i^L$ is the weak neighbor of $s_i^L$ in $L$, $w_i^L$ and $s_i^R$ are neighbors of $w_i^L$ and $s_i^L$ in $R$, respectively, while $\alpha_i$ is such a vertex (the existence is guaranteed by Lemma 2) that $w_i^L, s_i^R, \sigma_k, \omega_k \in E(R)$ (see Fig. 4(a)).

And let $C_n = (u_L, \alpha_n, s_L, w_L, w_R, s_R, u_k)$, where

$$C_n = \begin{cases} u_L = 0x_2 \ldots x_{n-2}x_{n-1}0, \\
\sigma_k = 0(x_2 + 1)x_3 \ldots x_{n-1}x_{n-1} \ldots x_{n-1}1, \\
w_L = 0(x_2 + 1)x_3 \ldots x_{n-1}x_{n-1} \ldots x_{n-1}1, \\
s_R = 1x_2 \ldots x_{n-2}x_{n-1}1, \\
w_R = 1x_2 \ldots x_{n-2}x_{n-1}1, \\
u_R = 1x_2 \ldots x_{n-2}x_{n-1}0, \\
u_k = 1x_2 \ldots x_{n-2}x_{n-1}0, \\
\end{cases}$$

where $w_R$ is the weak neighbor of $u_R$ in $R$, while $s_R$ is a strong neighbor of $w_R$ in $R$, $s_L$ and $w_L$ are neighbors of $s_R$ and $w_R$ in $L$, respectively, $\alpha_i$ is such a vertex that $u_i \omega_k, \alpha_i \sigma_k \in E(L)$ (see Fig. 4(b)).

Since $n > 4$, such constructed cycles of length 7 are well defined. For example, in $LTQ_n$, let $e = u_i u_k$, where $u_L = 0000$ and $u_R = 1000$. Two cycles $C_3$ and $C_4$ of length 7 containing the edge $e$ are as follows.

$$C_3 = \{0000, 0010, 0011, 1111, 1110, 1010, 1000\}.$$  
$$C_4 = \{0000, 0100, 0101, 0111, 1011, 1001, 1000\}.$$  

It is easy to see that the cycles $C_3, \ldots, C_{n-1}, C_n$ are as required, at least one of them is fault-free since $|F| = n - 3 < n - 2$.

Subcase 1.2.2. $x_n = 1$.

The required $n - 2$ cycles $C_3, \ldots, C_{n-1}, C_n$ are constructed by considering two cases depending on $n = 4$ or $n \geq 5$.

If $n = 4$ then $|F| = 1$ and $|E_C| = 8$, where

$$E_C = \{(0000, 1000), (0010, 1010), (0100, 1100), (0110, 1110), (0111, 1011), (0101, 1001), (0001, 1111), (0011, 1111)\}.$$  

For each edge of $E_C$, we find two cycles of length 7, which contain this edge and disjoint except this edge as follows.
Fig. 4. The illustrations of Subcase 1.2.1.

<table>
<thead>
<tr>
<th>Edge</th>
<th>Two cycles in common only the edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0000,1000)</td>
<td>(0000,0001,1101,1111,1110,1100,1000), (0000, 0010,0110,0101,1001,1000,1000)</td>
</tr>
<tr>
<td>(0010,1010)</td>
<td>(0010,0011,0101,0100,0000,1000,1010), (0010,0110,1111,1110,1011,1011,1011)</td>
</tr>
<tr>
<td>(0100,1100)</td>
<td>(0100,0111,0101,0100,1000,1001,1010,1010), (0100,0000,0011,1101,1111,1110,1100)</td>
</tr>
<tr>
<td>(0110,1110)</td>
<td>(0110,0111,0001,0000,1000,1010,1101,1111), (0110,0010,0111,0101,1001,1111,1110)</td>
</tr>
<tr>
<td>(0111,1011)</td>
<td>(0111,1111,0000,0010,0101,1111,1110,1010,1011,1011)</td>
</tr>
<tr>
<td>(0101,1001)</td>
<td>(0101,0111,0110,0000,0001,1101,1111,1110,1100,1100,1100)</td>
</tr>
<tr>
<td>(0001,1101)</td>
<td>(0001,0000,0100,0101,1111,1110,1100,1100,1100,1100)</td>
</tr>
<tr>
<td>(0011,1111)</td>
<td>(0011,0001,0000,0100,1001,1111,1110,1010,1011,1011,1011)</td>
</tr>
</tbody>
</table>

Now, we assume $n \geq 5$.

We construct $n - 2$ cycles $C_3, C_4, \ldots, C_{n-2}, C_n$ of length 7 as follows. For each $i = 3, 4, \ldots, n - 2$, let $C_i = (u_i, s_{i1}^R, w_{i1}^R, w_{i2}^R, o_i^R, s_{i1}^L, u_i)$, where

$$
\begin{align*}
    u_i &= 0x_2x_3 \ldots x_{n-1}1, \\
    s_{i1}^R &= 0x_2x_3 \ldots x_{n-1}x_{i-1} + 1x_{i-2} \ldots x_{n-1}1, \\
    w_{i1}^R &= 0x_2x_3 \ldots x_{i-1}x_{i-1} + 1x_{i-2} \ldots x_{n-1}0, \\
    o_i^R &= 1(x_2 + 1)x_3 \ldots x_{i-1}x_{i-1} + 1x_{i-2} \ldots x_{n-1}0, \\
    s_{i1}^L &= 1(x_2 + 1)x_3 \ldots x_{i-1}x_{i-1} + 1x_{i-2} \ldots x_{n-1}1, \\
    u_i &= 1(x_2 + 1)x_3 \ldots x_{n-1}1.
\end{align*}
$$

And $C_{n-1} = (u_{n-1}, s_{n-1}^{n-1}, w_{n-1}^{n-1}, w_{n-1}^r, o_{n-1}^{n-1}, s_{n-1}^L, u_{n-1})$, where

$$
\begin{align*}
    u_{n-1} &= 0x_2x_3 \ldots x_{n-2}x_{n-1}1, \\
    s_{n-1}^{n-1} &= 0x_2x_3 \ldots x_{n-2}x_{n-2}x_{n-1}1, \\
    w_{n-1}^{n-1} &= 0x_2x_3 \ldots x_{n-2}x_{n-1}0, \\
    o_{n-1}^{n-1} &= 1(x_2 + 1)x_3 \ldots x_{n-2}x_{n-1}0, \\
    s_{n-1}^L &= 1(x_2 + 1)x_3 \ldots x_{n-2}x_{n-1}1, \\
    u_{n-1} &= 1(x_2 + 1)x_3 \ldots x_{n-1}1.
\end{align*}
$$

And $C_n = (u_n, o_n, s_n, w_n, w_n, s_n, u_n)$, where

$$
\begin{align*}
    u_n &= 0x_2x_3 \ldots x_{n-2}x_{n-1}1, \\
    o_n &= 0x_2x_3 \ldots x_{n-2}x_{n-1}0, \\
    s_n &= 0(x_2 + 1)x_3 \ldots x_{n-2}x_{n-1}0, \\
    w_n &= 0(x_2 + 1)x_3 \ldots x_{n-2}x_{n-1}0, \\
    s_n &= 1(x_2 + 1)x_3 \ldots x_{n-2}x_{n-1}0, \\
    u_n &= 1(x_2 + 1)x_3 \ldots x_{n-2}x_{n-1}1.
\end{align*}
$$
Since $n \geq 5$, such constructed cycles of length 7 are well defined. It is easy to check that these $(n - 2)$ cycles are disjoint except a common edge $u_i u_R$. Since $|F| = n - 3 < n - 2$, among them there exists a fault-free $u_i u_R$-cycle of length $\ell = 7$ containing edge $e$.

Now assume $8 \leq \ell \leq 2^n - |F|$. If $n = 4$ then let $\ell = \ell' + 1$, where $2^{\ell' - 1} - 1 = 7 \leq \ell' \leq 2^4 - f_R - 1$. By Lemma 5, there is a fault-free $u_i u_R$-path $P$ of length $\ell'$ in $L_{TQ_R}$. Then $P + u_i u_R$ is a fault-free cycle of length $\ell$ containing the edge $e = u_i u_R$.

Assume $n \geq 5$ below. Let $s_i$ be a fault-free strong neighbor of $u_i$ in $L$. Then $u_R s_i$ in $R$.

If $8 \leq \ell \leq 2^n - 1 - |F|^2 - 1$, then let $\ell = \ell' + 2$, where $6 \leq \ell' \leq 2^n - |F|^2 - 3$. By the induction hypothesis, there is a fault-free cycle $C$ of length $\ell'$ containing the edge $u_i s_i$ in $L$. Then $C = C - u_i s_i + s_i R + s_R u_R + u_i u_R$ is a fault-free cycle of length $\ell$ containing the edge $e = u_i u_R$.

If $2^n - 1 - |F|^2 - 1 \leq \ell \leq 2^n - f_R$, then let $\ell = \ell_1 + \ell_2 + 1$, where $2^{\ell_1} - 2^{\ell_2} - 1 = \ell' \leq 2^n - |F|^2 - 3$ and $2^{\ell_1} - 1 \leq \ell_2 \leq 2^{\ell_1} - 1$. Since $2^n - 2^{\ell_1} - 1 > 6$ for $n \geq 5$, by the induction hypothesis, let $C_1$ be a cycle of length $\ell_1$ containing the edge $u_i s_i$ in $L$ and, by Lemma 5, let $P$ be a fault-free $s_R u_R$-path of length $\ell_2$ in $R$. Then $C = C_1 - u_i s_i + s_i R + P + u_R u_i + u_i u_R$ is a fault-free cycle of length $\ell$ containing the edge $e = u_i u_R$ in $L_{TQ_R}$ (see Fig. 5).

Case 2. $|F| = n - 3$. In this case $|F_2| = |F_1| = 0$.

Let $e = u v$ be a fault-free edge in $L_{TQ_R}$. Let $\ell = \ell' + 1$. If $2^n - 1 - \ell' \leq 2^n - f_R - 1$ then, by Lemma 5, there exists a fault-free $u v$-path $P$ of length $\ell'$ in $L_{TQ_R}$. Thus, $P + u v$ is a fault-free cycle of length $\ell$ containing the edge $e$ in $L_{TQ_R}$. Thus, we only need to consider $\ell$ with $6 \leq \ell \leq 2^n - 1$. The proof of the theorem is complete. □

5. Conclusions and remarks

As one of the most fundamental networks for parallel and distributed computation, a cycle is suitable for developing simple algorithms with low communication cost. Edge and/or vertex failures are inevitable when a large parallel computer system is put in use. Therefore, the fault-tolerant capacity of a network is a critical issue in parallel computing. The fault-tolerant pancyclicity of an interconnection network is a measure of its capability of implementing ring-structured parallel algorithms in a communication-efficient fashion in the presence of faults.

Fig. 5. The illustration of Subcase 1.2 for $\ell = \ell_1 + \ell_2 + 1$. 

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The locally twisted cube $LTQ_n$, as a variation of the hypercube $Q_n$, not only retains some favorable properties of $Q_n$ but also possesses some embedding properties that $Q_n$ does not. For example, the diameter of $LTQ_n$ is only about half of the diameter of $Q_n$. Yang, Megson and Evans [40] proved that $LTQ_n$ contains cycles of all lengths from 4 to $2^n$. In particular, Chang, Ma and Xu [4] further improved the above results by proving that $LTQ_n$ contains fault-free cycles of all lengths from 4 to $2^n - f_e$ provided $f_e + f_f < n - 2$. In this paper, we improve this result by proving that if $f_e + f_f < n - 3$ then for any fault-free edge $e$ in $LTQ_n$ a cycle of length $\ell$ containing the edge $e$.

In view of the fact that the hypercube network $Q_n$ contains only even cycles, $LTQ_n$ is superior to $Q_n$ in fault-tolerant pancyclicity. This shows that, when the locally twisted cube is used to model the topological structure of a large-scale parallel processing system, our result implies that the system has larger capability of implementing ring-structured parallel algorithms in a communication-efficient fashion in the hybrid presence of edge and vertex failures than one of the hypercube network.

Our result is optimal in the following sense.

(1) Consider the edge $uv$, where $u = 00 \ldots 00$ and $v = 00 \ldots 01$. Then both $u$ and $v$ are in $L$ if we write $LTQ_n = L \oplus R$, and $v$ is the weak neighbor of $u$. By Lemma 2, it is easy to see that the edge $uv$ is contained in one and only cycle of length 4. If the cycle contains only one weak neighbor except vertices $u$, $v$, and the edge $uv$, then there are no fault-free cycles of length 4 containing the edge $uv$ in LTQ. For example, in $LTQ_4$, $u = 0000$ and $v = 0001$, the only cycle of length 4 containing the edge $uv$ is $C = (0000, 0001, 0011, 0010)$ (see Fig. 1). If a faulty element is any vertex except $u$ and $v$, or any edge except $uv$ in $C$, then there are no fault-free cycles of length 4 containing the edge $uv$ in $LTQ_4$.

(2) Consider the edge $u_iu_R$, where $u_i \in L$ and $u_R \in R$ if we write $LTQ_n = L \oplus R$. By Lemma 2, there are only two distinct cycles of length 5 containing the edge $u_iu_R$, which are obtained by the weak neighbor $o_i$ of $u_i$ in $L$ and the weak neighbor $o_R$ of $u_R$ in $R$, respectively. It is easy to see that the two cycles contain $o_i$ and $o_R$. If one of $o_i$ and $o_R$ is faulty, then there are no fault-free cycles of length 5 containing the edge $u_iu_R$ in $LTQ_n$. For example, in $LTQ_4$, taking $u_i = 0011$ and $u_R = 1111$, the only two cycles $C_1$ and $C_2$ of length 5 containing the edge $uv$ are as follows:

$$C_1 = (0011, 0010, 1010, 1111, 1111, 0011)$$
$$C_2 = (0011, 0010, 1010, 1111, 1111, 0011)$$

If the vertex $x = [0010]$ is faulty, then there are no fault-free cycles of length 5 containing the edge $uv$ in $LTQ_4$.

(3) As for the condition $f_e + f_f < n - 3$, we can say that it can be not improved as $n - 2$ at least when $n$ is small. In fact, if so, in $LTQ_n$, let $[010]$ be a faulty vertex, then there are no fault-free cycles of length 6 containing edge $[000, 001]$. Our proof for Theorem 1 uses induction on $n \geq 3$. The induction is based upon $n = 3$, which does not hold for $f_e + f_f = n - 2$ by the above example. The induction steps strongly depend on Lemma 5 which holds only when $f_e + f_f < n - 3$. Thus, our method can not improve $n - 3$ as $n - 2$. However, as our further work, we must make it clear whether or not $n - 3$ can be improved as $n - 2$ for more general integer $n$.

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References