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On the complexity of the bondage and reinforcement problems☆

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ABSTRACT

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is a dominating set if every vertex not in D is adjacent to a vertex in D. A dominating set D is called a total dominating set if every vertex in D is adjacent to a vertex in D. The domination (resp. total domination) number of G is the smallest cardinality of a dominating (resp. total dominating) set of G. The bondage (resp. total bondage) number of a nonempty graph G is the smallest number of edges whose removal from G results in a graph with larger domination (resp. total domination) number of G. The reinforcement (resp. total reinforcement) number of G is the smallest number of edges whose addition to G results in a graph with smaller domination (resp. total domination) number. This paper shows that the decision problems for the bondage, total bondage, reinforcement and total reinforcement numbers are all NP-hard.

1. Introduction

In this paper, we follow Xu [19] for graph-theoretical terminology and notation. A graph $G = (V, E)$ always means a finite, undirected and simple graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of G.

A subset $D \subseteq V$ is a *dominating set* of G if every vertex not in D is adjacent to a vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set D is called a γ*-set* of G if $|D| = \gamma(G)$. The *bondage number* of G, denoted by $b(G)$, is the

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minimum number of edges whose removal from G results in a graph with larger domination number of G. The *reinforcement number* of G, denoted by $r(G)$, is the smallest number of edges whose addition to G results in a graph with smaller domination number of G. Domination is a classical concept in graph theory. The bondage number and the reinforcement number were introduced by Fink et al. [3] and Kok, Mynhardt [13], respectively, in 1990. The reinforcement number for digraphs has been studied by Huang et al. [12]. Domination as well as related topics is now well studied in graph theory. The bondage number and the reinforcement number are two important parameters for measuring the vulnerability and stability of the network domination under link failure and link addition. The literature on these subjects has been, in detail, surveyed in the two excellent domination books by Haynes et al. [6,7].

Theory of domination has been applied in many research fields. For different applications, many variations of dominations were proposed in the research literature by adding some restricted conditions to dominating sets, for example, the total domination and the restrained domination.

A dominating set D is called a *total dominating set* if every vertex in D is adjacent to another vertex in D. The *total domination number*, denoted by $\gamma_t(G)$, of G is the minimum cardinality of a total dominating set of G. Use the symbol D_t to denote a total dominating set. A total dominating set D_t is called a *γ_t-set* of G if $|D_t| = \gamma_t(G)$. The *total bondage number* of G, denoted by $b_t(G)$, is the minimum number of edges whose removal from G results in a graph with larger total domination number of G. The *total reinforcement number* of G, denoted by $r_t(G)$, is the smallest number of edges whose addition to G results in a graph with smaller total domination number of G. The total domination was introduced by Cockayne et al. [1]. Total domination in graphs has been extensively studied in the literature. The recent results on the total domination is surveyed in Henning [5]. The total bondage number of a graph was first studied by Kulli and Patwari [14] and further studied by Sridharan et al. [16], and Huang and Xu [11]. The total reinforcement number of a graph was first studied by Sridharan et al. [17] and further studied by Henning et al. [8].

Analogously, a dominating set D is called a *restrained dominating set* if every vertex not in D is adjacent to another vertex not in D. The *restrained domination number*, denoted by $\gamma_r(G)$, of G is the minimum cardinality of a restrained dominating set of G. The *restrained bondage number* of G, denoted by $b_r(G)$, is the minimum number of edges whose removal from G results in a graph with larger restrained domination number of G. The restrained domination was introduced by Telle and Proskurowski [18], and the restrained bondage number was defined by Hattingh and Plummer [10].

Why a graph-theoretical parameter is proposed at once is to determine the exact value of this parameter for all graphs, that is, giving either an explicit expression in terms of other graph-theoretical parameters, or a polynomial algorithm for computing these parameters. However, the problem determining domination for general graphs has been proved to be NP-complete (see GT2 in Appendix in Garey and Johnson [4]); the problems determining total domination and restrained domination for general graphs have been also proved to be NP-complete by Laskar et al. [15], and by Domke et al. [2], respectively.

As regards the bondage problem, Hattingh and Plummer [10] showed that the restrained bondage problem is NP-complete even for bipartite graphs. For the general bondage problem, from the algorithmic point of view, Hartnell et al. [9] designed a linear time algorithm to compute the bondage number of a tree. However, the complexity of this problem is still unknown for other classes of graphs.

In this paper, we will show that the decision problems for the bondage, total bondage, reinforcement and total reinforcement numbers are all NP-hard. In other words, there are no polynomial algorithms to compute these parameters unless $P = NP$. The proofs are in Sections 3–5, respectively.

2. 3-satisfiability problem

Following Garey and Johnson's techniques for proving NP-hardness [4], we prove our results by describing a polynomial transformation from the known NP-complete problem: 3-satisfiability problem. To state the 3-satisfiability problem, in this section, we first recall some terms.

Let U be a set of Boolean variables. A *truth assignment* for U is a mapping $t : U \to \{T, F\}$. If $t(u) = T$, then u is said to be "true" under t; if $t(u) = F$, then u is said to be "false" under t. If u is a variable in U, then u and $\bar{u}$ are *literals* over U. The literal u is true under t if and only if the variable u is true under t; the literal $\bar{u}$ is true if and only if the variable u is false.

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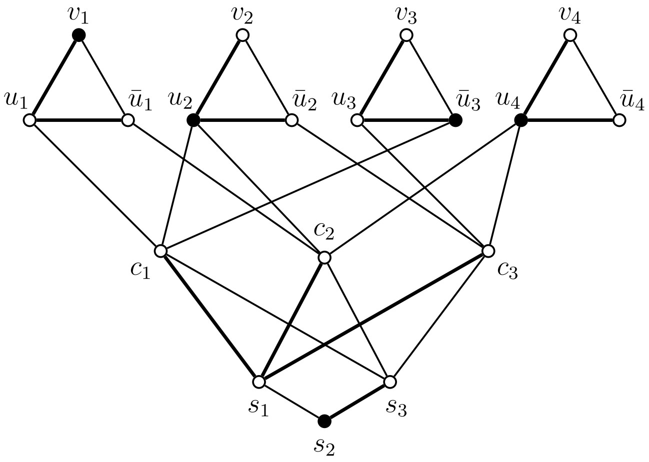

Fig. 1. An instance of the bondage problem resulting from an instance of the 3-satisfiability problem, in which $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{\{u_1, u_2, \bar{u_3}\}, \{\bar{u_1}, u_2, u_4\}, \{\bar{u_2}, u_3, u_4\}\}$. Here $k = 1$ and $\gamma = 5$, where the set of bold points is a γ-set.

A *clause* over U is a set of literals over U. It represents the disjunction of these literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathscr{C}$ of clauses over U is *satisfiable* if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathscr{C}$. Such a truth assignment is called a *satisfying truth assignment* for $\mathscr{C}$. The 3-satisfiability problem is specified as follows.

3-satisfiability problem:

Instance: A collection $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ *of clauses over a finite set U of variables such that* $|C_j| = 3$ *for* $j = 1, 2, \ldots, m$.

Question: Is there a truth assignment for U that satisfies all the clauses in $\mathscr{C}$?

Theorem 2.1 (*Theorem 3.1 in [4]*). *The 3-satisfiability problem is NP-complete.*

3. NP-hardness of bondage

In this section, we will show that the problem determining the bondage numbers of general graphs is NP-hard. We first state the problem as the following decision problem.

Bondage problem:

Instance: A nonempty graph G and a positive integer k.

Question: Is $b(G) \leq k$?

Theorem 3.1. *The bondage problem is NP-hard.*

Proof. We show the NP-hardness of the bondage problem by transforming the 3-satisfiability problem to it in polynomial time.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-satisfiability problem. We will construct a graph G and take a positive integer k such that $\mathscr{C}$ is satisfiable if and only if $b(G) \leq k$. Such a graph G can be constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a triangle T_i with vertex-set $\{u_i, \bar{u_i}, v_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{x_j, y_j, z_j\} \in \mathscr{C}$, associate a single vertex c_j and add an edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Finally, add a path $P = s_1s_2s_3$, join s_1 and s_3 to each vertex c_j with $1 \leq j \leq m$ and set $k = 1$.

Fig. 1 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u_3}\}$, $C_2 = \{\bar{u_1}, u_2, u_4\}$, $C_3 = \{\bar{u_2}, u_3, u_4\}$.

To prove that this is indeed a transformation, we must show that $b(G) = 1$ if and only if there is a truth assignment for U that satisfies all the clauses in $\mathscr{C}$. This aim can be obtained by proving the following four claims.

Claim 3.1. *$\gamma(G) \geq n + 1$. Moreover, if $\gamma(G) = n + 1$, then for any γ-set D in G, $D \cap V(P) = \{s_2\}$ and $|D \cap V(T_i)| = 1$ for each $i = 1, 2, \ldots, n$, while $c_j \notin D$ for each $j = 1, 2, \ldots, m$.*

Proof. Let D be a γ-set of G. By the construction of G, the vertex s_2 can be dominated only by vertices in P, which implies $|D \cap V(P)| \geq 1$; for each $i = 1, 2, \ldots, n$, the vertex v_i can be dominated only by vertices in T_i, which implies $|D \cap V(T_i)| \geq 1$. It follows that $\gamma(G) = |D| \geq n + 1$.

Suppose that $\gamma(G) = n + 1$. Then $|D \cap V(P)| = 1$ and $|D \cap V(T_i)| = 1$ for each $i = 1, 2, \ldots, n$. Consequently, $c_j \notin D$ for each $j = 1, 2, \ldots, m$. If $s_1 \in D$, then $|D \cap V(P)| = 1$ implies that $D \cap V(P) = \{s_1\}$, and so s_3 could not be dominated by D, a contradiction. Hence $s_1 \notin D$. Similarly $s_3 \notin D$ and, thus, $D \cap V(P) = \{s_2\}$ since $|D \cap V(P)| = 1$. □

Claim 3.2. *$\gamma(G) = n + 1$ if and only if $\mathscr{C}$ is satisfiable.*

Proof. Suppose that $\gamma(G) = n + 1$ and let D be a γ-set of G. By Claim 3.1, for each $i = 1, 2, \ldots, n$, $|D \cap V(T_i)| = 1$, it follows that $D \cap V(T_i) = \{u_i\}$ or $D \cap V(T_i) = \{\bar{u}_i\}$ or $D \cap V(T_i) = \{v_i\}$. Define a mapping $t : U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } u_i \in D \text{ or } v_i \in D, \\ F & \text{if } \bar{u}_i \in D, \end{cases} \quad i = 1, 2, \ldots, n. \tag{3.1}$$

We will show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t. To this end, we arbitrarily choose a clause $C_j \in \mathscr{C}$ with $1 \leq j \leq m$. Since the corresponding vertex c_j in G is adjacent to neither s_2 nor v_i for any i with $1 \leq i \leq n$, there exists some i with $1 \leq i \leq n$ such that c_j is dominated by $u_i \in D$ or $\bar{u}_i \in D$. Suppose that c_j is dominated by $u_i \in D$. Since u_i is adjacent to c_j in G, the literal u_i is in the clause C_j by the construction of G. Since $u_i \in D$, it follows that $t(u_i) = T$ by (3.1), which implies that the clause C_j is satisfied by t. Suppose that c_j is dominated by $\bar{u}_i \in D$. Since $\bar{u}_i$ is adjacent to c_j in G, the literal $\bar{u}_i$ is in the clause C_j. Since $\bar{u}_i \in D$, it follows that $t(u_i) = F$ by (3.1). Thus, t assigns $\bar{u}_i$ the truth value T, that is, t satisfies the clause C_j. By the arbitrariness of j with $1 \leq j \leq m$, we show that t satisfies all the clauses in $\mathscr{C}$, that is, $\mathscr{C}$ is satisfiable.

Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. Construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$, then put the vertex u_i in D'; if $t(u_i) = F$, then put the vertex $\bar{u}_i$ in D'. Clearly, $|D'| = n$. Since t is a satisfying truth assignment for $\mathscr{C}$, for each $j = 1, 2, \ldots, m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex in D' since c_j is adjacent to each literal in C_j by the construction of G. Thus $D' \cup \{s_2\}$ is a dominating set of G, and so $\gamma(G) \leq |D' \cup \{s_2\}| = n + 1$. By Claim 3.1, $\gamma(G) \geq n + 1$, and so $\gamma(G) = n + 1$. □

Claim 3.3. *$\gamma(G - e) \leq n + 2$ for any $e \in E(G)$.*

Proof. Let $E_1 = \{s_2s_3, s_1c_j, u_i\bar{u}_i, u_iv_i, : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ (induced by heavy edges in Fig. 1) and let $E_2 = E(G) \setminus E_1$. Assume $e \in E_2$. Let $D' = \{u_1, u_2, \ldots, u_n, s_1, s_2\}$. Clearly, D' is a dominating set of $G - e$ since every vertex not in D' is incident with some vertex in D' via an edge in E_1. Hence, $\gamma(G - e) \leq |D'| = n + 2$. Now assume $e \in E_1$. Let $D'' = \{u_1, u_2, \ldots, u_n, s_2, s_3\}$. If e is either s_2s_3 or incident with the vertex s_1, then D'' is a dominating set of $G - e$, clearly. If e is either $u_i\bar{u}_i$ or u_iv_i for some i $(1 \leq i \leq n)$, then we use the vertex either v_i or $\bar{u}_i$ instead of u_i in D'' to obtain D''', and hence D''' is a dominating set of $G - e$. These facts imply that $\gamma(G - e) \leq n + 2$. □

Claim 3.4. *$\gamma(G) = n + 1$ if and only if $b(G) = 1$.*

Proof. Assume $\gamma(G) = n + 1$ and consider the edge $e = s_1s_2$. Suppose $\gamma(G) = \gamma(G - e)$. Let D' be a γ-set in $G - e$. It is clear that D' is also a γ-set of G. By Claim 3.1, we have $c_j \notin D'$ for each $j = 1, 2, \ldots, m$

and $D' \cap V(P) = \{s_2\}$. But then s_1 is not dominated by D', a contradiction. Hence, $\gamma(G) < \gamma(G - e)$, and so $b(G) = 1$.

Now, assume $b(G) = 1$. By Claim 3.1, we have that $\gamma(G) \geq n + 1$. Let e' be an edge such that $\gamma(G) < \gamma(G - e')$. By Claim 3.3, we have that $\gamma(G - e') \leq n + 2$. Thus, $n + 1 \leq \gamma(G) < \gamma(G - e') \leq n + 2$, which yields $\gamma(G) = n + 1$. □

By Claims 3.2 and 3.4, we prove that $b(G) = 1$ if and only if there is a truth assignment for U that satisfies all the clauses in $\mathscr{C}$. Since the construction of the bondage instance is straightforward from a 3-satisfiability instance, the size of the bondage instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial transformation.

The theorem follows. □

4. NP-hardness of total bondage

In this section, we will show that the problem determining the total bondage numbers of general graphs is NP-hard. We first state it as the following decision problem.

Total bondage problem:

Instance: A nonempty graph G and a positive integer k.

Question: Is $b_t(G) \leq k$?

Theorem 4.1. *The total bondage problem is NP-hard.*

Proof. We show the NP-hardness of the total bondage problem by reducing the 3-satisfiability problem to it in polynomial time.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-satisfiability problem. We will construct a graph G and take an integer k such that $\mathscr{C}$ is satisfiable if and only if $b_t(G) \leq k$. Such a graph G can be constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a graph H_i with vertex-set $V(H_i) = \{u_i, \bar{u}_i, v_i, v_i'\}$ and edge-set $E(H_i) = \{v_iu_i, u_i\bar{u}_i, \bar{u}_iv_i, v_iv_i'\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{x_j, y_j, z_j\} \in \mathscr{C}$, associate a single vertex c_j and add an edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$, $1 \leq j \leq m$. Finally, add a graph H with vertex-set $V(H) = \{s_1, s_2, s_3, s_4, s_5\}$ and edge-set $E(H) = \{s_1s_2, s_1s_4, s_2s_3, s_2s_4, s_4s_5\}$, join s_1 and s_3 to each vertex c_j, $1 \leq j \leq m$ and set $k = 1$.

Fig. 2 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u_3}\}$, $C_2 = \{\bar{u_1}, u_2, u_4\}$ and $C_3 = \{\bar{u_2}, u_3, u_4\}$.

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $\mathscr{C}$ is satisfiable if and only if $b_t(G) = 1$. This aim can be obtained by proving the following four claims.

Claim 4.1. *$\gamma_t(G) \geq 2n + 2$. For any γ_t-set D_t of G, $s_4 \in D_t$ and $v_i \in D_t$ for each $i = 1, 2, \ldots, n$. Moreover, if $\gamma_t(G) = 2n + 2$, then $D_t \cap V(H) = \{s_2, s_4\}$ and $|D_t \cap V(H_i)| = 2$ for each $i = 1, 2, \ldots, n$, while $c_j \notin D_t$ for each $j = 1, 2, \ldots, m$.*

Proof. Let D_t be a γ_t-set of G. By the construction of G, it is clear that v_i is certainly in D_t to dominate v_i', and v_i can be dominated only by another vertex in H_i. It follows that $v_i \in D_t$ and $|D_t \cap V(H_i)| \geq 2$ for each $i = 1, 2, \ldots, n$. It is also clear that s_4 is certainly in D_t to dominate s_5, and s_4 can be dominated only by another vertex in H. This fact implies that $s_4 \in D_t$ and $|D_t \cap V(H)| \geq 2$. Thus, $\gamma_t(G) = |D_t| \geq 2n + 2$.

Suppose that $\gamma_t(G) = 2n + 2$. Then $|D_t \cap V(H_i)| = 2$ for each $i = 1, 2, \ldots, n$, and $|D_t \cap V(H)| = 2$. Consequently, $c_j \notin D_t$ for each $j = 1, 2, \ldots, m$. As a result, s_3 can be dominated only by the vertex s_2 in S, that is, $s_2 \in D_t$. Noting $s_4 \in D_t$ and $|D_t \cap V(H)| = 2$, we have $D_t \cap V(H) = \{s_2, s_4\}$. □

Claim 4.2. *$\gamma_t(G) = 2n + 2$ if and only if $\mathscr{C}$ is satisfiable.*

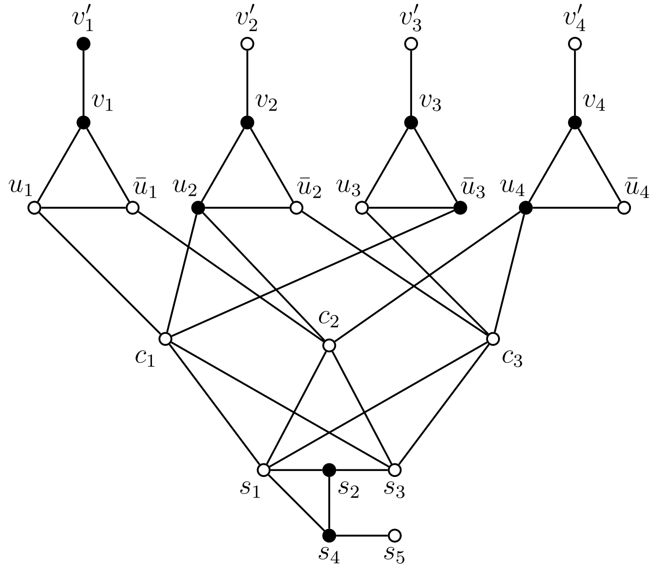

Fig. 2. An instance of the total bondage problem resulting from an instance of the 3-satisfiability problem, in which $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{\{u_1, u_2, \bar{u}_3\}, \{\bar{u}_1, u_2, u_4\}, \{\bar{u}_2, u_3, u_4\}\}$. Here $k = 1$ and $\gamma_t = 10$, where the set of bold points is a γ_t-set.

Proof. Suppose that $\gamma_t(G) = 2n+2$ and let D_t be a γ_t-set of G. By Claim 4.1, $D_t \cap V(H) = \{s_2, s_4\}$ and for each $i = 1, 2, \ldots, n$, $|D_t \cap V(H_i)| = 2$, it follows that $D_t \cap V(H_i) = \{u_i, v_i\}$ or $\{\bar{u}_i, v_i\}$ or $\{v_i, v_i'\}$. Define a mapping $t : U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } u_i \in D_t \text{ or } v_i' \in D_t, \\ F & \text{if } \bar{u}_i \in D_t, \end{cases} \quad i = 1, 2, \ldots, n. \tag{4.1}$$

We will show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that t satisfies every clause in $\mathscr{C}$. To this end, we arbitrarily choose a clause $C_j \in \mathscr{C}$. Since the corresponding vertex c_j is not adjacent to any member of $\{s_2, s_4\} \cup \{v_i, v_i' : 1 \le i \le n\}$, there exists some i with $1 \le i \le n$ such that c_j is dominated by $u_i \in D_t$ or $\bar{u}_i \in D_t$.

Suppose that c_j is dominated by $u_i \in D_t$. Then u_i is adjacent to c_j in G, that is, the literal u_i is in the clause C_j by the construction of G. Since $u_i \in D_t$, we have $t(u_i) = T$ by (4.1), which implies that t satisfies the clause C_j.

Suppose that c_j is dominated by $\bar{u}_i \in D_t$. Then $\bar{u}_i$ is adjacent to c_j in G, that is, the literal $\bar{u}_i$ is in the clause C_j. Since $\bar{u}_i \in D_t$, we have $t(u_i) = F$ by (4.1), which implies that $\bar{u}_i$ is assigned the truth value T by t, so the clause C_j is satisfied by t.

The arbitrariness of j with $1 \le j \le m$ shows that all the clauses in $\mathscr{C}$ is satisfied, that is, $\mathscr{C}$ is satisfiable.

Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. Construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$, then put the vertex u_i in D'; if $t(u_i) = F$, then put the vertex $\bar{u}_i$ in D'. Clearly, $|D'| = n$. Since t is a satisfying truth assignment for $\mathscr{C}$, for each $j = 1, 2, \ldots, m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex in D' since c_j is adjacent to each literal in C_j by the construction of G. Let $D_t' = D' \cup \{s_2, s_4, v_1, \ldots, v_n\}$. Clearly, D_t' is a dominating set of G and $|D_t'| = 2n+2$. Since s_2 and s_4 are dominated by each other, u_i and $\bar{u}_i$ are dominated by $v_i \in D_t'$ for each $i = 1, 2, \ldots, n$, D_t' is also a total dominating set of G. Hence, $\gamma_t(G) \le |D_t'| = 2n+2$. By Claim 4.1, $\gamma(G) \ge 2n+2$. Therefore, $\gamma_t(G) = 2n+2$. □

Claim 4.3. *For any $e \in E(G)$, $\gamma_t(G - e) \le 2n + 3$.*

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Proof. We first assume $e = s_2s_3$ or $e = v_i\bar{u}_i$ for some i with $1 \le i \le n$, and let $D'_t = (\cup_{i=1}^{n}\{u_i, v_i\}) \cup \{c_1, s_1, s_4\}$. It is easy to see that D'_t is a total dominating set of $G - e$. Second, assume $e = s_1c_j$ for some j with $1 \le j \le m$, and let $D'_t = (\cup_{i=1}^{n}\{u_i, v_i\}) \cup \{s_2, s_3, s_4\}$. Then D'_t is a total dominating set of $G-e$. Otherwise, let $D'_t = (\cup_{i=1}^{n}\{v_i, \bar{u}_i\}) \cup \{s_1, s_2, s_4\}$. Then D'_t is a total dominating set of $G-e$. Hence, $\gamma_t(G-e) \le |D'_t| = 2n+3$. □

Claim 4.4. $\gamma_t(G) = 2n+2$ *if and only if* $b_t(G) = 1$.

Proof. Assume $\gamma_t(G) = 2n+2$ and take $e = s_2s_4$. Suppose that $\gamma_t(G-e) = \gamma_t(G)$. Let D'_t be a γ_t-set of $G-e$. As D'_t is also a γ_t-set of G, by Claim 4.1 we have $c_j \notin D'_t$ for every j and $D'_t \cap V(H) = \{s_2, s_4\}$, which contradicts the fact that s_2 and s_4 could not be dominated by each other in $G - e$. This contradiction shows that $\gamma_t(G-e) > \gamma_t(G)$, whence $b_t(G) = 1$.

Now, assume $b_t(G) = 1$. By Claim 4.1, we have that $\gamma_t(G) \ge 2n+2$. Let e' be an edge such that $\gamma_t(G-e') > \gamma_t(G)$. By Claim 4.3, we have that $\gamma_t(G-e) \le 2n+3$. Thus, $2n+2 \le \gamma_t(G) < \gamma_t(G-e') \le 2n+3$, which yields $\gamma_t(G) = 2n+2$. □

It follows from Claims 4.2 and 4.4 that $b_t(G) = 1$ if and only if $\mathscr{C}$ is satisfiable. The theorem follows. □

5. NP-hardness of reinforcement

In this section, we will show that the problem determining the reinforcements and total reinforcements of general graphs are NP-hard. We first state them as the following decision problem.

(Total) Reinforcement problem:

Instance: A graph G and a positive integer k.

Question: Is $(r_t(G))\, r(G) \le k$?

Theorem 5.1. *The reinforcement problem is NP-hard.*

Proof. We show the NP-hardness of the reinforcement problem by reducing the 3-satisfiability problem to it in polynomial time.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-satisfiability problem. We will construct a graph G and take an integer k such that $\mathscr{C}$ is satisfiable if and only if $r(G) \le k$. Such a graph G can be constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a triangle T_i with vertex-set $\{u_i, \bar{u}_i, v_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{x_j, y_j, z_j\}$, associate a single vertex c_j and add edges (c_j, x_j), (c_j, y_j) and (c_j, z_j), $1 \le j \le m$. Finally, add a vertex s and join s to every vertex c_j and set $k = 1$.

Fig. 3 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u_3}\}$, $C_2 = \{\bar{u_1}, u_2, u_4\}$, $C_3 = \{\bar{u_2}, u_3, u_4\}$.

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $\mathscr{C}$ is satisfiable if and only if $r(G) = 1$. To this aim, we first prove the following two claims.

Claim 5.1.1. $\gamma(G) = n+1$.

Proof. Use the symbol $N[s]$ to denote the closed-neighborhood of s in G, that is, $N[s] = \{u \in V(G) : us \in E\} \cup \{s\}$. On the one hand, let D be a γ-set of G, then $\gamma(G) = |D| \ge n+1$ since $|D \cap V(T_i)| \ge 1$ and $|D \cap N[s]| \ge 1$. On the other hand, $D' = \{s, u_1, u_2, \ldots, u_n\}$ is a dominating set of G, which implies that $\gamma(G) \le |D'| = n+1$. It follows that $\gamma(G) = n+1$. □

Claim 5.1.2. *If there exists an edge $e \in E(\bar{G})$ such that $\gamma(G+e) = n$, and let D_e be a γ-set of $G+e$, then $|D_e \cap V(T_i)| = 1$ for each $i = 1, 2, \ldots, n$, while $c_j \notin D_e$ for each $j = 1, 2, \ldots, m$.*

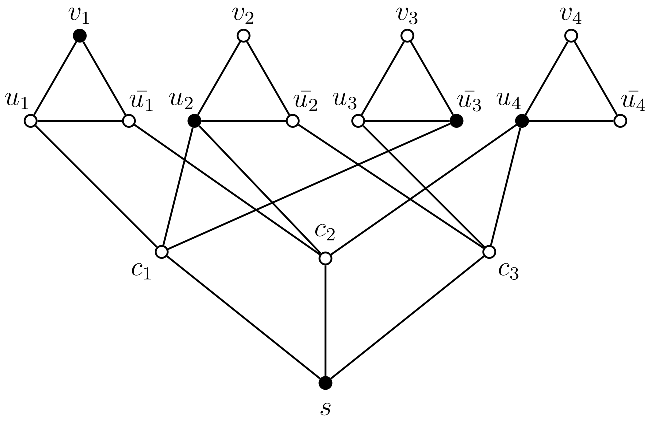

Fig. 3. An instance of the reinforcement problem resulting from an instance of the 3-satisfiability problem, in which $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{\{u_1, u_2, \bar{u}_3\}, \{\bar{u}_1, u_2, u_4\}, \{\bar{u}_2, u_3, u_4\}\}$. Here $k = 1$ and $\gamma = 5$, where the set of bold points is a γ-set.

Proof. Suppose to the contrary that $|D_e \cap V(T_{i_0})| = 0$ for some i_0 with $1 \le i_0 \le n$. Then one end-vertex of the edge e should be v_{i_0} since D_e dominates it via the edge e in $G + e$, and for every $i \ne i_0$, $|D_e \cap V(T_i)| \ge 1$ since D_e dominates v_i. By the hypotheses, two literals u_{i_0} and $\bar{u}_{i_0}$ do not simultaneously appear in the same clause in $\mathscr{C}$, they are not incident with the same vertex c_j in G for some j. Since u_{i_0} and $\bar{u}_{i_0}$ should be dominated by D_e, there exist two distinct vertices $c_j, c_l \in D_e$ such that c_j dominates u_{i_0} and c_l dominates $\bar{u}_{i_0}$. Thus, $|D_e| \ge n + 1$, a contradiction. Hence, $|D_e \cap V(T_i)| = 1$ for each $i = 1, 2, \ldots, n$, and $c_j \notin D_e$ for every j since $|D_e| = n$. □

We now show that $\mathscr{C}$ is satisfiable if and only if $r(G) = 1$.

Suppose that $\mathscr{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. We construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$ then put the vertex u_i in D'; if $t(u_i) = F$ then put the vertex $\bar{u}_i$ in D'. Then $|D'| = n$. Since t is a satisfying truth assignment for $\mathscr{C}$, for each $j = 1, 2, \ldots, m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex in D' since c_j is adjacent to each literal in C_j by the construction of G. Without loss of generality let $t(u_1) = T$, then D' is a dominating set of $G + su_1$, and hence $\gamma(G + su_1) \le |D'| = n$. By Claim 5.1.1, we have $\gamma(G) = n + 1$. It follows that $\gamma(G + su_1) \le n < n + 1 = \gamma(G)$, which implies $r(G) = 1$.

Conversely, assume $r(G) = 1$. Then there exists an edge e in $\bar{G}$ such that $\gamma(G + e) = n$. Let D_e be a γ-set of $G + e$. By Claim 5.1.2, $|D_e \cap V(T_i)| = 1$ for each $i = 1, 2, \ldots, n$, and $c_j \notin D_e$ for each $j = 1, 2, \ldots, m$. Define $t : U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } u_i \in D_e \text{ or } v_i \in D_e, \\ F & \text{if } \bar{u}_i \in D_e, \end{cases} \quad i = 1, 2, \ldots, n. \tag{5.1}$$

We will show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t.

Consider arbitrary clause $C_j \in \mathscr{C}$ with $1 \le j \le m$. By Claim 5.1.2, the corresponding vertex c_j in G is dominated by u_i or $\bar{u}_i$ in D_e for some i. Suppose that c_j is dominated by $u_i \in D_e$. Then u_i is adjacent to c_j in G, that is, the literal u_i is in the clause C_j by the construction of G. Since $u_i \in D_e$, we have $t(u_i) = T$ by (5.1), which implies that C_j is satisfied by t. Suppose that c_j is dominated by $\bar{u}_i \in D_e$. Then $\bar{u}_i$ is adjacent to c_j in G, that is, the literal $\bar{u}_i$ is in the clause C_j. Since $\bar{u}_i \in D_e$, we have $t(u_i) = F$ by (5.1), which implies that $\bar{u}_i$ is assigned the truth value T by t, so the clause C_j is satisfied. The arbitrariness of j with $1 \le j \le m$ shows that all the clauses in $\mathscr{C}$ are satisfied by t, that is, $\mathscr{C}$ is satisfiable.

The theorem follows. □

By using an analogous argument as in the proof of Theorem 5.1, we can prove that total reinforcement problem is also NP-hard. Here, we give an outline of the proof of it, details are omitted.

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Theorem 5.2. *The total reinforcement problem is NP-hard.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-satisfiability problem. We will construct a graph G and take an integer k such that $\mathscr{C}$ is satisfiable if and only if $r_t(G) \le k$. Such a graph G can be constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a graph H_i with vertex-set $V(H_i) = \{u_i, \bar{u}_i, v_i, v_i'\}$ and edge-set $E(H_i) = \{v_iu_i, u_i\bar{u}_i, \bar{u}_iv_i, v_iv_i'\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{x_j, y_j, z_j\} \in \mathscr{C}$, associate a single vertex c_j and add an edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$, $1 \le j \le m$. Finally, add a path $P = s_1s_2s_3$ and join s_1 to each vertex c_j, $1 \le j \le m$ and set $k = 1$.

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $\mathscr{C}$ is satisfiable if and only if $r_t(G) = 1$.

Claim 5.2.1. $\gamma_t(G) = 2n + 2$.

Claim 5.2.2. *If there exists an edge $e \in E(\bar{G})$ such that $\gamma_t(G+e) < 2n+2$, and let D_e be a γ_t-set of $G+e$, then $|D_e \cap V(H_i)| = 2$ for each $i = 1, 2, \ldots, n$, while $s_1 \notin D_e$ and $c_j \notin D_e$ for each $j = 1, 2, \ldots, m$.*

Using the same methods as in Theorem 5.1, we can show that $\mathscr{C}$ is satisfiable if and only if $r_t(G) = 1$. □

6. Conclusions

Domination is a very important concept in graph theory. It is well known that the domination number is NP-complete. The (total) bondage number and the (total) reinforcement number are concepts in graph theory related to domination number, and are also important parameters for measuring the vulnerability and stability of the (total) domination under edge failure and edge addition, and have received much research attention. First of all, we should know whether or not there are explicit expressions in terms of other graph-theoretical parameters or polynomial algorithms for computing these parameters. However, these problems have not been answered to general graphs in the recent literature.

In this paper, we investigate the computational complexity of these problems and prove that they are NP-hard by reductions from 3-SAT. These results show that there are neither explicit expressions in terms of other graph-theoretical parameters nor polynomial algorithms to compute these parameters unless $P = NP$. At the same time, these results also show that the following study is of important significance.

- Find approximation polynomial algorithms with performance ratio as small as possible.
- Find the lower and upper bounds with difference as small as possible.
- Determine exact values for some graphs, specially well-known networks.

Unfortunately, we cannot prove whether or not determining the bondage and the reinforcement is NP-problem, since for any subset $B \subset E(G)$, it is not clear that there is a polynomial algorithm to verify $\gamma(G - B) > \gamma(G)$ (or $\gamma(G + B) < \gamma(G)$). Since the problem of determining the domination number is NP-complete, we conjecture that they are not in *NP*. We will focus on this work for further study.

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