Information Processing Letters 111 (2011) 1037–1043

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Conditional fault tolerance of arrangement graphs ☆

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ARTICLE INFO

Article history:
Received 26 December 2010
Received in revised form 20 July 2011
Accepted 20 July 2011
Available online 6 August 2011
Communicated by A.A. Bertossi

Keywords:
Fault tolerance
Conditional connectivity
(n,k)-Arrangement graph

ABSTRACT

Fault tolerance is especially important for interconnection networks, since the growing size of the networks increases its vulnerability to component failures. A classic measure for the fault tolerance of a network in the case of vertex failures is its connectivity. Given a network based on a graph G and a positive integer h, the R^h-connectivity of G is the minimum cardinality of a set of vertices in G, if any, whose deletion disconnects G, and every remaining component has minimum degree at least h. This paper investigates the R^h-connectivity of the (n,k)-arrangement graph $A_{n,k}$ for $h=1$ and $h=2$, and determines that $\kappa^1(A_{n,k}) = (2k-1)(n-k)-1$ and $\kappa^2(A_{n,k}) = (3k-2)(n-k)-2$, respectively.

1. Introduction

The study of interconnection networks has been an important research area for parallel and distributed computer systems. Network reliability is one of the major factors in designing the topology of an interconnection network. A network can be modeled as a graph $G = G(V,E)$. The traditional connectivity $\kappa(G)$ is an important parameter to measure the fault tolerance of the network. However, there is an obvious deficiency in the definition of $\kappa(G)$, it is tacitly assumed that all vertices incident with a vertex can potentially fail at the same time. To compensate for this shortcoming, it is natural to generalize the classical connectivity by introducing some conditions or restrictions on the separating set S and/or the components of $G-S$.

Motivated by the shortcomings of the traditional connectivity measure, Harary [11] introduced the concept of conditional connectivity by requiring some property for disconnected components of $G-S$. Esfahanian [10] studied the fault tolerance of graphs by using the notion of forbidden sets. Although a forbidden faulty set R can be chosen as any subset of vertices of G, some choices will be more useful than others. In what follows, we consider R as such a set

$$R = \{N(x)\colon \forall x \in V(G)\},$$

where $N(x)$ denotes the set of neighbors of x, i.e., $N(x) = \{y\colon xy \in E(G)\}$. In other words, the set $N(x)$ of neighbors of any vertex x is considered as a forbidden faulty set. This implies that all processors which are directly connected to any processor cannot fail at the same time, i.e., each vertex in G has at least one healthy neighbor. Such a consideration is specially true since the probability that all faulty vertices are neighbors of one vertex is very small.

For a given R, a non-empty and proper subset S of $V(G)$ is called a *restricted vertex cut* of G, *R-vertex cut* for short, if S contains no element in R and $G-S$ is disconnected. The *restricted connectivity* with respect to R, *R-connectivity* for short, of G, denoted by $\kappa_R(G)$, is defined as

$$\kappa_R(G) = \min\{|S|\colon S \text{ is an } R\text{-vertex cut of } G\}.$$

From the choice of forbidden faulty vertex sets R of G, it is clear that a vertex cut S of G is an R-vertex cut if and only if $G-S$ contains no isolated vertices. Motivated

☆ This work was partly supported by National Natural Science foundation of China (Nos. 61072080, 11071233) and the Key Project of Fujian Provincial Universities services to the western coast of the straits-Information Technology Research Based on Mathematics.

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0020-0190/$ – see front matter
doi:10.1016/j.ipl.2011.07.017

by this simple observation, Latifi et al. [15] generalized the concept of $\kappa_R(G)$ to more general case. Let G be a d-regular graph, h an integer with $0 \leqslant h < d$, and let

$$\mathscr{A}^h(x) = \{X \subset N_G(x)\colon |X| > d - h\} \quad \text{for } x \in V(G).$$

A subset of vertices of G is called a forbidden faulty set of G if and only if it belongs to $\mathscr{A}^h(x)$ for some $x \in V(G)$. In other words, each vertex in G has at least h healthy neighbors. We use the symbol $\mathscr{A}^h$ to denote the collection of such forbidden faulty sets. It is clear that a vertex cut S of G is an R^h-vertex cut if and only if $G - S$ contains no vertices of degree less than h. The *R^h-restricted connectivity* with respect to $\mathscr{A}^h$, *R^h-connectivity* for short, of G, denoted by $\kappa_R^h(G)$, is defined as

$$\kappa_R^h(G) = \min\{|S|\colon\ S \text{ is an } R^h\text{-vertex cut of } G\}.$$

Clearly, for $h = 0$ and 1, we have $\kappa_R^0(G) = \kappa(G)$ and $\kappa_R^1(G) = \kappa'(G)$ clearly.

This new measurement $\kappa_R^h(G)$ in conjunction with $\kappa(G)$ can provide a more accurate measure for fault tolerance of a large-scale parallel processing system G. This gives rise to a fundamental question that for a given graph G how can $\kappa_R^h(G)$ be computed? However, no polynomial-time algorithm for the computation of $\kappa_R^h(G)$ on a general graph is known, nor do we know any tight upper bound for $\kappa_R^h(G)$. Thus, one is interested in determining precise values of $\kappa_R^h(G)$ for some particular classes of graphs. For the n-dimensional hypercube Q_n, Esfahanian [10] proved that $\kappa_R^1(Q_n) = 2n - 2$, Latifi et al. [15], Oh and Choi [17] independently determined $\kappa_R^h(Q_n) = (n-h)2^h$ for $1 \leqslant h \leqslant \lfloor \frac{1}{2}n \rfloor$. For the n-dimensional star graph S_n, Hu and Yang [14] proved that $\kappa_R^1(S_n) = 2n - 4$.

To simplify the computation of $\kappa_R^h(G)$, Wan and Zhang [18] proposed a kind of conditional connectivity by placing some requirements on the components of $G - F$ only. Given a network based on a graph G and a positive integer h, the R^h-connectivity of G, denoted by $\kappa^h(G)$, is the minimum cardinality of a set of vertices in G, if any, whose deletion disconnects G, and every remaining component has minimum degree at least h. Wan and Zhang [18] determined $\kappa^2(S_n) = 6(n-3)$ for $n \geqslant 4$. For the (n,k)-star graphs $S_{n,k}$, Yang et al. [20] proved that $\kappa^1(S_{n,k}) = n + k - 3$, and $\kappa^2(S_{n,k}) = n + 2k - 5$ for $2 \leqslant k \leqslant n - 2$. For the n-dimensional alternating group graph AG_n, Cheng et al. [4] gave a detailed characterization of fault tolerance of the 2-tree generated networks which has AG_n as a special case, and obtained that $\kappa^1(AG_4) = 4$ and $\kappa^1(AG_n) = 4n - 11$ for $n \geqslant 5$. Zhang et al. [21] proved $\kappa^2(AG_4) = 4$ and $\kappa^2(AG_n) = 6n - 18$ for $n \geqslant 5$.

In this paper, we consider the (n,k)-arrangement graph $A_{n,k}$, and determine that $\kappa^1(A_{n,k}) = (2k-1)(n-k) - 1$ and $\kappa^2(A_{n,k}) = (3k-2)(n-k) - 2$. In Section 2, we recall some definitions, notations and the structure of (n,k)-arrangement graph $A_{n,k}$. The proofs of our results are in Section 3.

2. Arrangement graphs

For notation and terminology not defined here we follow [22]. Specifically, we use a graph $G = G(V, E)$ to represent an interconnection network, where a vertex $u \in V$ represents a processor and an edge $(u, v) \in E$ represents a link between vertices u and v. If at least one end of an edge is faulty, the edge is said to be faulty; otherwise, the edge is said to be fault-free. Let S be a subset of $V(G)$. The subgraph of G induced by S, denoted by $G[S]$, is the graph with the vertex-set S and the edge-set $\{(u,v) \mid (u,v) \in E(G), u, v \in S\}$. For a vertex u in G, $N(u)$ denotes the set of all neighbors of u, i.e., $N(u) = \{v \mid (u,v) \in E\}$. Let H be a subgraph of G or a subset of $V(G)$, and let $N(S) = \bigcup_{u \in S}(u) \setminus S$. We use K_n to denote the complete graph of order n, and $d(u,v)$ to denote the distance between u and v, the length of a shortest path between u and v in G. The diameter of G is defined as the maximum distance between any two vertices in G.

For any subset $F \subset V$, the notation $G - F$ denotes a graph obtained by removing all vertices in F from G and deleting those edges with at least one end-vertex in F, simultaneously. If $G - F$ is disconnected, F is called a *separating set*. A separating set F is called a *k-separating set* if $|F| = k$. The maximal connected subgraphs of $G - F$ are called *components*. The *connectivity* $\kappa(G)$ of G is defined as the minimum k for which G has a k-separating set; otherwise $\kappa(G)$ is defined $n - 1$ if $G = K_n$. A graph G is called to be *k-connected* if $\kappa(G) \geqslant k$. A k-separating set is called to be *minimum* if $k = \kappa(G)$.

Network reliability is one of the major factors in designing the topology of an interconnection network. The well-known hypercube is the first major class of interconnection networks. As another topology of an interconnection network, Akers and Krishnamurthy [1] proposed the star graph S_n, which has smaller degree, diameter, and average distance than the comparable hypercube, while reserving symmetry properties and desirable fault-tolerant characteristics. As a result, the star graph has been recognized as an alternative to the hypercube. However, the star graph is less flexible in adjusting its sizes. With the restriction on the number of vertices, there is a large gap between $n!$ and $(n+1)!$ for expanding an S_n to S_{n+1}. To relax the restriction of the numbers of vertices $n!$ in S_n. The arrangement graph was proposed by Day and Tripathi [7] as a generalization of the star graph S_n. It is more flexible in its size than S_n.

Definition 2.1. Given two positive integers n and k with $n > k$, let $\langle n \rangle$ denote the set $\{1, 2, \ldots, n\}$, and let $P_{n,k}$ be a set of arrangements of k elements in $\langle n \rangle$. The *(n,k)-arrangement graph*, denoted by $A_{n,k}$, has vertex-set $V(A_{n,k}) = P_{n,k}$ and edge-set $E(A_{n,k}) = \{(p,q) \mid p \text{ and } q \text{ differ in exactly one position}\}$.

The graph shown in Fig. 1 is a $(4,2)$-arrangement graph $A_{4,2}$.

Clearly, $A_{n,k}$ is a $k(n-k)$-regular graph with $\frac{n!}{(n-k)!}$ vertices. It was showed by Day and Tripathi [7] that $A_{n,k}$ is vertex-symmetric and edge-symmetric and has the diameter of $\lfloor \frac{3k}{2} \rfloor$. Day and Tripathi [8] showed that the connectivity is $\kappa(A_{n,k}) = k(n-k)$.

Moreover, $A_{n,1}$ is isomorphic to the complete graph K_n, and $A_{n,n-1}$ is isomorphic to the n-dimensional star graph

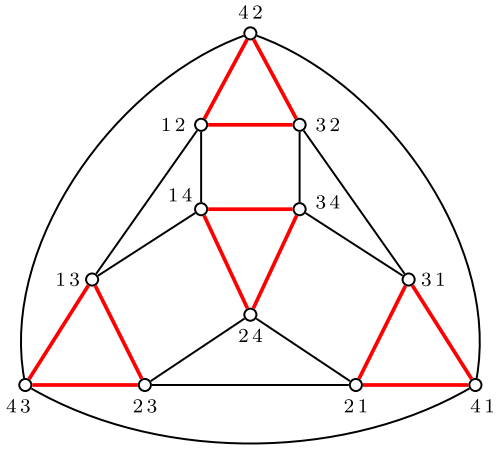

Fig. 1. The structure of $A_{4,2}$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

S_n. Chiang and Chen [6] showed that $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n.

For two distinct i and j in $\langle n\rangle$, let $V_{n,k}^{j:i}$ be the set of all vertices in $A_{n,k}$ with the jth position being i, that is,

$$V_{n,k}^{j:i} = \{p \mid p = p_1 \cdots p_j \cdots p_k \in P_{n,k} \text{ and } p_j = i\}.$$

For a fixed position $j \in \langle n\rangle$, $\{V_{n,k}^{j:i} \mid 1 \leqslant i \leqslant n\}$ forms a partition of $V(A_{n,k})$. Let $A_{n,k}^{j:i}$ denote the subgraph of $A_{n,k}$ induced by $V_{n,k}^{j:i}$. Then for each $j \in \langle n\rangle$, $A_{n,k}^{j:i}$ is isomorphic to $A_{n-1,k-1}$. For example, a partition of $A_{4,2}$ is shown in Fig. 1, where red triangles are $A_{4,2}^{2:i}$'s with $i \in \langle 4\rangle$, which is isomorphic to $A_{3,1} = K_3$.

Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. It is easy to check that each $A_{n,k}^{j:i}$ is a subgraph of $A_{n,k}$, and we say that $A_{n,k}$ is decomposed into n subgraphs $A_{n,k}^{j:i}$'s according to the jth position. For simplicity, by the symmetry of $A_{n,k}$ we shall take j as the last position k, and use $A_{n,k}^{i}$ to denote $A_{n,k}^{k:i}$.

Let $E(i,j)$ be the set of edges between $A_{n,k}^{i}$ and $A_{n,k}^{j}$, that is,

$$E(i,j) = \left\{(p,q) \in E(A_{n,k}) \;\middle|\; p \in V\left(A_{n,k}^{i}\right) \text{ and } q \in \left(A_{n,k}^{j}\right)\right\}.$$

Clearly, $E(i,j)$ is a perfect matching (a set of edges in which any two edges have no common end-vertex) between $A_{n,k}^{i}$ and $A_{n,k}^{j}$, and

$$\left|E(i,j)\right| = \frac{(n-2)!}{(n-k-1)!}. \tag{2.1}$$

Let $u \in V(A_{n,k}^{i})$ for some $i \in \langle n\rangle$ and I be a subset of $\langle n\rangle$. We use $N^{I}(u)$ to denote the set of all neighbors of u in some $A_{n,k}^{j}$ with $j \in I$. Particularly, we use $N^{\bar{i}}(u)$ and $N^{i}(u)$ as an abbreviation of $N^{\langle n\rangle-\{i\}}(u)$ and $N^{\{i\}}(u)$, respectively, and call vertices in $N^{\bar{i}}(u)$ the *outer neighbors* of u. Obviously, every vertex u of $A_{n,k}^{i}$ has $n-k$ outer neighbors, and two arbitrary outer neighbors of u are distributed in distinct subgraphs. It follows from the definitions that

$$\left|N^{i}(u)\right| = (k-1)(n-k) \quad \text{and} \quad \left|N^{\bar{i}}(u)\right| = n-k, \tag{2.2}$$

and $N^{\bar{i}}(u) \cap N^{\bar{i}}(v) = \emptyset$ for any two distinct u and v in $A_{n,k}^{i}$.

We say that one vertex u is adjacent to some subgraph $A_{n,k}^{j}$ if u has an outer neighbor in $A_{n,k}^{j}$. Let

$$V_i = \big\{u_1u_2\cdots u_{i-1}xu_{i+1}\cdots u_k \;\big|\; x \in \langle n\rangle \setminus \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k\}\big\}.$$

Then the graph induced by V_i is a complete graph of order $n-k+1$ and a subgraph of $A_{n,k}^{u_k}$, which implies that any two adjacent vertices have exactly $(n-k-1)$ common neighbors. Thus, by the edge-transitivity of $A_{n,k}$, for any edge e,

$$\begin{aligned}\left|N(e)\right| &= 2k(n-k) - (n-k-1) - 2 \\ &= (2k-1)(n-k) - 1.\end{aligned} \tag{2.3}$$

Other fault tolerant properties of the arrangement graph have received considerable attention in the literature. First, Day and Tripathi [9] showed the existence of pancyclicity. Hsieh et al. [12] investigated the existence of hamiltonian cycle in $A_{n,k}$ with faulty vertices, Lo and Chen [16] studied hamiltonian connectedness of $A_{n,k}$ with faulty edges. Hsu et al. [13] further obtained an optimal result that the graph $A_{n,k}$ $(n \geqslant k+2)$ is $(k(n-k)-2)$-hamiltonian and $(k(n-k)-3)$-hamiltonian connected in $G - F$ for any $F \subset V(G) \cup E(G)$ with $|F| \leqslant f$. Teng et al. [19] showed that $A_{n,k}$ is panpositionable hamiltonian and panconnected if $k > 1$ and $n \geqslant k+2$. In addition, Bai et al. [2] proposed a distributed algorithm with optimal time complexity and without message redundancy for one-to-all broadcasting in one-port communication model on the fault-free arrangement graphs, and also developed a fault tolerant broadcasting algorithm with less than $k(n-k)$ faulty edges. Chen et al. [3] presented efficient one/all-to-all broadcasting algorithms on the arrangement graphs by constructing $n-k$ spanning trees, where the height of each tree is $2n-1$.

3. Conditional connectivity of the arrangement graph

We first consider the conditional connectivity of $(4,2)$-arrangement graph $A_{4,2}$. For any edge e of $A_{4,2}$, $N(e)$ is an R^1-vertex cut with $|N(e)| = (2k-1)(n-k)-1 = 5$, but the R^1-connectivity of $A_{4,2}$ is 4. So $N(e)$ is not the minimum R^1-vertex cut coinciding with the R^1-connectivity of $A_{4,2}$. In the following, we consider the R^1-connectivity of $A_{n,k}$ for $k \geqslant 3$, $n \geqslant k+1$.

Lemma 3.1. *Let $F \subset V(A_{n,k})$ be a faulty set of arrangement graph $A_{n,k}$. Let I be a subset of $\langle n\rangle$ such that $f_i \geqslant (k-1)(n-k)$ for any $i \in I$ and let*

$$A_{n,k}^{I} = \bigcup_{i\in I} A_{n,k}^{i}, \qquad F_I = \bigcup_{i\in I} F_i.$$

Then

(1) *If $|F| \leqslant (2k-1)(n-k)-1$ with $k \geqslant 3$, then $|I| \leqslant 2$, and $A_{n,k} - (A_{n,k}^{I} \cup F)$ is connected.*
(2) *If $|F| \leqslant (3k-2)(n-k)-2$ with $k \geqslant 4$, then $|I| \leqslant 3$, and $A_{n,k} - (A_{n,k}^{I} \cup F)$ is connected.*

Proof. (1) It is easy to see that $|I| \leqslant 2$ because of $3((k-1)(n-k)) > (2k-1)(n-k) - 2 \geqslant |F|$.

Suppose that $f_j \leqslant (k-1)(n-k) - 1$ for any $j \in \langle n\rangle - I$. Then $A^j_{n,k} - F_j$ is connected since $A^j_{n,k}$ is $(k-1)(n-k)$-connected. To show that $A_{n,k} - (A^I_{n,k} \cup F)$ is connected, it suffices to show that $A^x_{n,k}$ and $A^y_{n,k}$ are connected in $A_{n,k} - F$ for any two distinct $x, y \in \langle n\rangle - I$. For $k \geqslant 3$ and $n \geqslant k+1$, we have that

$$\begin{aligned}\big|E(x,y)\big| &= \frac{(n-2)!}{(n-k-1)!} \\ &= (n-2)(n-3)\cdots(n-k) \\ &\quad \times \begin{cases} > 2((k-1)(n-k)-1) \\ \quad \text{if } k \geqslant 4 \text{ or } k = 3 \text{ and } n \geqslant 6; \\ = 2(2n-7) \\ \quad \text{if } k = 3 \text{ and } n \in \{4,5\}. \end{cases}\end{aligned} \tag{3.1}$$

When either $k \geqslant 4$ or $k = 3$ and $n \geqslant 6$, there is a fault-free edge e in $E(x,y)$ since $f_x + f_y \leqslant 2((k-1)(n-k)-1) < |E(x,y)|$, so $A^x_{n,k} - F_x$ and $A^y_{n,k} - F_y$ are connected by the fault-free edge e in $E(x,y)$.

When $k = 3$ and $n \in \{4,5\}$, if there is a fault-free edge e in $E(x,y)$, then $A^x_{n,k} - F_x$ and $A^y_{n,k} - F_y$ are connected by the fault-free edge e in $E(x,y)$, we are done; otherwise, we assume, without loss of generality, that all the edges in $E(x,y)$ are faulty. Then by (3.1), $f_x = f_y = 2n-7$ for $n \in \{4,5\}$. In this case, $|F| \leqslant 7n-24$, $|\langle n\rangle - I| \geqslant 3$. For any three distinct $x, y, z \in \langle n\rangle - I$,

$$\begin{aligned}|F| - (f_x + f_y) &\leqslant (7n-24) - 2(2n-7) \\ &= 3n - 10 \\ &= \begin{cases} 5 < |E(x,z)| = |E(z,y)| & \text{if } n = 5; \\ 2 = |E(x,z)| = |E(z,y)| & \text{if } n = 4. \end{cases}\end{aligned} \tag{3.2}$$

If $n = 5$ then, by (3.2), there are a fault-free edge e_1 in $E(x,z)$ and a fault-free edge e_2 in $E(z,y)$. Then $A^x_{5,3}$ and $A^y_{5,3}$ can be connected in $A_{5,3} - F$ by $A^z_{5,3}$ and the fault-free edges e_1 and e_2.

If $n = 4$, then $f_x = f_y = 1$, and every vertex in $A^i_{4,3}$ has only one outer neighbor for each $i \in \{1,2,3,4\}$. Thus, by (3.2), there are a fault-free edge e_1 in $E(x,z)$ and a fault-free edge e_2 in $E(z,y)$. Then $A^x_{4,3}$ and $A^y_{4,3}$ can be connected in $A_{4,3} - F$ by $A^z_{4,3}$ and the fault-free edges e_1 and e_2.

(2) It is easy to see that $|I| \leqslant 3$ because of $4((k-1)(n-k)) > (3k-1)(n-k) - 3 \geqslant |F|$.

By the hypothesis, for any $j \in \langle n\rangle \setminus I$, $f_j \leqslant (k-1)(n-k) - 1$, that is, $A^j_{n,k} - F_j$ is connected since $A^j_{n,k}$ is $(k-1)(n-k)$-connected. To show that $A_{n,k} - (A^I_{n,k} \cup F)$ is connected, we only need to show that $A^x_{n,k}$ and $A^y_{n,k}$ are connected in $A_{n,k} - F$ for any two distinct $x, y \in \langle n\rangle - I$. Consider any two distinct $A^x_{n,k}$ and $A^y_{n,k}$, where $x, y \in \langle n\rangle \setminus I$. Note that, for $k \geqslant 4$, and $n \geqslant k+2$,

$$\begin{aligned}\big|E(x,y)\big| &= \frac{(n-2)!}{(n-k-1)!} \\ &= (n-2)(n-3)\cdots(n-k) \\ &\geqslant (n-2)(n-3)(n-4) \\ &> 2\big((k-1)(n-k)-1\big) \\ &> f_x + f_y,\end{aligned}$$

there is a fault-free edge in $E(x,y)$ that connects $A^x_{n,k}$ and $A^y_{n,k}$ in $A_{n,k} - F$. By the arbitrariness of x and y, $A_{n,k} - (A^I_{n,k} \cup F)$ is connected. □

Lemma 3.2. *$\kappa^1(A_{n,k}) \leqslant (2k-1)(n-k) - 1$ for $k \geqslant 3$, $n \geqslant k+1$.*

Proof. When $k = 3$, let $e = (u,v)$ be a j-dimensional edge of $A_{n,k}$. By the edge symmetry of $A_{n,k}$, we decompose $A_{n,k}$ along some dimension $i_1 \neq j$ such that e is totally contained in $A^{i_1}_{n,k}$. We further decompose $A^{i_1}_{n,k}$ along another dimension such that e is contained in some subgraph $A^{i_1,i_2}_{n,k}$, which is isomorphic to $K_{n-k+1} = K_{n-2}$. Obviously, $A^{i_1,i_2}_{n,k} \setminus \{u,v\} \subseteq N(e)$. As there are $\frac{(n-2)!}{(n-k-1)!}$ matching edges between two subgraphs $A^x_{n,k}$ and $A^y_{n,k}$ for any $x \neq y \in \langle n\rangle \setminus \{i_1\}$, $A_{n,k} - A^{i_1}_{n,k}$ is still connected. Obviously, $A^{i_1}_{n,k} - A^{i_1,i_2}_{n,k}$ is still connected, and every vertex of $A^{i_1}_{n,k} - A^{i_1,i_2}_{n,k}$ has $n-k$ outer neighbors in $A_{n,k} - A^{i_1}_{n,k}$ by (2.2), so $A_{n,k} - A^{i_1,i_2}_{n,k}$ is still connected. From the discussion above, $|N(e)| = (2k-1)(n-k) - 1$ by (2.3), and $A_{n,k} - e$ has exactly two connected components, every vertex of $A_{n,k} - N(e)$ has degree at least one.

When $k \geqslant 4$, we decompose $A_{n,k}$ continuously $k-1$ times such that e is totally contained in some subgraph $A^{i_1,i_2,\ldots,i_{k-1}}_{n,k}$ which is isomorphic to K_{n-k+1}. $|N(e)| = (2k-1)(n-k) - 1$, and $A_{n,k} - e$ has exactly two connected components, every vertex of $A_{n,k} - N(e)$ has degree at least 1. □

Theorem 3.3. *$\kappa^1(A_{n,k}) = (2k-1)(n-k) - 1$ for $k \geqslant 3$, $n \geqslant k+1$.*

Proof. By Lemma 3.2, it suffices to show that $\kappa^1(A_{n,k}) \geqslant (2k-1)(n-k) - 1$ by contradiction. We assume that F is an R^1-vertex cut of $A_{n,k}$ with $|F| \leqslant (2k-1)(n-k) - 2$, and denote $F_i = A^i_{n,k} \cap F$, $f_i = |F_i|$ for $1 \leqslant i \leqslant n$. Then $\sum_{i=1}^n f_i = |F| \leqslant (2k-1)(n-k) - 2$.

If $|I| = 0$, $A_{n,k} - F$ is connected, which contradicts that F is an R^1-vertex cut. To complete the proof of this theorem, we consider two cases as follows

Case 1. $|I| = 1$.

Since F is an R^1-vertex cut, $A_{n,k} - F$ is disconnected. Then the subgraph $A_{n,k} - (A^i_{n,k} \cup F)$ is connected by Lemma 3.1(1). Let H be a connected component which has no vertices in $A^j_{n,k}$, $j \in \langle n\rangle \setminus \{i\}$, H is totally contained in $A^i_{n,k} - F_i$ by Lemma 3.1(1). Obviously, $|H| \geqslant 2$ because every vertex of $A_{n,k} - F$ has at least one fault-free neighbor.

Pick an edge $e = uv$ in H. Denote $T = A_{n,k}[u,v]$.

$$N_{A^i_{n,k}}(T) - (H - T) \subseteq F_i. \tag{3.3}$$

By (3.3), we have

$$\begin{aligned}|F_i| &\geqslant \left|N_{A^i_{n,k}}(T) - (H-T)\right| \\ &\geqslant \left|N_{A^i_{n,k}}(T)\right| - \left|(H-T)\right| \\ &\geqslant \left|N_{A^i_{n,k}}(T)\right| - |H| + |T| \\ &\geqslant 2(k-1)(n-k) - (n-k-1) - 2 - |H| + 2 \\ &\geqslant (2k-3)(n-k) - |H| + 1.\end{aligned}$$

Since $N^i(H) \subseteq F - F_i$, $|F - F_i| \geqslant |H|(n-k)$. Thus we have

$$\begin{aligned}&(2k-1)(n-k) - 2 \\ &\quad \geqslant |F| = |F_i| + |F - F_i| \\ &\quad \geqslant (2k-3)(n-k) - |H| + 1 + |H|(n-k) \\ &\quad \geqslant \big(2k - 3 + |H|\big)(n-k) - |H| + 1.\end{aligned}$$

That is $(|H|-2)(n-k) \leqslant |H| - 3$, a contradiction.

Case 2. $|I| = 2$, and let $I = \{i, j\}$.

By Lemma 3.1(1), $A_{n,k} - (A^i_{n,k} \cup A^j_{n,k} \cup F)$ is connected. Under our hypothesis,

$$\begin{aligned}\left|F \setminus (F_i \cup F_j)\right| &\leqslant (2k-1)(n-k) - 2 - 2\big((k-1)(n-k)\big) \\ &= n - k - 2. \qquad (3.4)\end{aligned}$$

Since every vertex in $A^i_{n,k} - F_i$ has exactly $(n-k)$ outer neighbors and any two distinct vertices in $A^i_{n,k} - F_i$ have no common outer neighbors, $A^i_{n,k} - F_i$ has at most one vertex, say u_i, isolated from $A_{n,k} - (A^i_{n,k} \cup A^j_{n,k} \cup F)$. Similarly, $A^j_{n,k} - F_j$ has at most one vertex, say u_j, isolated from $A_{n,k} - (A^i_{n,k} \cup A^j_{n,k} \cup F)$. If no of u_i or u_j exists, $A_{n,k} - F$ is connected, a contradiction; otherwise, since every vertex of $A_{n,k} - F$ has at least one fault-free neighbor, u_i and u_j must occur at the same time, and $(u_i, u_j) \in E(A_{n,k})$. Since the subgraph $A_{n,k}[u_i, u_j]$ has exactly $|N(u_i) \cap N(u_j)| = n - k - 1$ outer neighbors in $A_{n,k} - A^I_{n,k}$, while $|N(u_i) \cap N(u_j)| > n - k - 2 \geqslant |F \setminus (F_i \cup F_j)|$, the subgraph $A_{n.k}[u_i, u_j]$ has at least one fault-free outer neighbor in $A_{n,k} - (A^i_{n,k} \cup A^j_{n,k} \cup F)$, so $A_{n,k} - F$ is still connected, a contradiction.

So, the proof is complete. □

Since $A_{n,n-1}$ is isomorphic to a star graph S_n and $A_{n,n-2}$ is isomorphic to the alternating group graph AG_n, we have the following corollaries by Theorem 3.3, immediately.

Corollary 3.4. *(See S.C. Hu and C.B. Yang [14], and Cheng and Lipman [5].)* $\kappa^1(S_n) = 2n - 4$ *for* $n \geqslant 4$.

Corollary 3.5. *(See E. Cheng et al. [4], and Z. Zhang et al. [21].)* $\kappa^1(AG_n) = 4n - 11$ *for* $n \geqslant 5$.

As the star graph S_n is a special case of $A_{n,k}$ with $n = k+1$, and Wan, Zhang [18] have obtained $\kappa^2(S_n) = 6(n-3)$ with $n \geqslant 5$, we only need to consider $\kappa^2(A_{n,k})$ with $k \geqslant 4$, $n \geqslant k+2$.

Lemma 3.6. $\kappa^2(A_{n,k}) \leqslant (3k-2)(n-k) - 2$ *for* $k \geqslant 4$, $n \geqslant k+2$.

Proof. Let K_3 be a triangle of $A_{n,k}$. K_3 must be in some complete subgraph K_{n-k+1}, and the three edges of K_3 are some j-dimensional edges. When $k = 3$, we decompose $A_{n,k}$ along some dimension $i_1 \neq j$ such that K_3 is contained totally in $A^{i_1}_{n,k}$. We further decompose $A^{i_1}_{n,k}$ along another dimension $i_2 \neq j$ such that K_3 is contained in some subgraph $A^{i_1,i_2}_{n,k}$, which is isomorphic to $K_{n-k+1} = K_{n-2}$. Obviously, $A^{i_1,i_2}_{n,k} - K_3 \subseteq N(K_3)$. As there are $\frac{(n-2)!}{(n-k-1)!}$ matching edges between two subgraphs $A^x_{n,k}$ and $A^y_{n,k}$ for any $x \neq y \in \langle n\rangle \setminus \{i_1\}$, $A_{n,k} - A^{i_1}_{n,k}$ is still connected, and every vertex has $(k-1)(n-k) \geqslant 2$ neighbors in $A_{n,k} - A^{i_1}_{n,k}$. Obviously, $A^{i_1}_{n,k} - A^{i_1,i_2}_{n,k}$ is still connected, and every vertex of $A^{i_1}_{n,k} - A^{i_1,i_2}_{n,k}$ has $n - k \geqslant 2$ outer neighbors in $A_{n,k} - A^{i_1}_{n,k}$, so $A_{n,k} - A^{i_1,i_2}_{n,k}$ is still connected. From the discussion above, $|N(K_3)| = (3k-2)(n-k) - 2$, and $A_{n,k} - K_3$ has exactly two connected components, every vertex of $A_{n,k} - N(K_3)$ has degree at least two.

When $k \geqslant 4$, similarly, we decompose $A_{n,k}$ continuously $k-1$ times such that K_3 is totally contained in some subgraph $A^{i_1,i_2,\ldots,i_{k-1}}_{n,k}$ which is isomorphic to K_{n-k+1}. $|N(K_3)| = (3k-2)(n-k) - 2$, and $A_{n,k} - K_3$ has exactly two connected components, every vertex of $A_{n,k} - N(K_3)$ has at least two fault-free neighbors. □

Theorem 3.7. $\kappa^2(A_{n,k}) = (3k-2)(n-k) - 2$ *for* $k \geqslant 4$, $n \geqslant k+2$.

Proof. By Lemma 3.6, it is only need to show that $\kappa^2(A_{n,k}) \geqslant (3k-2)(n-k) - 2$. We assume, by contradiction, that F is an R^2-vertex cut of $A_{n,k}$ with $|F| \leqslant (3k-2)(n-k) - 3$, and denote $F_i = A^i_{n,k} \cap F$, and $f_i = |F_i|$ for $1 \leqslant i \leqslant n$. Obviously, $\sum_{i=1}^n f_i = |F| \leqslant (3k-2)(n-k) - 3$.

If $|I| = 0$, $A_{n,k} - F$ is connected, which contradicts that F is a restricted vertex cut. To complete the proof of this theorem, we consider three cases as follows.

Case 1. $|I| = 1$, and let $I = \{i\}$.

Since $A_{n,k} - F$ is disconnected and $A_{n,k} - (A^i_{n,k} \cup F)$ is connected by Lemma 3.1(2), $A^i_{n,k} - F_i$ is disconnected. Let H be a connected component which has no vertices in $A^j_{n,k}$, $j \in \langle n\rangle \setminus \{i\}$, H is totally contained in $A^i_{n,k} - F_i$ by Lemma 3.1(2). Obviously, $N(H) \subseteq F$. Since every vertex of H has at least two neighbors, $|H| \geqslant 3$. Let $h = |H|$ and we choose a subset $T \subseteq H$ such that $|T| = 3$. Obviously,

$$N_{A^i_{n,k}}(T) - (H - T) \subseteq F_i. \qquad (3.5)$$

When $n \geqslant k+2$, one edge of the induced graph by T is contained in a complete K_{n-k+1}, that is, every vertex in $K_{n-k+1} - T$ is a common neighbor of vertices in T, and the induced subgraph by T has at most three edges. It follows that

$$\left|N_{A^i_{n,k}}(T)\right| \geqslant 3(k-1)(n-k) - 6 - 2(n-k-2).$$

By (3.5), we have

$$\begin{aligned} f_i &\geqslant \left|N_{A^i_{n,k}}(T) \setminus (H-T)\right| \\ &\geqslant \left|N_{A^i_{n,k}}(T)\right| - (h-3) \\ &\geqslant 3(k-1)(n-k) - 6 - 2(n-k-2) - (h-3) \\ &= (3k-5)(n-k) - h + 1, \end{aligned}$$

that is,

$$f_i \geqslant (3k-5)(n-k) - h + 1. \tag{3.6}$$

By Lemma 3.1(2), $|F| - f_i \geqslant h(n-k)$, and so

$$f_i \leqslant (3k-2)(n-k) - 1 - h(n-k). \tag{3.7}$$

Then, combining (3.6) with (3.7) yields

$$(h-3)(n-k) \leqslant h-6,$$

a contradiction.

Case 2. $|I| = 2$, and let $I = \{i, j\}$.

Since $A_{n,k} - F$ is disconnected and $A_{n,k} - (A^I_{n,k} \cup F)$ is connected by Lemma 3.1(2). We divide the proof into three subcases.

Subcase 2.1. $A^i_{n,k} - F_i$ and $A^j_{n,k} - F_j$ both are connected.

Since, for any $k \geqslant 4$,

$$\begin{aligned} \frac{(n-1)!}{(n-k-1)!} &\geqslant (n-1)(n-2)\cdots(n-k) \\ &> (3k-2)(n-k) - 3 \geqslant |F|, \end{aligned}$$

we have that, for any $i \in I$,

$$\left|V\left(A^i_{n,k} - F_i\right)\right| = \frac{(n-1)!}{(n-k-1)!} - |F_i| > |F - F_i|.$$

Thus, there exists at least one fault-free edge connecting $A^i_{n,k} - F_i$ to $A_{n,k} - (A^I_{n,k} \cup F)$, $A_{n,k} - F$ is connected, a contradiction.

Subcase 2.2. Exactly one of $A^i_{n,k} - F_i$ and $A^j_{n,k} - F_j$ is connected.

By an argument similar to case 1, we arrive at a contradiction.

Subcase 2.3. Both of $A^i_{n,k} - F_i$ and $A^j_{n,k} - F_j$ are disconnected.

Let H be a component of $A_{n,k} - F$ which is contained in $A^I_{n,k} - F_I$. Obviously, H has no fault-free neighbor in $A_{n,k} - (A^I_{n,k} \cup F)$. Each vertex of $(A^i_{n,k} - F_i) \cap H$ has at least one fault-free neighbor in $A^i_{n,k} - F_i$ since it has at most one fault-free neighbor in $A^j_{n,k} - F_j$. Thus, $|F_i| \geqslant (2k-3)(n-k) - 1$ by Theorem 3.3. $A^j_{n,k} - F_j$ has the similar property, and so $|F_j| \geqslant (2k-3)(n-k) - 1$.

$$\begin{aligned} (3k-2)(n-k) - 3 &\geqslant |F| \geqslant |F_i| + |F_j| \\ &\geqslant 2\big((2k-3)(n-k) - 1\big), \end{aligned}$$

which induces $(k-4)(n-k) \leqslant -1$ with $k \geqslant 4$ and $n \geqslant k+2$, a contradiction.

Case 3. $|I| = 3$, and let $I = \{i, j, l\}$.

Under our hypothesis,

$$\begin{aligned} &\left|F \setminus (F_i \cup F_j \cup F_l)\right| \\ &\quad \leqslant (3k-2)(n-k) - 3 - 3\big((k-1)(n-k)\big) \\ &\quad = n-k-3. \end{aligned} \tag{3.8}$$

By Lemma 3.1(2), $A_{n,k} - A^i_{n,k} \cup A^j_{n,k} \cup A^h_{n,k} \cup F$ is connected. Since every vertex in $A^i_{n,k} - F_i$ has exactly $(n-k)$ outer neighbors and any two outer neighbors are in distinct subgraphs, every vertex of $A^i_{n,k} \cup A^j_{n,k} \cup A^h_{n,k}$ has at least $n-k-2$ neighbors outside $A^i_{n,k} \cup A^j_{n,k} \cup A^h_{n,k}$. Thus, any vertex of $A^i_{n,k} - F_i$ is connected to $A_{n,k} - A^i_{n,k} \cup A^j_{n,k} \cup A^h_{n,k} \cup F$. The similar result applies to $A^j_{n,k} - F_j$ and $A^h_{n,k} - F_h$. Thus, $A_{n,k} - F$ is connected, a contradiction. □

When $n = k+2$, the arrangement graph $A_{n,k}$ is isomorphic to the alternating group graph AG_n [21]. Thus, we obtain the following result.

Corollary 3.8. *(See Z. Zhang et al. [21].)* $\kappa^2(AG_n) = 6n - 18$ *for* $n \geqslant 6$.

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