Trees with unique minimum p-dominating sets *

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Abstract

Let p be a positive integer and $G = (V, E)$ a simple graph. A p-dominating set of G is a subset S of V such that every vertex not in S is dominated by at least p vertices in S. The p-domination number $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. In this paper, for $p \geq 2$, we give three equivalent conditions for trees with unique minimum p-dominating sets and also give a constructive characterization of such trees.

Key words: domination, p-domination, tree

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1 Introduction

For notation and graph theory terminology we follow [3, 10, 11]. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of a vertex $v \in V(G)$ are denoted by $N_G(v) =$

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$\{u \in V(G) | uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$ and $deg_G(v) = |N_G(v)|$, respectively. The maximum degree $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. For $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. For a pair of vertices $u, v \in V(G)$, the distance $d_G(u, v)$ of u and v is the length of the shortest uv-paths in G. The diameter of G is $d(G) = \max\{d_G(u, v) : u, v \in V(G)\}$.

Let D be a subset of $V(G)$ and p a positive integer. For any $x \in D$, a vertex y not in D is called a *p-private neighbor* of x with regard to D if y is a neighbor of x and $|D \cap N_G(y)| = p$. The *p-private neighborhood* of x with regard to D, denoted by $PN_p(x, D, G)$, is the set of all p-private neighbors of x with regard to D in G. If the graph G is clear from the context, we will simply use $PN_p(x, D)$ instead of $PN_p(x, D, G)$.

Let T be a tree and $p \geq 2$ a positive integer. A *p-leaf* of T is a vertex with degree at most $p-1$ in T. Denote the set of p-leaves of T by $L_p(T)$ and let $X_p(T) = V(T) - L_p(T)$. Then, for $x \in X_p(T)$, $deg_T(x) \geq p$. Note that the 2-leaves are the usual leaves and $L_2(T)$ is the set of leaves of T. Therefore, we also denote $L_2(T)$ by $L(T)$. If T is a rooted tree T, then, for every $v \in V(T)$, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and define $D[v] = D(v) \cup \{v\}$.

In [6], Fink and Jacobson introduced the concept of p-domination. Let p be a positive integer. A subset S of $V(G)$ is a *p-dominating set* of G if, for every $v \in V(G) - S$, $|S \cap N_G(v)| \geq p$. The p-domination number $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Any p-dominating set of G with cardinality $\gamma_p(G)$ will be called a γ_p-set of G. Note that the γ_1-set is the classic minimum dominating set. For any $S, T \subseteq V(G)$, S p-dominates T in G if, for every $v \in T - S$, $|S \cap N_G(v)| \geq p$.

Unique domination in graphs has been investigated in many papers (see, for example, [2, 4, 5, 7, 8, 9]). In [8], Gunther et al. characterized all trees with unique minimum dominating sets. In this paper, for $p \geq 2$, we first give three equivalent conditions for trees with unique minimum p-dominating sets, and then we give a constructive characterization of such trees.

2 Equivalent conditions for trees with unique γ_p-sets

Lemma 1. *([1]) Every p-dominating set of a graph G contains any vertex of degree at most $p-1$.*

Theorem 2. *Suppose that T is a tree and $p \geq 2$ is a positive integer. Let D be a subset of $V(T)$. Then the following conditions are equivalent:*

(i) *D is a unique γ_p-set of T;*

(ii) *D is a γ_p-set of T satisfying either $|D \cap N_T(x)| \leq p-2$ or $|PN_p(x, D)| \geq 2$ for every $x \in D \cap X_p(T)$;*

(iii) *D is a p-dominating set of T satisfying either $|D \cap N_T(x)| \leq p-2$ or $|PN_p(x, D)| \geq 2$ for every $x \in D \cap X_p(T)$.*

Proof. We will prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). (ii) $\Rightarrow$ (iii) is obvious.

(i) $\Rightarrow$ (ii) : Suppose to the contrary that there exists some $x \in D \cap X_p(T)$ such that $|D \cap N_T(x)| \geq p-1$ and $|PN_p(x, D)| \leq 1$.

If $|PN_p(x, D)| = 0$, then $|D \cap N_T(x)| = p-1$ (Otherwise, $D-\{x\}$ is a p-dominating set of T, which contradicts that D is a γ_p-set of T). Thus, there exists a neighbor, denoted by y, of x which is not in D since $deg_T(x) \geq p$. Let $D' = (D - \{x\}) \cup \{y\}$, then D' is a γ_p-set of T different from D, a contradiction.

If $|PN_p(x, D)| = 1$, then we denote $PN_p(x, D)$ by $\{y\}$ and let $D' = (D-\{x\}) \cup \{y\}$. Since $|D \cap N_T(x)| \geq p-1$, $|D' \cap N_T(x)| = |(D \cap N_T(x)) \cup \{y\}| \geq p$. Hence D' is a γ_p-set of T different from D, a contradiction.

(iii) $\Rightarrow$ (i) : Assume that there is a tree T which has a p-dominating set D satisfying the condition of (iii) but D is not a unique γ_p-set of T. Let T be such a counterexample of minimum order. Then, by Lemma 1, $D \cap X_p(T) \neq \emptyset$. Let S be an arbitrary γ_p-set of T. In the following, we only need prove that $S = D$, which contradicts the assumption that D is not a unique γ_p-set of T.

If $d(T)=2$, then T is a star, and so $|D\cap X_p(T)|\leq 1$. By $D\cap X_p(T)\neq\emptyset$, we can denote $D\cap X_p(T)$ by $\{a\}$. By Lemma 1 and $PN_p(a,D)=\emptyset$, $deg_T(a)=|D\cap N_T(a)|\leq p-2$, which contradicts with $deg_T(a)\geq p$. If $d(T)=3$, then T contains exact two vertices with degree at least 2. For every $b\in D\cap X_p(T)$, we have $deg_T(b)\geq p$ and $|PN_p(b,D)|\leq 1$. From D fulfils (iii), we can derive that $|D\cap N_T(b)|\leq p-2$. So $deg_T(b)=|D\cap N_T(b)|+1\leq p-1$, a contradiction. Hence $d(T)\geq 4$. Let $P=uvwx\cdots r$ be a longest path in T. We root T at r. By Lemma 1, $D(v)\subseteq D$ and $D(v)\subseteq S$.

Case 1. $deg_T(v)\leq p-1$.

By Lemma 1, $D[v]\subseteq D$. Let $T'=T-u$, then $D\cap V(T')=D-\{u\}$ is a p-dominating set of T'. Note that $(D\cap V(T'))\cap N_{T'}(z)=D\cap N_T(z)$ and $PN_p(z,D\cap V(T'),T')=PN_p(z,D,T)$ for every $z\in(D\cap V(T'))\cap X_p(T')$. Hence T' is a tree whose p-dominating set $D\cap V(T')$ fulfils (iii) since T and D fulfil (iii). By our assumption that T is the counterexample of minimum order, $D\cap V(T')$ is a unique γ_p-set of T'. So $\gamma_p(T')=|D\cap V(T')|=|D|-1\geq\gamma_p(T)-1$.

By Lemma 1, $D[v]\subseteq S$. Hence $S\cap V(T')=S-\{u\}$ is a p-dominating set of T' with $|S\cap V(T')|=|S|-1=\gamma_p(T)-1\leq\gamma_p(T')$. That is, $S\cap V(T')$ is also a γ_p-set of T'. Hence $S\cap V(T')=D\cap V(T')$. Thus

$$S=(S\cap V(T'))\cup\{u\}=(D\cap V(T'))\cup\{u\}=D.$$

Case 2. $deg_T(v)=p$.

For every $v'\in D(w)\cap X_p(T)$ $(=C(w)\cap X_p(T))$, we have $|D\cap N_T(v')|\geq|D(v')|=deg_T(v')-1\geq p-1$, and so $v'\notin D$ since T and D fulfil (iii). Then, by Lemma 1, $D\cap D(w)=L_p(T)\cap D(w)$. By $deg_T(v)=p$, $v\notin D$. Since D p-dominates v, we have $w\in D$.

Let $T'=T-D(w)$, then $D\cap V(T')$ is a p-dominating set of T'. Since $w\in D$ and $deg_{T'}(w)=1$, for every $z\in(D\cap V(T'))\cap X_p(T')$, $(D\cap V(T'))\cap N_{T'}(z)=D\cap N_T(z)$ and $PN_p(z,D\cap V(T'),T')=PN_p(z,D,T)$. Hence T' is a tree whose p-dominating set $D\cap V(T')$ fulfils (iii) since T and D fulfil (iii). By our assumption that T is the counterexample of minimum order, $D \cap V(T')$ is a unique γ_p-set of T'. So $|D \cap V(T')| = \gamma_p(T')$ and $\gamma_p(T) \leq |D| = |D \cap D(w)| + |D \cap V(T')| = |D \cap D(w)| + \gamma_p(T')$.

Now we prove $S = D$. Suppose that $v \in S$, then, by the definition of γ_p-set and Lemma 1, $w \notin S$ and $(S \cap D(w)) \cap X_p(T) = \{v\}$ (Assume that $(S \cap D(w)) \cap X_p(T)$ contains another vertex v', then v' is a neighbor of w in $D(w)$. Note that $D(v') \subseteq S$ by Lemma 1 and $|D(v')| \geq p - 1$. We can replace v, v' by w in S and get a p-dominating set of T of order $|S| - 1$, a contradiction). Hence $S \cap D(w) = (L_p(T) \cap D(w)) \cup \{v\} = (D \cap D(w)) \cup \{v\}$. Since $(S \cap V(T')) \cup \{w\}$ is a p-dominating set of T' with

$$|(S \cap V(T')) \cup \{w\}| = |S| - |S \cap D(w)| + 1 = \gamma_p(T) - |D \cap D(w)| \leq \gamma_p(T'),$$

we know that $(S \cap V(T')) \cup \{w\}$ is a γ_p-set of T'. Hence $(S \cap V(T')) \cup \{w\} = D \cap V(T')$. By $w \notin S$ and $w \in D$, we have $deg_T(w) \geq p$ and $w \in D \cap X_p(T)$. Note that

$$\begin{aligned} |D \cap N_T(w)| &= |(D \cap D(w)) \cap N_T(w)| + |D \cap \{x\}| \\ &= |((S \cap D(w)) - \{v\}) \cap N_T(w)| + |S \cap \{x\}| \\ &= |(S \cap D(w)) \cap N_T(w)| - 1 + |S \cap \{x\}| = |S \cap N_T(w)| - 1 \geq p - 1. \end{aligned}$$

Hence $|PN_p(w, D, T)| \geq 2$ since T and D satisfy (iii). By $w \notin S$ and $(S \cap D(w)) \cap X_p(T) = \{v\}$, $D(w) \cap X_p(T)$ $(= C(w) \cap X_p(T))$ contains a unique vertex v of degree p. Thus $PN_p(w, D, T) \cap D(w) = \{v\}$. From $|PN_p(w, D, T)| \geq 2$, we know that

$$PN_p(w, D, T) = PN_p(w, D, T) \cap (D(w) \cup \{x\}) = \{v, x\}.$$

So $|D \cap N_T(x)| = p$ and $x \notin D \cap V(T') = (S \cap V(T')) \cup \{w\}$. Further, $x \notin S$. To p-dominate x, $|S \cap N_T(x)| \geq p$, which contradicts with

$$\begin{aligned} |S \cap N_T(x)| &= |(S \cap V(T')) \cap N_T(x)| \\ &= |(D \cap N_T(x)) - \{w\}| = |D \cap N_T(x)| - 1 = p - 1. \end{aligned}$$

Hence $v \notin S$.

To p-dominate v, $w \in S$ and, by the definition of γ_p-set, $S \cap D(w) = L_p(T) \cap D(w) = D \cap D(w)$. Then $S \cap V(T')$ is a p-dominating set of T' with $|S \cap V(T')| =$

$|S| - |S \cap D(w)| = \gamma_p(T) - |D \cap D(w)| \leq \gamma_p(T')$, which implies that $S \cap V(T')$ is also a γ_p-set of T'. Hence $S \cap V(T') = D \cap V(T')$, and so

$$S = (S \cap V(T')) \cup (S \cap D(w)) = (D \cap V(T')) \cup (D \cap D(w)) = D.$$

Case 3. $deg_T(v) \geq p + 1$.

Note that $|D \cap N_T(v)| \geq |D(v)| = deg_T(v) - 1 \geq p$. We have $v \notin D$ since T and D fulfil (iii). Let $T' = T - D[v]$, then $D \cap V(T')$ is a p-dominating set of T'. Since $v \notin D$ and $D \cap N_T(v) = D(v) \cup (D \cap \{w\})$, $(D \cap V(T')) \cap N_{T'}(z) = D \cap N_T(z)$ and $PN_p(z, D \cap V(T'), T') = PN_p(z, D, T)$ for every $z \in (D \cap V(T')) \cap X_p(T')$. Hence T' is a tree whose p-dominating set $D \cap V(T')$ fulfils (iii) since T and D fulfil (iii). As T is the counterexample of minimum order, $D \cap V(T')$ is the unique γ_p-set of T'. So $|D \cap V(T')| = \gamma_p(T')$ and $\gamma_p(T) \leq |D| = |D(v)| + |D \cap V(T')| = |D(v)| + \gamma_p(T')$.

Now we prove that $S = D$. Suppose that $v \in S$, then by the definition of γ_p-set, $w \notin S$. Thus $(S \cap V(T')) \cup \{w\}$ is a p-dominating set of T' with

$$|(S \cap V(T')) \cup \{w\}| = |S| - |D[v]| + 1 = \gamma_p(T) - |D(v)| \leq \gamma_p(T'),$$

which implies that $(S \cap V(T')) \cup \{w\}$ is a γ_p-set of T'. Hence $(S \cap V(T')) \cup \{w\} = D \cap V(T')$. So $w \in D$. By $w \notin S$, $v \notin D$ and $v \in S$, we have $w \in D \cap X_p(T)$ and

$$\begin{aligned} |D \cap N_T(w)| &= |(D \cap V(T')) \cap N_T(w)| \\ &= |(S \cap V(T')) \cap N_T(w)| = |S \cap N_T(w)| - 1 \geq p - 1. \end{aligned}$$

Hence $|PN_p(w, D, T)| \geq 2$ since T and D fulfil (iii). Thus we can choose a vertex y in $V(T')$ from $PN_p(w, D, T)$. Clearly, $|D \cap N_T(y)| = p$, $N_T[y] \subseteq V(T')$ and $y \notin D \cap V(T') - \{w\} = S \cap V(T')$. So $y \notin S$ and, to p-dominate y, $|S \cap N_T(y)| \geq p$. But

$$\begin{aligned} |S \cap N_T(y)| &= |(S \cap V(T')) \cap N_T(y)| \\ &= |(D \cap V(T')) \cap N_T(y) - \{w\}| = |(D \cap N_T(y))| - 1 = p - 1, \end{aligned}$$

a contradiction. Therefore, $v \notin S$ and $S \cap V(T')$ is a p-dominating set of T' with $|S \cap V(T')| = |S| - |D(v)| = \gamma_p(T) - |D(v)| \leq \gamma_p(T')$. Then $S \cap V(T')$ is also a γ_p-set of T', and so $S \cap V(T') = D \cap V(T')$. Hence

$$S = (S \cap V(T')) \cup D(v) = (D \cap V(T')) \cup D(v) = D.$$

$\square$

Now we will establish the third equivalent condition for trees with unique minimum p-dominating sets.

Lemma 3. *Suppose that T is a tree and p is a positive integer. Let D be a γ_p-set of T and $x \in D$.*

(a) *If $\gamma_p(T - x) > \gamma_p(T)$, then $|PN_p(x, D)| \geq 2$.*

(b) *If D is a unique γ_p-set of T, then $\gamma_p(T - x) = \gamma_p(T) + |PN_p(x, D)| - 1$.*

Proof. Let $T_1, \cdots, T_k$ be all components of $T - x$. Then $k = deg_T(x) \geq 1$. For $i = 1, \cdots, k$, we denote the neighbor of x in $V(T_i)$ by y_i and let $D_i = D \cap V(T_i)$. We claim that,

$$|D_i| \leq \gamma_p(T_i) \leq |D_i| + 1, \text{ for every } i = 1, \cdots, k. \tag{1}$$

In fact, for $i = 1, \cdots, k$, it is obvious that $\gamma_p(T_i) \leq |D_i| + 1$ since $D_i \cup \{y_i\}$ is a p-dominating set of T_i. Suppose that there exists some $j \in \{1, \cdots, k\}$ such that $\gamma_p(T_j) \leq |D_j| - 1$. Let D'_j be a γ_p-set of T_j. Since $x \in D$, $(D - D_j) \cup D'_j$ is a p-dominating set of T with $|(D - D_j) \cup D'_j| = |D| - |D_j| + \gamma_p(T_j) \leq \gamma_p(T) - 1$, a contradiction. The claim holds.

Let $|PN_p(x, D)| = t$, then $0 \leq t \leq k$. Since T is a tree, $|PN_p(x, D) \cap V(T_i)| \leq 1$ for every $i = 1, \cdots, k$. So, without loss of generality, we can assume that $PN_p(x, D) = \{y_1, \cdots, y_t\}$. By the definition of p-dominating set, D_i $(i = t + 1, \cdots, k)$ is a p-dominating set of T_i since $y_i \notin PN_p(x, D)$, and so $|D_i| \geq \gamma_p(T_i)$. Applying inequality (1), we have

$$|D_i| = \gamma_p(T_i), \text{ for every } i = t + 1, \cdots, k. \tag{2}$$

Now we prove (a). Suppose that $|PN_p(x, D)| \leq 1$. Then, by inequality (1) and equality (2),

$$\gamma_p(T-x)=\sum_{i=1}^{k}\gamma_p(T_i)\le\sum_{i=1}^{k}|D_i|+1=\gamma_p(T),$$

which contradicts with $\gamma_p(T-x)>\gamma_p(T)$. (a) is true.

To the end, we prove (b). We claim that, for every $i=1,\cdots,t$, $|D_i|+1=\gamma_p(T_i)$ if D is a unique γ_p-set of T. To the contrary, by inequality (1), there exists some $j\in\{1,\cdots,t\}$ such that $|D_j|=\gamma_p(T_j)$. Since $x\in D$ and $y_j\in PN_p(x,D)$, $|D_j\cap N_{T_j}(y_j)|=|(D\cap N_T(y_j))-\{x\}|=p-1$, and so D_j is not a p-dominating set of T_j. Let D''_j be a γ_p-set of T_j, then $(D-D_j)\cup D''_j$ is a p-dominating set of T different from D and $|(D-D_j)\cup D''_j|=|D|-|D_j|+|D''_j|=\gamma_p(T)$. Hence, $(D-D_j)\cup D''_j$ is a γ_p-set of T, which contradicts that D is the unique γ_p-set of T. The claim holds. Hence, by equality (2) and $x\in D$,

$$\gamma_p(T-x)=\sum_{i=1}^{k}\gamma_p(T_i)=\sum_{i=1}^{t}(|D_i|+1)+\sum_{i=t+1}^{k}|D_i|=\gamma_p(T)+|PN_p(x,D)|-1.$$

□

Theorem 4. *Suppose that T is a tree and $p\ge 2$ is a positive integer. Let D be a subset of $V(T)$. Then D is a unique γ_p-set of T if and only if D is a γ_p-set of T satisfying either $|D\cap N_T(x)|\le p-2$ or $\gamma_p(T-x)>\gamma_p(T)$ for every $x\in D\cap X_p(T)$.*

Proof. If D is a γ_p-set of T satisfying, for every $x\in D\cap X_p(T)$, either $|D\cap N_T(x)|\le p-2$ or $\gamma_p(T-x)>\gamma_p(T)$, then, by Lemma 3 (a), T is a tree whose γ_p-set D satisfies (ii) of Theorem 2. By Theorem 2 (i)$\Leftrightarrow$(ii), D is a unique γ_p-set of T.

Conversely, by Theorem 2 (i) $\Leftrightarrow$ (ii), D is a γ_p-set of T satisfying, for every $x\in D\cap X_p(T)$, either $|D\cap N_T(x)|\le p-2$ or $|PN_p(x,D,T)|\ge 2$. By Lemma 3 (b), $\gamma_p(T-x)=\gamma_p(T)+|PN_p(x,D)|-1$ for every $x\in D$. For every $x\in D\cap X_p(T)$, if $|D\cap N_T(x)|\ge p-1$, then $|PN_p(x,D)|\ge 2$, and so $\gamma_p(T-x)=\gamma_p(T)+|PN_p(x,D)|-1>\gamma_p(T)$. The proof is completed. □

3 A constructive characterization of trees with unique γ_p-sets

In this section, we will give a constructive characterization of all trees with unique minimum p-dominating sets for $p \geq 2$.

A vertex is a central vertex of a star $K_{1,t}$ $(t \geq 1)$ if either $t \geq 2$ and it is the support vertex or $t = 1$ and it is one of the two leaves. For convenience, an isolated vertex itself is also called its central vertex.

We first introduce a family $\mathcal{T}_p$.

For any $T \in \mathcal{T}_p$, T is obtained from a sequence $T_1, T_2, \cdots, T_k$ $(k \geq 1)$ of trees, where $T_1 = K_{1,m}$ $(m \geq p)$, $T = T_k$, and, for $k \geq 2$, T_{i+1} $(1 \leq i \leq k-1)$ is obtained from T_i by one of the operations listed below. Let $A(T_1) = L(T_1)$.

• **Operation** $\mathcal{O}_1$: Attach h (≥ 0) stars $K_{1,p-1}$, denoted by $\{H_1, \cdots, H_h\}$, and t (≥ 0) isolated vertices, denoted by $\{v_1, \cdots, v_t\}$, to T_i by adding $h+t$ edges from their central vertices to a leaf w of T_i, where h, t and w must fulfil one of the following conditions.

(a) $h = 0$ and $t \leq p-2$;

(b) $h = 1$ and $t \leq p-3$;

(c) $h = 1$, $t = p-2$ and the support vertex of w in T_i isn't in $A(T_i)$;

(d) $h = 1$, $t \geq p-1$ and the support vertex of w is a p-private vertex of w with regard to $A(T_i)$ in T_i;

(e) $h \geq 2$.

Let $A(T_{i+1}) = A(T_i) \cup (\cup_{j=1}^{h} L(H_j)) \cup \{v_1, \cdots, v_t\}$.

• **Operation** $\mathcal{O}_2$: Attach a star $K_{1,t}$ $(t \geq p)$ to T_i by adding an edge from its central vertex to a vertex w of T_i satisfying either $deg_{T_i}(w) \neq p-1$ or $N_{T_i}(w) \nsubseteq A(T_i)$. Let $A(T_{i+1}) = A(T_i) \cup L(K_{1,t})$.

Lemma 5. *For any $T \in \mathcal{T}_p$, $A(T)$ is a unique γ_p-set of T.*

Proof. We claim that $A(T)$ is a p-dominating set of T and, for every vertex $x \in A(T) \cap X_p(T)$, $|A(T) \cap N_T(x)| \leq p-2$ or $|PN_p(x, A(T), T)| \geq 2$. Then, by (i) $\Leftrightarrow$ (iii) of Theorem 2, $A(T)$ is a unique γ_p-set of T. Assume that T is obtained from a sequence $T_1, \cdots, T_k$ $(k \geq 1)$ of trees constructed recursively from T_1 by Operation $\mathcal{O}_1$ and $\mathcal{O}_2$, where $T_1 = K_{1,m}$ $(m \geq p)$ and $T = T_k$. We can prove easily the claim by induction on the length k of the sequence $T_1, \cdots, T_k$. We omit the proof. $\square$

From Lemma 1 and the definition of Operation $\mathcal{O}_1$ (a), it is easy to see that the following lemma is true.

Lemma 6. *Let T be a tree containing a unique vertex with degree at least p. Then $T \in \mathcal{T}_p$.*

Lemma 7. *Let T be a tree containing exact two vertices with degree at least p. If T has a unique γ_p-set, then $T \in \mathcal{T}_p$.*

Proof. Denote the two vertices with degree at least p by u and v. We distinguish the following two cases.

Case 1. If $uv \in E(T)$, then $deg_T(u) \geq p+1$ and $deg_T(v) \geq p+1$ since T has a unique γ_p-set. Let $T_1 = T[N_T[v] - \{u\}]$, then T_1 is a star $K_{1,m}$ $(m = deg_T(v) - 1 \geq p)$. Hence $T_2 = T[N_T[u] \cup N_T[v]]$ is obtained from T_1 by Operation $\mathcal{O}_2$ by attaching a star $K_{1,t} = T[N_T[u] - \{v\}]$ $(t = deg_T(u) - 1 \geq p)$ to v, and so $T_2 \in \mathcal{T}_p$. Since every vertex of $V(T) - V(T_2)$ has degree at most $p-1$, T can be obtained recursively from T_2 by Operation $\mathcal{O}_1$ satisfying (a). So $T \in \mathcal{T}_p$.

Case 2. If $uv \notin E(T)$, then we root T at u and denote the father of v by w.

If $deg_T(v) = p$, let $T' = T - D(w)$. Obviously, T' is a tree containing a unique vertex with degree at least p and w is a leaf of T'. By Lemma 6, $T' \in \mathcal{T}_p$. Since $deg_T(v) = p$, $T[N_T[v] - \{w\}] = K_{1,p-1}$. By $deg_T(w) \leq p-1$, $|C(w) - \{v\}| \leq p-3$. Let $T'' = T[V(T') \cup (N_T[v] - \{w\}) \cup (C(w) - \{v\})]$, then T'' is obtained from T' by Operation $\mathcal{O}_1$ satisfying (b), and so $T'' \in \mathcal{T}_p$. Since every vertex of $V(T) - V(T'')$ has degree at most $p-1$, T can be obtained recursively from T'' by Operation $\mathcal{O}_1$ satisfying (a). So $T \in \mathcal{T}_p$.

If $deg_T(v) \geq p+1$, let $T' = T - D[v]$. Since T has exact two vertices u and v with degree at least p, T' is a tree containing a unique vertex u with degree at least p and $deg_T(w) \leq p-2$. By Lemma 6, $T' \in \mathcal{T}_p$. Let $T'' = T[V(T') \cup (C(v) \cup \{v\})]$, then T'' is obtained from T' by Operation $\mathcal{O}_2$ by attaching a star $K_{1,t} = T[C(v) \cup \{v\}]$ ($t = deg_T(v) - 1 \geq p$) to w. So $T'' \in \mathcal{T}_p$. Since every vertex of $V(T) - V(T'')$ has degree at most $p-1$, T can be obtained recursively from T'' by Operation $\mathcal{O}_1$ satisfying (a). So $T \in \mathcal{T}_p$. □

Theorem 8. *Let T be a tree and $p \geq 2$ a positive integer. Then T has a unique γ_p-set if and only if $\Delta(T) \leq p-1$ or $T \in \mathcal{T}_p$.*

Proof. If $\Delta(T) \leq p-1$ or $T \in \mathcal{T}_p$, by Lemmas 1 and 5, T has a unique γ_p-set.

Conversely, let T be a tree with a unique γ_p-set. We will prove $\Delta(T) \leq p-1$ or $T \in \mathcal{T}_p$ by induction on the order n of T.

If $n \in \{1, 2\}$, then $\Delta(T) \leq p-1$. This establishes the base case. Assume that, if tree T' with order $2 \leq |V(T')| < n$ has a unique γ_p-set, then $\Delta(T') \leq p-1$ or $T' \in \mathcal{T}_p$.

If $d(T) = 2$, then T has at most one vertex with degree at least p. By Lemma 6, the result holds. If $d(T) = 3$, then T has at most two vertices with degree at least p. By Lemmas 6 and 7, the result holds. In the following, we can assume that $\Delta(T) \geq p$ and $d(T) \geq 4$.

Let $p = uvwx \cdots r$ be a longest path in T such that the degree of v is as large as possible. We root T at r and denote the unique γ_p-set of T by D. By Theorem 2, T and D fulfil (ii) of Theorem 2.

We claim that, if there exists a vertex $v' \in C(w)$ with $2 \leq deg_T(v') \leq p-1$, then $T \in \mathcal{T}_p$.

In fact, by the choice of path P and $v' \in C(w)$, $T[D(v')]$ consists of $|D(v')|$ isolated vertices and $|D(v')| = deg_T(v') - 1 \leq p-2$. By Lemma 1, $D[v'] \subseteq D$. Let $T' =$

$T - D(v')$, then $D \cap V(T')$ is a p-dominating set of T'. Since $v' \in D$ and v' is a leaf of T', $(D \cap V(T')) \cap N_{T'}(z) = D \cap N_T(z)$ and $PN_p(z, D \cap V(T'), T') = PN_p(z, D, T)$ for every $z \in (D \cap V(T')) \cap X_p(T')$. Hence T' is a tree whose p-dominating set $D \cap V(T')$ fulfils (iii) of Theorem 2 since T and D fulfil (ii) of Theorem 2. By Theorem 2 (i)$\Leftrightarrow$(iii), $D \cap V(T')$ is a unique γ_p-set of T'. Applying the induction on T', $\Delta(T') \leq p - 1$ or $T' \in \mathcal{T}_p$. If $\Delta(T') \leq p - 1$, then $\Delta(T) \leq p - 1$, which contradicts with $\Delta(T) \geq p$. If $T' \in \mathcal{T}_p$, then T is obtained from T' by Operation $\mathcal{O}_1$ satisfying (a) by attaching $|D(v')|$ isolated vertices to the leaf v' of T'. Hence $T \in \mathcal{T}_p$. The claim holds.

By claim, we only need consider the case that every vertex of $C(w)$ has degree 1 or at least p. Since $v \in C(w)$ and $deg_T(v) \geq 2$, $deg_T(v) \geq p$.

Case 1. $deg_T(v) = p$.

By the choice of path P, for every vertex $v' \in C(w)$, $deg_T(v') \leq deg_T(v) = p$, and so $deg_T(v') = p$ or 1. Let h and t be the number of vertices with degree p and 1, respectively, in $C(w)$. Then $h \geq 1$ and $T[D(w)]$ consists of h stars $K_{1,p-1}$ and t isolated vertices. Since $deg_T(v) = p$, $|D \cap N_T(v)| \geq |D(v)| = deg_T(v) - 1 = p - 1$. By (ii) of Theorem 2 , $v \notin D$. To p-dominate v, $w \in D$. Let $T' = T - D(w)$, then $D \cap V(T')$ is a p-dominating set of T'. Since $w \in D$ and $deg_{T'}(w) = 1$, for every $z \in (D \cap V(T')) \cap X_p(T')$, $(D \cap V(T')) \cap N_{T'}(z) = D \cap N_T(z)$ and $PN_p(z, D \cap V(T'), T') = PN_p(z, D, T)$. Hence T' is a tree whose p-dominating set $D \cap V(T')$ fulfils (iii) of Theorem 2 since T and D fulfil (ii) of Theorem 2. By Theorem 2 (i)$\Leftrightarrow$(iii), $D \cap V(T')$ is a unique γ_p-set of T'. Applying the induction on T', $\Delta(T') \leq p - 1$ or $T' \in \mathcal{T}_p$.

Subcase 1.1. $\Delta(T') \leq p - 1$.

If $h = 1$, then every vertex of $D(w) - \{v\}$ is a leaf of T. Hence all vertices of $V(T) - \{v, w\}$ have degree at most $\Delta(T')$ ($\leq p - 1$) in T. By Lemmas 6 and 7, $T \in \mathcal{T}_p$.

If $h \geq 2$, then, by the definition of Operation $\mathcal{O}_1$ and $\mathcal{O}_2$, we can check easily that $T'' = T[D[w] \cup \{x\}] \in \mathcal{T}_p$. Since all vertices of $V(T) - (D[w] \cup \{x\})$ ($\subset V(T')$) have degree at most $\Delta(T')$ ($\leq p-1$) in T and x is a leaf of T'', T can be obtained recursively from T'' by Operation $\mathcal{O}_1$ satisfying condition (a). So $T \in \mathcal{T}_p$.

Subcase 1.2. $T' \in \mathcal{T}_p$.

When $h \geq 2$, then T is obtained from T' by Operation $\mathcal{O}_1$ satisfying condition (e). Hence $T \in \mathcal{T}_p$. Now we assume that $h = 1$.

If $deg_T(w) \leq p - 1$, then $t = deg_T(w) - 2 \leq p - 3$. Thus T is obtained from T' by $\mathcal{O}_1$ satisfying condition (b). Hence $T \in \mathcal{T}_p$.

If $deg_T(w) = p$, then $t = deg_T(w) - 2 = p - 2$. Note that x is the support vertex of w in T'. We claim that $x \notin D \cap V(T') = A(T')$. Otherwise, by $w \in D$, $(D - \{w\}) \cup \{v\}$ is a γ_p-set of T different from D, a contradiction. Hence T is obtained from T' by Operation $\mathcal{O}_1$ satisfying condition (c). Thus $T \in \mathcal{T}_p$.

If $deg_T(w) \geq p + 1$, then $t = deg_T(w) - 2 \geq p - 1$. Note that $w \in D \cap X_p(T)$ and $|D \cap N_T(w)| \geq t \geq p - 1$. By (ii) of Theorem 2, w has at least two p-private vertices with regard to D in T. Then we can see easily that $PN_p(w, D, T) = \{v, x\}$. So $PN_p(w, A(T'), T') = PN_p(w, D \cap V(T'), T') = \{x\}$. Hence T is obtained from T' by $\mathcal{O}_1$ satisfying condition (d), and so $T \in \mathcal{T}_p$.

Case 2. $deg_T(v) \geq p + 1$.

Let $T' = T - D[v]$. Since $deg_T(v) \geq p+1$ and $|D \cap N_T(v)| \geq |D(v)| = deg_T(v) - 1 \geq p$, by (ii) of Theorem 2, $v \notin D$. Hence $D \cap V(T')$ is a p-dominating set of T'. Since $v \notin D$ and $D \cap N_T(v) = D(v) \cup (D \cap \{w\})$, $(D \cap V(T')) \cap N_{T'}(z) = D \cap N_T(z)$ and $PN_p(z, D \cap V(T'), T') = PN_p(z, D, T)$ for every $z \in (D \cap V(T')) \cap X_p(T')$. Hence T' is a tree whose p-dominating set $D \cap V(T')$ fulfils (iii) of Theorem 2 since T and D fulfil (ii) of Theorem 2. By Theorem 2 (i)$\Leftrightarrow$(iii), $D \cap V(T')$ is a unique γ_p-set of T'. Applying the induction on T', $\Delta(T') \leq p - 1$ or $T' \in \mathcal{T}_p$.

If $\Delta(T') \leq p-1$, then all vertices of $T - \{v, w\}$ have degree at most $\Delta(T')$ ($\leq p-1$) in T. By Lemmas 6 and 7, $T \in \mathcal{T}_p$.

If $T' \in \mathcal{T}_p$, then we claim that $deg_T(w) \neq p - 1$ or $N_T(w) \nsubseteq D$. Suppose that

$deg_T(w) = p-1$ and $N_T(w) \subseteq D$, then, by Lemma 1, $w \in D$. It is easy to see that $(D-\{w\}) \cup \{v\}$ is a γ_p-set of T different from D, a contradiction. Hence T is obtained from T' by Operation $\mathcal{O}_2$ by attaching a star $K_{1,t}$ $(= T[D[v]],\ t = deg_T(v) - 1 \geq p)$ to w of T'. So $T \in \mathcal{T}_p$. $\square$

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