Trees with unique minimum $p$-dominating sets *

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Abstract

Let $p$ be a positive integer and $G = (V, E)$ a simple graph. A $p$-dominating set of $G$ is a subset $S$ of $V$ such that every vertex not in $S$ is dominated by at least $p$ vertices in $S$. The $p$-domination number $\gamma_p(G)$ is the minimum cardinality among the $p$-dominating sets of $G$. In this paper, for $p \geq 2$, we give three equivalent conditions for trees with unique minimum $p$-dominating sets and also give a constructive characterization of such trees.

Key words: domination, $p$-domination, tree

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1 Introduction

For notation and graph theory terminology we follow [3, 10, 11]. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of a vertex $v \in V(G)$ are denoted by $N_G(v) =$

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\{u \in V(G)|uv \in E(G)\}, \ N_G[v] = N_G(v) \cup \{v\} \text{ and } deg_G(v) = |N_G(v)|, \text{ respectively. The maximum degree } \Delta(G) = \max\{deg_G(v) : v \in V(G)\}. \text{ For } S \subseteq V(G), \text{ the subgraph induced by } S \text{ is denoted by } G[S]. \text{ For a pair of vertices } u, v \in V(G), \text{ the distance } d_G(u, v) \text{ of } u \text{ and } v \text{ is the length of the shortest } uv\text{-paths in } G. \text{ The diameter of } G \text{ is } d(G) = \max\{d_G(u, v) : u, v \in V(G)\}.

Let } D \text{ be a subset of } V(G) \text{ and } p \text{ a positive integer. For any } x \in D, \text{ a vertex } y \text{ not in } D \text{ is called a } p\text{-private neighbor of } x \text{ with regard to } D \text{ if } y \text{ is a neighbor of } x \text{ and } |D \cap N_G(y)| = p. \text{ The } p\text{-private neighborhood of } x \text{ with regard to } D, \text{ denoted by } P{\text{N}_p}(x, D, G), \text{ is the set of all } p\text{-private neighbors of } x \text{ with regard to } D \text{ in } G. \text{ If the graph } G \text{ is clear from the context, we will simply use } P{\text{N}_p}(x, D) \text{ instead of } P{\text{N}_p}(x, D, G).

Let } T \text{ be a tree and } p \geq 2 \text{ a positive integer. A } p\text{-leaf of } T \text{ is a vertex with degree at most } p-1 \text{ in } T. \text{ Denote the set of } p\text{-leaves of } T \text{ by } L_p(T) \text{ and let } X_p(T) = V(T) - L_p(T). \text{ Then, for } x \in X_p(T), \text{ } deg_T(x) \geq p. \text{ Note that the } 2\text{-leaves are the usual leaves and } L_2(T) \text{ is the set of leaves of } T. \text{ Therefore, we also denote } L_2(T) \text{ by } L(T). \text{ If } T \text{ is a rooted tree } T, \text{ then, for every } v \in V(T), \text{ we let } C(v) \text{ and } D(v) \text{ denote the set of children and descendants, respectively, of } v, \text{ and define } D[v] = D(v) \cup \{v\}.

In [6], Fink and Jacobson introduced the concept of } p\text{-domination. Let } p \text{ be a positive integer. A subset } S \text{ of } V(G) \text{ is a } p\text{-dominating set of } G \text{ if, for every } v \in V(G) - S, |S \cap N_G(v)| \geq p. \text{ The } p\text{-domination number } \gamma_p(G) \text{ is the minimum cardinality among the } p\text{-dominating sets of } G. \text{ Any } p\text{-dominating set of } G \text{ with cardinality } \gamma_p(G) \text{ will be called a } \gamma_p\text{-set of } G. \text{ Note that the } \gamma_1\text{-set is the classic minimum dominating set. For any } S, T \subseteq V(G), \text{ } S \text{ } p\text{-dominates } T \text{ in } G \text{ if, for every } v \in T - S, |S \cap N_G(v)| \geq p.

Unique domination in graphs has been investigated in many papers (see, for example, [2, 4, 5, 7, 8, 9]). In [8], Gunther et al. characterized all trees with unique minimum dominating sets. In this paper, for } p \geq 2, \text{ we first give three equivalent conditions for trees with unique minimum } p\text{-dominating sets, and then we give a constructive characterization of such trees.
2 Equivalent conditions for trees with unique $\gamma_p$-sets

Lemma 1. ([1]) Every $p$-dominating set of a graph $G$ contains any vertex of degree at most $p - 1$.

Theorem 2. Suppose that $T$ is a tree and $p \geq 2$ is a positive integer. Let $D$ be a subset of $V(T)$. Then the following conditions are equivalent:

(i) $D$ is a unique $\gamma_p$-set of $T$;
(ii) $D$ is a $\gamma_p$-set of $T$ satisfying either $|D \cap N_T(x)| \leq p - 2$ or $|PN_p(x, D)| \geq 2$ for every $x \in D \cap X_p(T)$;
(iii) $D$ is a $p$-dominating set of $T$ satisfying either $|D \cap N_T(x)| \leq p - 2$ or $|PN_p(x, D)| \geq 2$ for every $x \in D \cap X_p(T)$.

Proof. We will prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (i). (ii) ⇒ (iii) is obvious.

(i) ⇒ (ii) : Suppose to the contrary that there exists some $x \in D \cap X_p(T)$ such that $|D \cap N_T(x)| \geq p - 1$ and $|PN_p(x, D)| \leq 1$.

If $|PN_p(x, D)| = 0$, then $|D \cap N_T(x)| = p - 1$ (Otherwise, $D \setminus \{x\}$ is a $p$-dominating set of $T$, which contradicts that $D$ is a $\gamma_p$-set of $T$). Thus, there exists a neighbor, denoted by $y$, of $x$ which is not in $D$ since $deg_T(x) \geq p$. Let $D' = (D \setminus \{x\}) \cup \{y\}$, then $D'$ is a $\gamma_p$-set of $T$ different from $D$, a contradiction.

If $|PN_p(x, D)| = 1$, then we denote $PN_p(x, D)$ by $\{y\}$ and let $D' = (D \setminus \{x\}) \cup \{y\}$. Since $|D \cap N_T(x)| \geq p - 1$, $|D' \cap N_T(x)| = |(D \cap N_T(x)) \cup \{y\}| \geq p$. Hence $D'$ is a $\gamma_p$-set of $T$ different from $D$, a contradiction.

(iii) ⇒ (i) : Assume that there is a tree $T$ which has a $p$-dominating set $D$ satisfying the condition of (iii) but $D$ is not a unique $\gamma_p$-set of $T$. Let $T$ be such a counterexample of minimum order. Then, by Lemma 1, $D \cap X_p(T) \neq \emptyset$. Let $S$ be an arbitrary $\gamma_p$-set of $T$. In the following, we only need prove that $S = D$, which contradicts the assumption that $D$ is not a unique $\gamma_p$-set of $T$. 

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If \( d(T) = 2 \), then \( T \) is a star, and so \( |D \cap X_p(T)| \leq 1 \). By \( D \cap X_p(T) \neq \emptyset \), we can denote \( D \cap X_p(T) \) by \( \{a\} \). By Lemma 1 and \( PN_p(a, D) = \emptyset \), \( deg_T(a) = |D \cap N_T(a)| \leq p - 2 \), which contradicts with \( deg_T(a) \geq p \). If \( d(T) = 3 \), then \( T \) contains exact two vertices with degree at least 2. For every \( b \in D \cap X_p(T) \), we have \( deg_T(b) \geq p \) and \( |PN_p(b, D)| \leq 1 \). From \( D \) fulfils (iii), we can derive that \( |D \cap N_T(b)| \leq p - 2 \). So \( deg_T(b) = |D \cap N_T(b)| + 1 \leq p - 1 \), a contradiction. Hence \( d(T) \geq 4 \). Let \( P = wwx \cdots r \) be a longest path in \( T \). We root \( T \) at \( r \). By Lemma 1, \( D(v) \subseteq D \) and \( D(v) \subseteq S \).

**Case 1.** \( deg_T(v) \leq p - 1 \).

By Lemma 1, \( D[v] \subseteq D \). Let \( T' = T - u \), then \( D \cap V(T') = D - \{u\} \) is a \( p \)-dominating set of \( T' \). Note that \( (D \cap V(T')) \cap N_T(z) = D \cap N_T(z) \) and \( PN_p(z, D \cap V(T'), T') = PN_p(z, D, T) \) for every \( z \in (D \cap V(T')) \cap X_p(T') \). Hence \( T' \) is a tree whose \( p \)-dominating set \( D \cap V(T') \) fulfils (iii) since \( T \) and \( D \) fulfil (iii). By our assumption that \( T \) is the counterexample of minimum order, \( D \cap V(T') \) is a unique \( \gamma_p \)-set of \( T' \). So \( \gamma_p(T') = |D \cap V(T')| = |D| - 1 \geq \gamma_p(T) - 1 \).

By Lemma 1, \( D[v] \subseteq S \). Hence \( S \cap V(T') = S - \{u\} \) is a \( p \)-dominating set of \( T' \) with \( |S \cap V(T')| = |S| - 1 = \gamma_p(T) - 1 \leq \gamma_p(T') \). That is, \( S \cap V(T') \) is also a \( \gamma_p \)-set of \( T' \). Hence \( S \cap V(T') = D \cap V(T') \). Thus

\[
S = (S \cap V(T')) \cup \{u\} = (D \cap V(T')) \cup \{u\} = D.
\]

**Case 2.** \( deg_T(v) = p \).

For every \( v' \in D(w) \cap X_p(T) (= C(w) \cap X_p(T)) \), we have \( |D \cap N_T(v')| \geq |D(v')| = deg_T(v') - 1 \geq p - 1 \), and so \( v' \notin D \) since \( T \) and \( D \) fulfil (iii). Then, by Lemma 1, \( D \cap D(w) = L_p(T) \cap D(w) \). By \( deg_T(v) = p \), \( v \notin D \). Since \( D \) \( p \)-dominates \( v \), we have \( w \in D \).

Let \( T' = T - D(w) \), then \( D \cap V(T') \) is a \( p \)-dominating set of \( T' \). Since \( w \in D \) and \( deg_T(w) = 1 \), for every \( z \in (D \cap V(T')) \cap X_p(T') \), \( (D \cap V(T')) \cap N_T(z) = D \cap N_T(z) \) and \( PN_p(z, D \cap V(T'), T') = PN_p(z, D, T) \). Hence \( T' \) is a tree whose \( p \)-dominating set \( D \cap V(T') \) fulfils (iii) since \( T \) and \( D \) fulfil (iii). By our assumption that \( T \) is the
counterexample of minimum order, \( D \cap V(T') \) is a unique \( \gamma_p \)-set of \( T' \). So \( |D \cap V(T')| = \gamma_p(T') \) and \( \gamma_p(T) \leq |D| = |D \cap D(w)| + |D \cap V(T')| = |D \cap D(w)| + \gamma_p(T'). \)

Now we prove \( S = D \). Suppose that \( v \in S \), then, by the definition of \( \gamma_p \)-set and Lemma 1, \( w \notin S \) and \((S \cap D(w)) \cap X_p(T) = \{v\} \) (Assume that \((S \cap D(w)) \cap X_p(T)\) contains another vertex \( v' \), then \( v' \) is a neighbor of \( w \) in \( D(w) \). Note that \( D(v') \subseteq S \) by Lemma 1 and \( |D(v')| \geq p - 1 \). We can replace \( v, v' \) by \( w \) in \( S \) and get a \( p \)-dominating set of \( T \) of order \( |S| - 1 \), a contradiction). Hence \( S \cap D(w) = (L_p(T) \cap D(w)) \cup \{v\} = (D \cap D(w)) \cup \{v\} \). Since \((S \cap V(T')) \cup \{w\}\) is a \( p \)-dominating set of \( T' \) with

\[
|\{w\}| = |S| - |S \cap D(w)| + 1 = \gamma_p(T) - |D \cap D(w)| \leq \gamma_p(T'),
\]

we know that \((S \cap V(T')) \cup \{w\}\) is a \( \gamma_p \)-set of \( T' \). Hence \((S \cap V(T')) \cup \{w\}\) is a \( p \)-dominating set of \( T' \). By \( w \notin S \) and \( w \in D \), we have \( \deg_T(w) \geq p \) and \( w \in D \cap X_p(T) \). Note that

\[
|D \cap N_T(w)| = |(D \cap D(w)) \cap N_T(w)| + |D \cap \{x\}|
\]

\[
= |((S \cap D(w)) \setminus \{v\}) \cap N_T(w)| + |S \cap \{x\}|
\]

\[
= |(S \cap D(w)) \cap N_T(w)| - 1 + |S \cap \{x\}| = |S \cap N_T(w)| - 1 \geq p - 1.
\]

Hence \( |PN_p(w, D, T)| \geq 2 \) since \( T \) and \( D \) satisfy (iii). By \( w \notin S \) and \((S \cap D(w)) \cap X_p(T) = \{v\}, D(w) \cap X_p(T) (= C(w) \cap X_p(T))\) contains a unique vertex \( v \) of degree \( p \). Thus \( PN_p(w, D, T) \cap D(w) = \{v\} \). From \( |PN_p(w, D, T)| \geq 2 \), we know that

\[
PN_p(w, D, T) = PN_p(w, D, T) \cap (D(w) \cup \{x\}) = \{v, x\}.
\]

So \( |D \cap N_T(x)| = p \) and \( x \notin D \cap V(T') = (S \cap V(T')) \cup \{w\} \). Further, \( x \notin S \). To \( p \)-dominate \( x \), \( |S \cap N_T(x)| \geq p \), which contradicts with

\[
|S \cap N_T(x)| = |(S \cap V(T')) \cap N_T(x)|
\]

\[
= |(D \cap N_T(x)) \setminus \{w\}| = |D \cap N_T(x)| - 1 = p - 1.
\]

Hence \( v \notin S \).

To \( p \)-dominate \( v \), \( w \in S \) and, by the definition of \( \gamma_p \)-set, \( S \cap D(w) = L_p(T) \cap D(w) = D \cap D(w) \). Then \( S \cap V(T') \) is a \( p \)-dominating set of \( T' \) with \( |S \cap V(T')| = \)
\[|S| - |S \cap D(w)| = \gamma_p(T) - |D \cap D(w)| \leq \gamma_p(T'),\] which implies that \(S \cap V(T')\) is also a \(\gamma_p\)-set of \(T'\). Hence \(S \cap V(T') = D \cap V(T')\), and so

\[S = (S \cap V(T')) \cup (S \cap D(w)) = (D \cap V(T')) \cup (D \cap D(w)) = D.\]

**Case 3.** \(\text{deg}_T(v) \geq p + 1\).

Note that \(|D \cap N_T(v)| \geq |D(v)| = \text{deg}_T(v) - 1 \geq p\). We have \(v \notin D\) since \(T\) and \(D\) fulfil (iii). Let \(T' = T - D[v]\), then \(D \cap V(T')\) is a \(p\)-dominating set of \(T'\). Since \(v \notin D\) and \(D \cap N_T(v) = D(v) \cup (D \cap \{w\})\), \((D \cap V(T')) \cap N_T(z) = D \cap N_T(z)\) and \(PN_p(z, D \cap V(T'), T') = PN_p(z, D, T)\) for every \(z \in (D \cap V(T')) \cap N_T(T')\). Hence \(T'\) is a tree whose \(p\)-dominating set \(D \cap V(T')\) fulfils (iii) since \(T\) and \(D\) fulfil (iii). As \(T\) is the counterexample of minimum order, \(D \cap V(T')\) is the unique \(\gamma_p\)-set of \(T'\). So \(|D \cap V(T')| = \gamma_p(T')\) and \(\gamma_p(T) \leq |D| = |D(v)| + |D \cap V(T')| = |D(v)| + \gamma_p(T')\).

Now we prove that \(S = D\). Suppose that \(v \in S\), then by the definition of \(\gamma_p\)-set, \(w \notin S\). Thus \((S \cap V(T')) \cup \{w\}\) is a \(p\)-dominating set of \(T'\) with

\[|(S \cap V(T')) \cup \{w\}| = |S| - |D[v]| + 1 = \gamma_p(T) - |D(v)| \leq \gamma_p(T'),\]

which implies that \((S \cap V(T')) \cup \{w\}\) is a \(\gamma_p\)-set of \(T'\). Hence \((S \cap V(T')) \cup \{w\} = D \cap V(T')\). So \(w \in D\). By \(w \notin S\), \(v \notin D\) and \(v \in S\), we have \(w \in D \cap X_p(T)\) and

\[|D \cap N_T(w)| = |(D \cap V(T')) \cap N_T(w)| = |(S \cap V(T')) \cap N_T(w)| = |S \cap N_T(w)| - 1 \geq p - 1.\]

Hence \(|PN_p(w, D, T)| \geq 2\) since \(T\) and \(D\) fulfil (iii). Thus we can choose a vertex \(y\) in \(V(T')\) from \(PN_p(w, D, T)\). Clearly, \(|D \cap N_T(y)| = p\), \(N_T[y] \subseteq V(T')\) and \(y \notin D \cap V(T') - \{w\} = S \cap V(T')\). So \(y \notin S\) and, to \(p\)-dominate \(y\), \(|S \cap N_T(y)| \geq p\). But

\[|S \cap N_T(y)| = |(S \cap V(T')) \cap N_T(y)| = |(D \cap V(T')) \cap N_T(y) - \{w\}| = |(D \cap N_T(y))| - 1 = p - 1,\]

a contradiction. Therefore, \(v \notin S\) and \(S \cap V(T')\) is a \(p\)-dominating set of \(T'\) with \(|S \cap V(T')| = |S| - |D(v)| = \gamma_p(T) - |D(v)| \leq \gamma_p(T')\). Then \(S \cap V(T')\) is also a \(\gamma_p\)-set.
of $T'$, and so $S \cap V(T') = D \cap V(T')$. Hence

$$S = (S \cap V(T')) \cup D(v) = (D \cap V(T')) \cup D(v) = D.$$\hfill \square

Now we will establish the third equivalent condition for trees with unique minimum $p$-dominating sets.

**Lemma 3.** Suppose that $T$ is a tree and $p$ is a positive integer. Let $D$ be a $\gamma_p$-set of $T$ and $x \in D$.

(a) If $\gamma_p(T - x) > \gamma_p(T)$, then $|PN_p(x, D)| \geq 2$.

(b) If $D$ is a unique $\gamma_p$-set of $T$, then $\gamma_p(T - x) = \gamma_p(T) + |PN_p(x, D)| - 1$.

**Proof.** Let $T_1, \ldots, T_k$ be all components of $T - x$. Then $k = \deg_T(x) \geq 1$. For $i = 1, \ldots, k$, we denote the neighbor of $x$ in $V(T_i)$ by $y_i$ and let $D_i = D \cap V(T_i)$. We claim that,

$$|D_i| \leq \gamma_p(T_i) \leq |D_i| + 1, \text{ for every } i = 1, \ldots, k. \quad (1)$$

In fact, for $i = 1, \ldots, k$, it is obvious that $\gamma_p(T_i) \leq |D_i| + 1$ since $D_i \cup \{y_i\}$ is a $p$-dominating set of $T_i$. Suppose that there exists some $j \in \{1, \ldots, k\}$ such that $\gamma_p(T_j) \leq |D_j| - 1$. Let $D_j'$ be a $\gamma_p$-set of $T_j$. Since $x \in D$, $(D - D_j) \cup D_j'$ is a $p$-dominating set of $T$ with $|(D - D_j) \cup D_j'| = |D| - |D_j| + \gamma_p(T_j) \leq \gamma_p(T) - 1$, a contradiction. The claim holds.

Let $|PN_p(x, D)| = t$, then $0 \leq t \leq k$. Since $T$ is a tree, $|PN_p(x, D) \cap V(T_i)| \leq 1$ for every $i = 1, \ldots, k$. So, without loss of generality, we can assume that $PN_p(x, D) = \{y_1, \ldots, y_t\}$. By the definition of $p$-dominating set, $D_i$ ($i = t + 1, \ldots, k$) is a $p$-dominating set of $T_i$ since $y_i \notin PN_p(x, D)$, and so $|D_i| \geq \gamma_p(T_i)$. Applying inequality (1), we have

$$|D_i| = \gamma_p(T_i), \text{ for every } i = t + 1, \ldots, k. \quad (2)$$

Now we prove (a). Suppose that $|PN_p(x, D)| \leq 1$. Then, by inequality (1) and
equality (2),
\[ \gamma_p(T - x) = \sum_{i=1}^{k} \gamma_p(T_i) \leq \sum_{i=1}^{k} |D_i| + 1 = \gamma_p(T), \]
which contradicts with \( \gamma_p(T - x) > \gamma_p(T) \). (a) is true.

To the end, we prove (b). We claim that, for every \( i = 1, \ldots, t \), \( |D_i| + 1 = \gamma_p(T_i) \) if \( D \) is a unique \( \gamma_p \)-set of \( T \). To the contrary, by inequality (1), there exists some \( j \in \{1, \ldots, t\} \) such that \( |D_j| = \gamma_p(T_j) \). Since \( x \in D \) and \( y_j \in PN_p(x, D) \), \( |D_j \cap N_T(y_j)| = |(D \cap N_T(y_j)) - \{x\}| = p - 1 \), and so \( D_j \) is not a \( p \)-dominating set of \( T_j \). Let \( D_j'' \) be a \( \gamma_p \)-set of \( T_j \), then \( (D - D_j) \cup D_j'' \) is a \( p \)-dominating set of \( T \) different from \( D \) and \( |(D - D_j) \cup D_j''| = |D| - |D_j| + |D_j''| = \gamma_p(T) \). Hence, \( (D - D_j) \cup D_j'' \) is a \( \gamma_p \)-set of \( T \), which contradicts that \( D \) is the unique \( \gamma_p \)-set of \( T \). The claim holds. Hence, by equality (2) and \( x \in D \),
\[ \gamma_p(T - x) = \sum_{i=1}^{k} \gamma_p(T_i) = \sum_{i=1}^{t} (|D_i| + 1) + \sum_{i=t+1}^{k} |D_i| = \gamma_p(T) + |PN_p(x, D)| - 1. \]
\( \square \)

**Theorem 4.** Suppose that \( T \) is a tree and \( p \geq 2 \) is a positive integer. Let \( D \) be a subset of \( V(T) \). Then \( D \) is a unique \( \gamma_p \)-set of \( T \) if and only if \( D \) is a \( \gamma_p \)-set of \( T \) satisfying either \( |D \cap N_T(x)| \leq p - 2 \) or \( \gamma_p(T - x) > \gamma_p(T) \) for every \( x \in D \cap X_p(T) \).

**Proof.** If \( D \) is a \( \gamma_p \)-set of \( T \) satisfying, for every \( x \in D \cap X_p(T) \), either \( |D \cap N_T(x)| \leq p - 2 \) or \( \gamma_p(T - x) > \gamma_p(T) \), then, by Lemma 3 (a), \( T \) is a tree whose \( \gamma_p \)-set \( D \) satisfies (ii) of Theorem 2. By Theorem 2 (i) \( \Leftrightarrow \) (ii), \( D \) is a unique \( \gamma_p \)-set of \( T \).

Conversely, by Theorem 2 (i) \( \Leftrightarrow \) (ii), \( D \) is a \( \gamma_p \)-set of \( T \) satisfying, for every \( x \in D \cap X_p(T) \), either \( |D \cap N_T(x)| \leq p - 2 \) or \( |PN_p(x, D, T)| \geq 2 \). By Lemma 3 (b), \( \gamma_p(T - x) = \gamma_p(T) + |PN_p(x, D)| - 1 \) for every \( x \in D \). For every \( x \in D \cap X_p(T) \), if \( |D \cap N_T(x)| \geq p - 1 \), then \( |PN_p(x, D)| \geq 2 \), and so \( \gamma_p(T - x) = \gamma_p(T) + |PN_p(x, D)| - 1 > \gamma_p(T) \). The proof is completed. \( \square \)
3 A constructive characterization of trees with unique \( \gamma_p \)-sets

In this section, we will give a constructive characterization of all trees with unique minimum \( p \)-dominating sets for \( p \geq 2 \).

A vertex is a central vertex of a star \( K_{1,t} \) \( (t \geq 1) \) if either \( t \geq 2 \) and it is the support vertex or \( t = 1 \) and it is one of the two leaves. For convenience, an isolated vertex itself is also called its central vertex.

We first introduce a family \( \mathcal{T}_p \).

For any \( T \in \mathcal{T}_p \), \( T \) is obtained from a sequence \( T_1, T_2, \cdots, T_k \) \( (k \geq 1) \) of trees, where \( T_1 = K_{1,m} \) \( (m \geq p) \), \( T = T_k \), and, for \( k \geq 2 \), \( T_{i+1} \) \( (1 \leq i \leq k - 1) \) is obtained from \( T_i \) by one of the operations listed below. Let \( A(T_1) = L(T_1) \).

- **Operation** \( O_1 \): Attach \( h \) \( (\geq 0) \) stars \( K_{1,p-1} \), denoted by \( \{H_1, \cdots, H_h\} \), and \( t \) \( (\geq 0) \) isolated vertices, denoted by \( \{v_1, \cdots, v_t\} \), to \( T_i \) by adding \( h + t \) edges from their central vertices to a leaf \( w \) of \( T_i \), where \( h, t \) and \( w \) must fulfil one of the following conditions.
  
  \begin{enumerate}
    \item \( h = 0 \) and \( t \leq p - 2 \);
    \item \( h = 1 \) and \( t \leq p - 3 \);
    \item \( h = 1, t = p - 2 \) and the support vertex of \( w \) in \( T_i \) isn’t in \( A(T_i) \);
    \item \( h = 1, t \geq p - 1 \) and the support vertex of \( w \) is a \( p \)-private vertex of \( w \) with regard to \( A(T_i) \) in \( T_i \);
    \item \( h \geq 2 \).
  \end{enumerate}

Let \( A(T_{i+1}) = A(T_i) \cup (\bigcup_{j=1}^{h} L(H_j)) \cup \{v_1, \cdots, v_t\} \).

- **Operation** \( O_2 \): Attach a star \( K_{1,t} \) \( (t \geq p) \) to \( T_i \) by adding an edge from its central vertex to a vertex \( w \) of \( T_i \) satisfying either \( \deg_{T_i}(w) \neq p - 1 \) or \( N_{T_i}(w) \not\subseteq A(T_i) \). Let \( A(T_{i+1}) = A(T_i) \cup L(K_{1,t}) \).

**Lemma 5.** For any \( T \in \mathcal{T}_p \), \( A(T) \) is a unique \( \gamma_p \)-set of \( T \).
Proof. We claim that $A(T)$ is a $p$-dominating set of $T$ and, for every vertex $x \in A(T) \cap X_p(T), |A(T) \cap N_T(x)| \leq p - 2$ or $|P_{N_p}(x, A(T), T)| \geq 2$. Then, by (i) $\iff$ (iii) of Theorem 2, $A(T)$ is a unique $\gamma_p$-set of $T$. Assume that $T$ is obtained from a sequence $T_1, \ldots, T_k$ $(k \geq 1)$ of trees constructed recursively from $T_1$ by Operation $O_1$ and $O_2$, where $T_1 = K_{1,m}$ $(m \geq p)$ and $T = T_k$. We can prove easily the claim by induction on the length $k$ of the sequence $T_1, \ldots, T_k$. We omit the proof. \hfill $\Box$

From Lemma 1 and the definition of Operation $O_1$ (a), it is easy to see that the following lemma is true.

**Lemma 6.** Let $T$ be a tree containing a unique vertex with degree at least $p$. Then $T \in \mathcal{T}_p$.

**Lemma 7.** Let $T$ be a tree containing exact two vertices with degree at least $p$. If $T$ has a unique $\gamma_p$-set, then $T \in \mathcal{T}_p$.

**Proof.** Denote the two vertices with degree at least $p$ by $u$ and $v$. We distinguish the following two cases.

**Case 1.** If $uv \in E(T)$, then $deg_T(u) \geq p + 1$ and $deg_T(v) \geq p + 1$ since $T$ has a unique $\gamma_p$-set. Let $T_1 = T[N_T[u] \cup \{v\}],$ then $T_1$ is a star $K_{1,m}$ $(m = deg_T(v) - 1 \geq p)$. Hence $T_2 = T[N_T[u] \cup N_T[v]]$ is obtained from $T_1$ by Operation $O_2$ by attaching a star $K_{1,t} =$ $T[N_T[u] \cup \{v\}]$ $(t = deg_T(u) - 1 \geq p)$ to $v$, and so $T_2 \in \mathcal{T}_p$. Since every vertex of $V(T) - V(T_2)$ has degree at most $p - 1$, $T$ can be obtained recursively from $T_2$ by Operation $O_1$ satisfying (a). So $T \in \mathcal{T}_p$.

**Case 2.** If $uv \notin E(T)$, then we root $T$ at $u$ and denote the father of $v$ by $w$.

If $deg_T(v) = p$, let $T' = T - D(w)$. Obviously, $T'$ is a tree containing a unique vertex with degree at least $p$ and $w$ is a leaf of $T'$. By Lemma 6, $T' \in \mathcal{T}_p$. Since $deg_T(v) = p$, $T[N_T[v] \cup \{w\}] = K_{1,p-1}$. By $deg_T(w) \leq p - 1$, $|C(w) - \{v\}| \leq p - 3$. Let $T'' = T[V(T') \cup (N_T[v] \cup \{w\}) \cup (C(v) - \{v\})]$, then $T''$ is obtained from $T'$ by Operation $O_1$ satisfying (b), and so $T'' \in \mathcal{T}_p$. Since every vertex of $V(T) - V(T'')$ has degree at most $p - 1$, $T$ can be obtained recursively from $T''$ by Operation $O_1$ satisfying
If \( \text{deg}_T(v) \geq p + 1 \), let \( T' = T - D[v] \). Since \( T \) has exact two vertices \( u \) and \( v \) with degree at least \( p \) and \( \text{deg}_T(w) \leq p - 2 \). By Lemma 6, \( T' \in \mathcal{T}_p \). Let \( T'' = T[V(T') \cup (C(v) \cup \{v\})] \), then \( T'' \) is obtained from \( T' \) by Operation \( O_2 \) by attaching a star \( K_{1,t} = T[C(v) \cup \{v\}] \) \((t = \text{deg}_T(v) - 1 \geq p)\) to \( w \). So \( T'' \in \mathcal{T}_p \). Since every vertex of \( V(T) - V(T'') \) has degree at most \( p - 1 \), \( T \) can be obtained recursively from \( T'' \) by Operation \( O_1 \) satisfying (a). So \( T \in \mathcal{T}_p \). □

**Theorem 8.** Let \( T \) be a tree and \( p \geq 2 \) a positive integer. Then \( T \) has a unique \( \gamma_p \)-set if and only if \( \Delta(T) \leq p - 1 \) or \( T \in \mathcal{T}_p \).

**Proof.** If \( \Delta(T) \leq p - 1 \) or \( T \in \mathcal{T}_p \), by Lemmas 1 and 5, \( T \) has a unique \( \gamma_p \)-set.

Conversely, let \( T \) be a tree with a unique \( \gamma_p \)-set. We will prove \( \Delta(T) \leq p - 1 \) or \( T \in \mathcal{T}_p \) by induction on the order \( n \) of \( T \).

If \( n \in \{1, 2\} \), then \( \Delta(T) \leq p - 1 \). This establishes the base case. Assume that, if tree \( T' \) with order \( 2 \leq |V(T')| < n \) has a unique \( \gamma_p \)-set, then \( \Delta(T') \leq p - 1 \) or \( T' \in \mathcal{T}_p \).

If \( d(T) = 2 \), then \( T \) has at most one vertex with degree at least \( p \). By Lemma 6, the result holds. If \( d(T) = 3 \), then \( T \) has at most two vertices with degree at least \( p \). By Lemmas 6 and 7, the result holds. In the following, we can assume that \( \Delta(T) \geq p \) and \( d(T) \geq 4 \).

Let \( p = uvwx \cdots r \) be a longest path in \( T \) such that the degree of \( v \) is as large as possible. We root \( T \) at \( r \) and denote the unique \( \gamma_p \)-set of \( T \) by \( D \). By Theorem 2, \( T \) and \( D \) fulfil (ii) of Theorem 2.

We claim that, if there exists a vertex \( v' \in C(w) \) with \( 2 \leq \text{deg}_T(v') \leq p - 1 \), then \( T \in \mathcal{T}_p \).

In fact, by the choice of path \( P \) and \( v' \in C(w) \), \( T[D(v')] \) consists of \( |D(v')| \) isolated vertices and \( |D(v')| = \text{deg}_T(v') - 1 \leq p - 2 \). By Lemma 1, \( D[v'] \subseteq D \). Let \( T' = \)
\( T - D(v') \), then \( D \cap V(T') \) is a \( p \)-dominating set of \( T' \). Since \( v' \in D \) and \( v' \) is a leaf of \( T' \), \( (D \cap V(T')) \cap N_T(z) = D \cap N_T(z) \) and \( PN_p(z, D \cap V(T'), T') = PN_p(z, D, T) \) for every \( z \in (D \cap V(T')) \cap X_p(T') \). Hence \( T' \) is a tree whose \( p \)-dominating set \( D \cap V(T') \) fulfils (iii) of Theorem 2 since \( T \) and \( D \) fulfil (ii) of Theorem 2. By Theorem 2 (i) \( \Leftrightarrow \) (iii), \( D \cap V(T') \) is a unique \( \gamma_p \)-set of \( T' \). Applying the induction on \( T' \), \( \Delta(T') \leq p - 1 \) or \( T' \in \mathcal{T}_p \). If \( \Delta(T') \leq p - 1 \), then \( \Delta(T) \leq p - 1 \), which contradicts with \( \Delta(T) \geq p \). If \( T' \in \mathcal{T}_p \), then \( T \) is obtained from \( T' \) by Operation \( \mathcal{O}_1 \) satisfying (a) by attaching \( |D(v')| \) isolated vertices to the leaf \( v' \) of \( T' \). Hence \( T \in \mathcal{T}_p \). The claim holds.

By claim, we only need consider the case that every vertex of \( C(w) \) has degree 1 or at least \( p \). Since \( v \in C(w) \) and \( \deg_T(v) \geq 2 \), \( \deg_T(v) \geq p \).

**Case 1.** \( \deg_T(v) = p \).

By the choice of path \( P \), for every vertex \( v' \in C(w) \), \( \deg_T(v') \leq \deg_T(v) = p \), and so \( \deg_T(v') = p \) or 1. Let \( h \) and \( t \) be the number of vertices with degree \( p \) and 1, respectively, in \( C(w) \). Then \( h \geq 1 \) and \( T[D(w)] \) consists of \( h \) stars \( K_{1,p-1} \) and \( t \) isolated vertices. Since \( \deg_T(v) = p \), \( |D \cap N_T(v)| \geq |D(v)| = \deg_T(v) - 1 = p - 1 \). By (ii) of Theorem 2, \( v \notin D \). To \( p \)-dominate \( v, w \in D \). Let \( T' = T - D(w) \), then \( D \cap V(T') \) is a \( p \)-dominating set of \( T' \). Since \( w \in D \) and \( \deg_T(w) = 1 \), for every \( z \in (D \cap V(T')) \cap X_p(T'), (D \cap V(T')) \cap N_T(z) = D \cap N_T(z) \) and \( PN_p(z, D \cap V(T'), T') = PN_p(z, D, T) \). Hence \( T' \) is a tree whose \( p \)-dominating set \( D \cap V(T') \) fulfils (iii) of Theorem 2 since \( T \) and \( D \) fulfil (ii) of Theorem 2. By Theorem 2 (i) \( \Leftrightarrow \) (iii), \( D \cap V(T') \) is a unique \( \gamma_p \)-set of \( T' \). Applying the induction on \( T' \), \( \Delta(T') \leq p - 1 \) or \( T' \in \mathcal{T}_p \).

**Subcase 1.1.** \( \Delta(T') \leq p - 1 \).

If \( h = 1 \), then every vertex of \( D(w) - \{v\} \) is a leaf of \( T \). Hence all vertices of \( V(T) - \{v, w\} \) have degree at most \( \Delta(T') (\leq p - 1) \) in \( T \). By Lemmas 6 and 7, \( T \in \mathcal{T}_p \).

If \( h \geq 2 \), then, by the definition of Operation \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), we can check easily that \( T'' = T[D[w] \cup \{x\}] \in \mathcal{T}_p \). Since all vertices of \( V(T) - (D[w] \cup \{x\}) (\subset V(T')) \) have degree at most \( \Delta(T') (\leq p - 1) \) in \( T \) and \( x \) is a leaf of \( T'' \), \( T \) can be obtained recursively
from $T''$ by Operation $\mathcal{O}_1$ satisfying condition (a). So $T \in \mathcal{T}_p$.

**Subcase 1.2.** $T' \in \mathcal{T}_p$.

When $h \geq 2$, then $T$ is obtained from $T'$ by Operation $\mathcal{O}_1$ satisfying condition (e). Hence $T \in \mathcal{T}_p$. Now we assume that $h = 1$.

If $\text{deg}_T(w) \leq p - 1$, then $t = \text{deg}_T(w) - 2 \leq p - 3$. Thus $T$ is obtained from $T'$ by $\mathcal{O}_1$ satisfying condition (b). Hence $T \in \mathcal{T}_p$.

If $\text{deg}_T(w) = p$, then $t = \text{deg}_T(w) - 2 = p - 2$. Note that $x$ is the support vertex of $w$ in $T'$. We claim that $x \notin D \cap V(T') = A(T')$. Otherwise, by $w \in D$, $(D - \{w\}) \cup \{v\}$ is a $\gamma_p$-set of $T$ different from $D$, a contradiction. Hence $T$ is obtained from $T'$ by Operation $\mathcal{O}_1$ satisfying condition (c). Thus $T \in \mathcal{T}_p$.

If $\text{deg}_T(w) \geq p + 1$, then $t = \text{deg}_T(w) - 2 \geq p - 1$. Note that $w \in D \cap X_p(T)$ and $|D \cap N_T(w)| \geq t \geq p - 1$. By (ii) of Theorem 2, $w$ has at least two $p$-private vertices with regard to $D$ in $T$. Then we can see easily that $PN_p(w, D, T) = \{v, x\}$. So $PN_p(w, A(T'), T') = PN_p(w, D \cap V(T'), T') = \{x\}$. Hence $T$ is obtained from $T'$ by $\mathcal{O}_1$ satisfying condition (d), and so $T \in \mathcal{T}_p$.

**Case 2.** $\text{deg}_T(v) \geq p + 1$.

Let $T' = T - D[v]$. Since $\text{deg}_T(v) \geq p + 1$ and $|D \cap N_T(v)| \geq |D(v)| = \text{deg}_T(v) - 1 \geq p$, by (ii) of Theorem 2, $v \notin D$. Hence $D \cap V(T')$ is a $p$-dominating set of $T'$. Since $v \notin D$ and $D \cap N_T(v) = D(v) \cup (D \cap \{w\})$, $(D \cap V(T')) \cap N_T(z) = D \cap N_T(z)$ and $PN_p(z, D \cap V(T'), T') = PN_p(z, D, T)$ for every $z \in (D \cap V(T')) \cap X_p(T')$. Hence $T'$ is a tree whose $p$-dominating set $D \cap V(T')$ fulfills (iii) of Theorem 2 since $T$ and $D$ fulfil (ii) of Theorem 2. By Theorem 2 (i) $\Leftrightarrow$ (iii), $D \cap V(T')$ is a unique $\gamma_p$-set of $T'$. Applying the induction on $T'$, $\Delta(T') \leq p - 1$ or $T' \in \mathcal{T}_p$.

If $\Delta(T') \leq p - 1$, then all vertices of $T - \{v, w\}$ have degree at most $\Delta(T') \leq p - 1$ in $T$. By Lemmas 6 and 7, $T \in \mathcal{T}_p$.

If $T' \in \mathcal{T}_p$, then we claim that $\text{deg}_T(w) \neq p - 1$ or $N_T(w) \notin D$. Suppose that
$\text{deg}_T(w) = p - 1$ and $N_T(w) \subseteq D$, then, by Lemma 1, $w \in D$. It is easy to see that $(D - \{w\}) \cup \{v\}$ is a $\gamma_p$-set of $T$ different from $D$, a contradiction. Hence $T$ is obtained from $T'$ by Operation $O_2$ by attaching a star $K_{1,t} (= T[D[v]], \ t = \text{deg}_T(v) - 1 \geq p)$ to $w$ of $T'$. So $T \in T_p$. □

References


