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Domination Numbers of Undirected Toroidal Mesh

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Abstract The (d, m)-domination number $\gamma_{d,m}$ is a new measure to characterize the reliability of resources-sharing in fault tolerant networks, in some sense, which can more accurately characterize the reliability of networks than the m-diameter does. In this paper, we study the $(d, 4)$-domination numbers of undirected toroidal mesh $C_{d_1} \times C_{d_2}$ for some special values of d, obtain that $\gamma_{d,4}(C_{d_1} \times C_3) = 2$ if and only if $d_4(G) - e_1 \le d < d_4(G)$ for $d_1 \ge 5$, $\gamma_{d,4}(C_{d_1} \times C_4) = 2$ if $d_4(G) - (2e_1 - \lfloor \frac{d_1+e_1}{2} \rfloor) \le d < d_4(G)$ for $d_1 \ge 24$, and $\gamma_{d,4}(C_{d_1} \times C_{d_2}) = 2$ if $d_4(G) - (e_1 - 2) \le d < d_4(G)$ for $d_1 = d_2 \ge 14$.

Keywords Undirected toroidal mesh, reliability, m-diameter, domination number

MR(2000) Subject Classification 05C40, 68M10, 68M15, 68R10

1 Introduction

We quote from [1] the terminology and notations not defined here, use graphs to represent networks and denote the length of a path P by $|P|$.

The n-dimensional undirected toroidal mesh, denoted by $C(d_1, d_2, \dots, d_n)$, has the vertex-set $\{(x_1, \dots, x_n) | 0 \le x_i < d_i \, (i = 1, 2, \dots, n)\}$. Each vertex $(x_1, x_2, \dots, x_n)$ is adjacent to $2n$ other vertices: $(x_1 \pm 1, x_2, \dots, x_n)$, $(x_1, x_2 \pm 1, \dots, x_n)$, $\dots$, $(x_1, x_2, \dots, x_n \pm 1)$, where additions are performed modulo $d_i \, (1 \le i \le n)$. It is well known that $C(d_1, d_2, \dots, d_n)$ is $2n$-regular and vertex-transitive. Its connectivity is $2n$. The toroidal mesh is widely used in network theory (see [2–4]).

In order to characterize the reliability of transmission delay in a real-time parallel processing system, Hsu and Lyuu [5] and Flandrin and Li [6] independently introduced m-diameter. For an m-connected graph G, the distance with width m from vertex x to y, denoted by $d_m(G; x, y)$, is the minimum number d for which there are m internally disjoint (x, y)-paths in G of length at most d. The diameter with width m, denoted by $d_m(G)$, is the maximum of $d_m(G; x, y)$ over all pairs (x, y) of vertices of G.

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Recently, Li and Xu in [7] defined a new parameter (d, m)-domination number in m-connected graphs, in some sense, which can more accurately characterize the reliability of networks than the wide-diameter does.

Definition *For an m-connected graph G and a given integer d, a nonempty and proper subset S of the vertex set of G is called a (d, m)-dominating set of G if for any vertex x of G but not in S there are at least m internally disjoint (x, S)-paths of length at most d. Denote a set of all (d, m)-dominating sets of G by $S_{d,m}(G)$. The parameter*

$$\gamma_{d,m}(G) = \min\{|S| : S \in S_{d,m}(G)\}$$

is called the (d, m)-domination number of G.

This notion not only generalizes that of the classical domination numbers of a graph but also gives a good measure of the problem of resources-sharing in fault tolerant networks. An important and practical problem is how to choose a (d, m)-dominating set S such that the number of vertices in S is as small as possible. Thus the (d, m)-domination numbers in conjunction with other well-known parameters, for example, the (d, m)-independence numbers (see [8]), can provide a more accurate analysis of fault tolerance for reliability and efficiency of networks of parallel architectures.

In general, to determine the (d, m)-domination number of a graph is NP-Complete since its special case of $(1, 1)$, the domination number of the graph, is NP-Complete (see [9]). Thus it is of interest to determine the (d, m)-domination number of some well-known networks for some special values of m and d. The $(1, 1)$-domination numbers of some graphs have been studied (see, for example, [10]).

Of course, if $d \geq d_m(G)$, then we obtain that $\gamma_{d,m}(G) = 1$. And we have $\gamma_{d,m}(G) \geq 2$ for $d < d_m(G)$. So it is of interest to determine values of $\gamma_{d,m}(G)$ for $d < d_m(G)$. Lu and Zhang [11] proved the $(d, 2n)$-domination number of $C(d_1, d_2, \ldots, d_n)\,(\neq C(3, 3, \ldots, 3))$ is 2 for $d = \mathrm{diam}(C(d_1, d_2, \ldots, d_n))$ $(n \geq 3, d_i \geq 3, i \in \{0, 1, \ldots, n\})$. Xie and Xu [12] proved the $(d, 2n)$-domination number of $C(3, 3, \ldots, 3)$ is 3 for $d = \mathrm{diam}(C(3, 3, \ldots, 3)) = n$. This paper obtains $\gamma_{d,4}(C_{d_1} \times C_3) = 2$ if and only if $d_4(G) - e_1 \leq d < d_4(G)$ for $d_1 \geq 5$, $\gamma_{d,4}(C_{d_1} \times C_4) = 2$ if $d_4(G) - (2e_1 - \lfloor \frac{d_1+e_1}{2} \rfloor) \leq d < d_4(G)$ for $d_1 \geq 24$, and $\gamma_{d,4}(C_{d_1} \times C_{d_2}) = 2$ if $d_4(G) - (e_1 - 2) \leq d < d_4(G)$ for $d_1 = d_2 \geq 14$.

2 Preliminaries

The undirected toroidal mesh $C(d_1, d_2, \ldots, d_n)$ can also be defined as the cartesian products $C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}$, where C_{d_i} is an undirected cycle for each $i = 1, 2, \ldots, n$. This graph is an n-dimensional orthogonal mesh with global edges. For $n = 2$, $C_{d_1} \times C_{d_2}$ is a mesh on a toroidal surface.

In this paper, we denote $\lfloor \frac{d_i}{2} \rfloor$ and $\lceil \frac{d_i}{2} \rceil$ by e_i and e'_i, define $\delta(x_i) = -1$ for $e_i < x_i \leq d_i - 1$ and $\delta(x_i) = 1$ for $0 < x_i \leq e_i$, respectively, where $i = 1, 2$.

Lemma 2.1 ([13]) *For any integers $n \geq 2$, $d_i \geq 3\,(1 \leq i \leq n)$,*

$$\mathrm{diam}(C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}) = \sum_{i=1}^{n} \left\lfloor \frac{d_i}{2} \right\rfloor.$$

We will use another lemma from [13] that relates to the m-diameter of a general toroidal mesh.

Lemma 2.2 ([13]) *Let $G = C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}$ for any $n \geq 2, d_1 \geq d_2 \geq \cdots \geq d_n \geq 3$. Then*

$$d_{2n}(G) = \begin{cases} 2m+1 = 2\text{diam}(G) - 1, & G = C_{2m+1} \times C_3,\ m \geq 2, \\ 2m+2 = 2\text{diam}(G) - 2, & G = C_{2m+2} \times C_3,\ m \geq 2, \\ e_1 + 4 = \text{diam}(G) + 2, & G = C_d \times C_4,\ d \geq 9 \text{ or} \\ & G = C_d \times C_5,\ 9 \leq d \leq 13, \\ 1 + \sum_{i=1}^{n} e_i = \text{diam}(G) + 1, & \text{otherwise.} \end{cases}$$

3 Main Results

Theorem 3.1 *Let $G = C_{d_1} \times C_3$, $d_1 \geq 5$. We have $\gamma_{d,4}(G) = 2$ if and only if $d_4(G) - e_1 \leq d < d_4(G)$.*

Proof We easily know $d_4(G) - e_1 = d_1 - e_1 = e_1' \,(\geq e_1)$ by Lemma 2.2.

First we prove $\gamma_{d,4}(G) = 2$ if $d_4(G) - e_1 \leq d < d_4(G)$. From the definition of the (d,m)-domination number, we have $\gamma_{d,4}(G) \geq 2$ for $d < d_4(G)$ and $\gamma_{d,4}(G) \leq \gamma_{d-1,4}(G)$, so we only need to prove $\gamma_{d,4}(G) \leq 2$ for $d = e_1'$. Let $S = \{o, e\}$, where $o = (0,0)$, $e = (e_1, 1)$. For any vertex $x = (x_1, x_2) \in V(G) - S$, $x_1 \in \{0, 1, \ldots, d_1 - 1\}$, $x_2 \in \{0, 1, 2\}$, we will prove S can $(d,4)$-dominate vertex x, we only need to consider the following cases by vertex-transitivity and symmetry:

Case 1 $x_2 = 0$, $x = (x_1, 0) \,(\neq o)$.

Subcase 1a $1 \leq x_1 \leq e_1 - 2$ or $e_1 + 2 \leq x_1 \leq d_1 - 1$.

We construct the four paths $P_i \,(1 \leq i \leq 4)$ as follows:

$P_1 : x = (x_1, 0) \to (x_1 - \delta(x_1), 0) \to \cdots \to (\delta(x_1), 0) \to o;$

$P_2 : x = (x_1, 0) \to (x_1, 1) \to (x_1 - \delta(x_1), 1) \to \cdots \to (\delta(x_1), 1) \to (0, 1) \to o;$

$P_3 : x = (x_1, 0) \to (x_1, 2) \to (x_1 - \delta(x_1), 2) \to \cdots \to (\delta(x_1), 2) \to (0, 2) \to o;$

$P_4 : x = (x_1, 0) \to (x_1 + \delta(x_1), 0) \to \cdots \to (e_1, 0) \to e.$

We can see $|P_1| = x_1$, $|P_2| = |P_3| = x_1 + 2 \leq e_1$ and $|P_4| = e_1 - x_1 + 1 \leq e_1$ if $1 \leq x_1 \leq e_1 - 2$. When $e_1 + 2 \leq x_1 \leq d_1 - 1$, we have $|P_1| = d_1 - x_1$, $|P_2| = |P_3| = d_1 - x_1 + 2 \leq d_1 - e_1 = e_1'$ and $|P_4| = x_1 - e_1 + 1 \leq d_1 - e_1 = e_1'$.

Subcase 1b $x_1 = e_1 - 1$ or $x_1 = e_1 + 1$.

We construct the four paths $P_i \,(1 \leq i \leq 4)$ as follows:

$P_1 : x = (e_1 - \delta(x_1), 0) \to (e_1 - 2\delta(x_1), 0) \to \cdots \to (\delta(x_1), 0) \to o;$

$P_2 : x = (e_1 - \delta(x_1), 0) \to (e_1, 0) \to e;$

$P_3 : x = (e_1 - \delta(x_1), 0) \to (e_1 - \delta(x_1), 1) \to e;$

$P_4 : x = (e_1 - \delta(x_1), 0) \to (e_1 - \delta(x_1), 2) \to (e_1, 2) \to e.$

The length of any path of the four is at most $\max\{x_1, d_1 - x_1\} = d_1 - e_1 - 1 = e_1' - 1$.

Subcase 1c $x_1 = e_1$.

We construct the four paths $P_i \,(1 \leq i \leq 4)$ as follows:

$P_1 : x = (e_1, 0) \to e;$

$P_2 : x = (e_1, 0) \to (e_1, 2) \to e$;
$P_3 : x = (e_1, 0) \to (e_1 + 1, 0) \to (e_1 + 1, 1) \to e$;
$P_4 : x = (e_1, 0) \to (e_1 - 1, 0) \to (e_1 - 1, 1) \to e$.
Of course, the lengths of paths are at most $3\,(\le e'_1)$.

Case 2 $x_2 = 1$, $x = (x_1, 1)\,(\ne e)$.

It is similar to Case 1 by vertex-transitivity.

Case 3 $x_2 = 2$, $x = (x_1, 2)$.

Subcase 3a $x_1 = 0$.

We can construct four paths between vertices o and x in the same way as Subcase 1c by vertex-transitivity.

Subcase 3b $1 \le x_1 \le d_1 - 1$ and $x_1 \ne e_1$.

We construct the four paths $P_i\,(1 \le i \le 4)$ as follows:
$P_1 : x = (x_1, 2) \to (x_1, 0) \to (x_1 - \delta(x_1), 0) \to \cdots \to (\delta(x_1), 0) \to o$;
$P_2 : x = (x_1, 2) \to (x_1 - \delta(x_1), 2) \to \cdots \to (\delta(x_1), 2) \to (0, 2) \to o$;
$P_3 : x = (x_1, 2) \to (x_1 + \delta(x_1), 2) \to \cdots \to (e_1, 2) \to e$;
$P_4 : x = (x_1, 2) \to (x_1, 1) \to (x_1 + \delta(x_1), 1) \to \cdots \to e$.

We can see $|P_1| = |P_2| = x_1 + 1 \le e_1$ and $|P_3| = |P_4| = e_1 - x_1 + 1 \le e_1$ if $1 \le x_1 \le e_1 - 1$. When $e_1 + 1 \le x_1 \le d_1 - 1$, we have $|P_1| = |P_2| = d_1 - x_1 + 1 \le d_1 - e_1 = e'_1$ and $|P_3| = |P_4| = x_1 - e_1 + 1 \le d_1 - e_1 = e'_1$.

Subcase 3c $x_1 = e_1$.

It is similar to Subcase 1c by vertex-transitivity.

From above, we know there exist four internally disjoint (x, S)-paths of length at most e'_1 in G for any vertex x. So S is a $(d, 4)$-dominating set of G, we have $\gamma_{d,4}(G) \le 2$.

Next we prove $d_4(G) - e_1 \le d < d_4(G)$ if $\gamma_{d,4}(G) = 2$. We only need to prove $\gamma_{d,4}(G) \ge 3$ if $d < d_4(G) - e_1 = e'_1$.

Assume $S = \{o, y\}$ is a $(d, 4)$-dominating set of G, where $o = (0, 0)$, $y = (y_1, y_2)$, $0 \le y_1 \le e_1$ and $0 \le y_2 \le 1$ without loss of generality.

If $y = (y_1, 0)$, then the distance between vertices y and $x = (y_1 + e'_1 - 1, 1)$ is $\min\{e_1 + 2, e'_1\} = e'_1(> d)$. If there exist four internally disjoint paths of length at most d between vertices o and x, we can assume that P_1 contains the vertices o, $(d_1 - 1, 0)$ and x, P_2 contains o, $(0, 1)$ and x, P_3 contains o, $(0, 2)$ and x. So the path P_4 must contain vertices o, $(1, 0)$ and $(y_1 + e'_1 - 2, 1)$. Of course, the length of P_4 is greater than or equal to $y_1 + e'_1 \ge 1 + e'_1(> d)$. It is a contradiction. S can not $(d, 4)$-dominate the vertex x.

If $y = (y_1, 1)$, then the distance between vertices y and $x = (y_1 + e'_1 - 1, 2)$ is at least $e'_1(> d)$. Similarly, we can easily know there do not exist four internally disjoint paths of length at most d between vertices o and $x = (y_1 + e'_1 - 1, 2)$, S can not $(d, 4)$-dominate the vertex $(y_1 + e'_1 - 1, 2)$.

It is a contradiction. So we have $\gamma_{d,4}(G) \ge 3$ for $d < e'_1$.

Thus, we have proven that $\gamma_{d,4}(G) = 2$ if and only if $d_4(G) - e_1 \le d < d_4(G)$. □

Theorem 3.2 *Let $G = C_{d_1} \times C_4$, $d_1 \ge 24$. We have $\gamma_{d,4}(G) = 2$ if $d_4(G) - (2e_1 - \lfloor \frac{d_1+e_1}{2} \rfloor) \le d < d_4(G)$.*

Proof By Lemma 2.2, we have $d_4(G)-(2e_1-\lfloor\frac{d_1+e_1}{2}\rfloor)=d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$. We only need to prove $\gamma_{d,4}(G)\le 2$ for $d=d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$.

Let $S=\{o,e\}$, where $o=(0,0),\ e=(e_1,0)$. We will prove S can $(d,4)$-dominate any vertex $x=(x_1,x_2)\in V(G)-S$, where $x_1\in\{0,1,\ldots,d_1-1\}$ and $x_2\in\{0,1,2,3\}$. We only consider the following cases by vertex-transitivity:

Case 1 $x_2=0, x=(x_1,0)\,(\ne o,e)$.

Subcase 1a $1\le x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil-4$ or $\lceil\frac{d_1+e_1}{2}\rceil+4\le x_1\le d_1-1$.

We construct the four paths $P_i\,(1\le i\le 4)$ as follows:

$P_1: x=(x_1,0)\to(x_1-\delta(x_1),0)\to\cdots\to(\delta(x_1),0)\to o;$ (1)

$P_2: x=(x_1,0)\to(x_1,1)\to(x_1-\delta(x_1),1)\to\cdots\to(\delta(x_1),1)\to(0,1)\to o;$ (2)

$P_3: x=(x_1,0)\to(x_1,-1)\to(x_1-\delta(x_1),-1)\to\cdots\to(\delta(x_1),-1)\to(0,-1)\to o;$ (3)

$P_4: x=(x_1,0)\to(x_1+\delta(x_1),0)\to(x_1+\delta(x_1),1)\to(x_1+\delta(x_1),2)\to(x_1,2)\to\cdots\to(\delta(x_1),2)\to(0,2)\to(-\delta(x_1),2)\to(-\delta(x_1),1)\to(-\delta(x_1),0)\to o.$ (4)

We can see $|P_1|=x_1$, $|P_2|=|P_3|=x_1+2$ and $|P_4|=x_1+8\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$ if $1\le x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil-4$. When $\lfloor\frac{d_1+e_1}{2}\rfloor+5\le x_1\le d_1-1$, we have $|P_1|=d_1-x_1$, $|P_2|=|P_3|=d_1-x_1+2$ and $|P_4|=d_1-x_1+8\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$.

Subcase 1b $d_1-\lceil\frac{d_1+e_1}{2}\rceil-3\le x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+2$ or $\lceil\frac{d_1+e_1}{2}\rceil-2\le x_1\le\lceil\frac{d_1+e_1}{2}\rceil+3$.

We construct the three paths $P_i\,(1\le i\le 3)$ as (1), (2), (3) and the one other path P_4 as follows:

$P_4: x=(x_1,0)\to(x_1+\delta(x_1),0)\to\cdots\to(e_1,0)=e.$ (5)

We can see $|P_1|=x_1$, $|P_2|=|P_3|=x_1+2\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$ and $|P_4|=e_1-x_1\le e_1-d_1+\lceil\frac{d_1+e_1}{2}\rceil+3$ if $d_1-\lceil\frac{d_1+e_1}{2}\rceil-3\le x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+2$. When $\lceil\frac{d_1+e_1}{2}\rceil-2\le x_1\le\lceil\frac{d_1+e_1}{2}\rceil+3$, we have $|P_1|=d_1-x_1$, $|P_2|=|P_3|=d_1-x_1+2\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$ and $|P_4|=x_1-e_1\le\lceil\frac{d_1+e_1}{2}\rceil-e_1+3$.

Subcase 1c $d_1-\lceil\frac{d_1+e_1}{2}\rceil+3\le x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$ or $\lceil\frac{d_1+e_1}{2}\rceil-4\le x_1\le\lceil\frac{d_1+e_1}{2}\rceil-3$.

We construct the two paths P_1,P_2 as (1), (5) and the other two paths P_3,P_4 as follows:

$P_3: x=(x_1,0)\to(x_1,1)\to(x_1+\delta(x_1),1)\to\cdots\to(e_1,1)\to e;$ (6)

$P_4: x=(x_1,0)\to(x_1,3)\to(x_1+\delta(x_1),3)\to\cdots\to(e_1,3)\to e.$ (7)

We can see $|P_1|=x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$, $|P_2|=e_1-x_1$, $|P_3|=|P_4|=e_1-x_1+2\le e_1-d_1+\lceil\frac{d_1+e_1}{2}\rceil-1$ if $d_1-\lceil\frac{d_1+e_1}{2}\rceil+3\le x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$. When $\lceil\frac{d_1+e_1}{2}\rceil-4\le x_1\le\lceil\frac{d_1+e_1}{2}\rceil-3$, we have $|P_1|=d_1-x_1\le d_1-\lceil\frac{d_1+e_1}{2}\rceil+4$, $|P_2|=x_1-e_1$ and $|P_3|=|P_4|=x_1-e_1+2\le\lceil\frac{d_1+e_1}{2}\rceil-e_1-1$.

Subcase 1d $d_1-\lceil\frac{d_1+e_1}{2}\rceil+5\le x_1\le e_1-1$ or $e_1+1\le x_1\le\lceil\frac{d_1+e_1}{2}\rceil-5$.

We construct the three paths P_1, P_2, P_3 as (5), (6), (7) and the one other path P_4 as follows:

$P_4: x=(x_1,0)\to(x_1-\delta(x_1),0)\to(x_1-\delta(x_1),1)\to(x_1-\delta(x_1),2)\to(x_1,2)\to\cdots\to(e_1+\delta(x_1),2)\to(e_1+\delta(x_1),1)\to(e_1+\delta(x_1),0)\to e.$

We can see $|P_1|=e_1-x_1$, $|P_2|=|P_3|=e_1-x_1+2$, $|P_4|=e_1-x_1+8=e_1-d_1+\lceil\frac{d_1+e_1}{2}\rceil+3$ if $d_1-\lceil\frac{d_1+e_1}{2}\rceil+5\le x_1\le e_1-1$. When $e_1+1\le x_1\le\lceil\frac{d_1+e_1}{2}\rceil-5$, we have $|P_1|=x_1-e_1$, $|P_2|=|P_3|=x_1-e_1+2$, $|P_4|=x_1-e_1+8=\lceil\frac{d_1+e_1}{2}\rceil-e_1+3$.

Case 2 $x_2=1$, $x=(x_1,1)$.

Subcase 2a $x=(0,1)$ or $x=(e_1,1)$.

We can easily construct four (x, S)-paths of length at most 3 in the same way as Subcase 1c of Theorem 3.1.

Subcase 2b $1 \le x_1 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil - 1$ or $\lceil \frac{d_1+e_1}{2} \rceil + 1 \le x_1 \le d_1 - 1$.

We construct the four paths $P_i\,(1 \le i \le 4)$ as follows:

$$P_1 : x = (x_1, 1) \to (x_1 - \delta(x_1), 1) \to \cdots \to (\delta(x_1), 1) \to (0, 1) \to o; \tag{8}$$

$$P_2 : x = (x_1, 1) \to (x_1, 0) \to (x_1 - \delta(x_1), 0) \to \cdots \to (\delta(x_1), 0) \to o; \tag{9}$$

$P_3 : x = (x_1, 1) \to (x_1, 2) \to (x_1 - \delta(x_1), 2) \to \cdots \to (\delta(x_1), 2) \to (0, 2) \to (-\delta(x_1), 2) \to (-\delta(x_1), 1) \to (-\delta(x_1), 0) \to o;$

$P_4 : x = (x_1, 1) \to (x_1 + \delta(x_1), 1) \to (x_1 + \delta(x_1), 2) \to (x_1 + \delta(x_1), 3) \to (x_1, 3) \to \cdots \to (\delta(x_1), 3) \to (0, 3) \to o.$

We can see $|P_1| = |P_2| = x_1 + 1$ and $|P_3| = |P_4| = x_1 + 5 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 4$ if $1 \le x_1 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil - 1$. When $\lceil \frac{d_1+e_1}{2} \rceil + 1 \le x_1 \le d_1 - 1$, we have $|P_1| = |P_2| = d_1 - x_1 + 1$, $|P_3| = |P_4| = d_1 - x_1 + 5 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 4$.

Subcase 2c $d_1 - \lceil \frac{d_1+e_1}{2} \rceil \le x_1 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 3$ or $\lceil \frac{d_1+e_1}{2} \rceil - 3 \le x_1 \le \lceil \frac{d_1+e_1}{2} \rceil$.

We construct the two paths P_1, P_2 as (8), (9) and the other two paths P_3, P_4 as follows:

$$P_3 : x = (x_1, 1) \to (x_1 + \delta(x_1), 1) \to \cdots \to (e_1, 1) \to e; \tag{10}$$

$P_4 : x = (x_1, 1) \to (x_1, 2) \to (x_1, 3) \to (x_1 + \delta(x_1), 3) \to \cdots \to (e_1, 3) \to e.$

We can see $|P_1| = |P_2| = x_1 + 1 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 4$, $|P_3| = e_1 - x_1 + 1$, $|P_4| = e_1 - x_1 + 3 \le e_1 - d_1 + \lceil \frac{d_1+e_1}{2} \rceil + 3$ if $d_1 - \lceil \frac{d_1+e_1}{2} \rceil \le x_1 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 3$. When $\lceil \frac{d_1+e_1}{2} \rceil - 3 \le x_1 \le \lceil \frac{d_1+e_1}{2} \rceil$, we have $|P_1| = |P_2| = d_1 - x_1 + 1 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 4$, $|P_3| = x_1 - e_1 + 1$, $|P_4| = x_1 - e_1 + 3 = \lceil \frac{d_1+e_1}{2} \rceil - e_1 + 3$.

Subcase 2d $d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 4 \le x_1 \le e_1 - 1$ or $e_1 + 1 \le x_1 \le \lceil \frac{d_1+e_1}{2} \rceil - 4$.

We construct the one path P_1 as (10) and the other three paths P_2, P_3, P_4 as follows:

$P_2 : x = (x_1, 1) \to (x_1, 0) \to (x_1 + \delta(x_1), 0) \to \cdots \to e;$

$P_3 : x = (x_1, 1) \to (x_1, 2) \to (x_1 + \delta(x_1), 2) \to \cdots \to (e_1 + \delta(x_1), 2) \to (e_1 + \delta(x_1), 1) \to (e_1 + \delta(x_1), 0) \to e;$

$P_4 : x = (x_1, 1) \to (x_1 - \delta(x_1), 1) \to (x_1 - \delta(x_1), 0) \to (x_1 - \delta(x_1), 3) \to (x_1, 3) \to \cdots \to (e_1, 3) \to e.$

We can see $|P_1| = |P_2| = e_1 - x_1 + 1$, $|P_3| = |P_4| = e_1 - x_1 + 5 = e_1 - d_1 + \lceil \frac{d_1+e_1}{2} \rceil + 1$ if $d_1 - \lceil \frac{d_1+e_1}{2} \rceil + 4 \le x_1 \le e_1 - 1$. When $e_1 + 1 \le x_1 \le \lceil \frac{d_1+e_1}{2} \rceil - 4$, we have $|P_1| = |P_2| = x_1 - e_1 + 1$, $|P_3| = |P_4| = x_1 - e_1 + 5 = \lceil \frac{d_1+e_1}{2} \rceil - e_1 + 1$.

Case 3 $x_2 = 2$, $x = (x_1, 2)$.

Subcase 3a $x = (0, 2)$ or $x = (e_1, 2)$.

We can easily construct four (x, S)-paths of length at most 4.

Subcase 3b $1 \le x_1 \le d_1 - \lceil \frac{d_1+e_1}{2} \rceil$ or $\lceil \frac{d_1+e_1}{2} \rceil \le x_1 \le d_1 - 1$.

We construct the four paths $P_i\,(1 \le i \le 4)$ as follows:

$P_1 : x = (x_1, 2) \to (x_1, 1) \to (x_1 - \delta(x_1), 1) \to \cdots \to (\delta(x_1), 1) \to (0, 1) \to o;$

$P_2 : x = (x_1, 2) \to (x_1, 3) \to (x_1 - \delta(x_1), 3) \to \cdots \to (\delta(x_1), 3) \to (0, 3) \to o;$

$P_3 : x = (x_1, 2) \to (x_1 + \delta(x_1), 2) \to (x_1 + \delta(x_1), 1) \to (x_1 + \delta(x_1), 0) \to (x_1, 0) \to \cdots \to (\delta(x_1), 0) \to o;$

$P_4 : x = (x_1, 2) \to (x_1 - \delta(x_1), 2) \to \cdots \to (\delta(x_1), 2) \to (0, 2) \to (-\delta(x_1), 2) \to (-\delta(x_1), 1) \to (-\delta(x_1), 0) \to o$.

We can see $|P_1| = |P_2| = x_1 + 2$ and $|P_3| = |P_4| = x_1 + 4 \le d_1 - \lceil\frac{d_1+e_1}{2}\rceil + 4$ if $1 \le x_1 \le d_1 - \lceil\frac{d_1+e_1}{2}\rceil$. When $\lceil\frac{d_1+e_1}{2}\rceil \le x_1 \le d_1 - 1$, we have $|P_1| = |P_2| = d_1 - x_1 + 2$, $|P_3| = |P_4| = d_1 - x_1 + 4 \le d_1 - \lceil\frac{d_1+e_1}{2}\rceil + 4$.

Subcase 3c $d_1 - \lceil\frac{d_1+e_1}{2}\rceil + 1 \le x_1 \le e_1 - 1$ or $e_1 + 1 \le x_1 \le \lceil\frac{d_1+e_1}{2}\rceil - 1$.

We construct the four paths $P_i\,(1 \le i \le 4)$ as follows:

$P_1 : x = (x_1, 2) \to (x_1, 1) \to (x_1 + \delta(x_1), 1) \to \cdots \to (e_1, 1) \to e$;

$P_2 : x = (x_1, 2) \to (x_1, 3) \to (x_1 + \delta(x_1), 3) \to \cdots \to (e_1, 3) \to e$;

$P_3 : x = (x_1, 2) \to (x_1 + \delta(x_1), 2) \to \cdots \to (e_1 + \delta(x_1), 2) \to (e_1 + \delta(x_1), 1) \to (e_1 + \delta(x_1), 0) \to e$;

$P_4 : x = (x_1, 2) \to (x_1 - \delta(x_1), 2) \to (x_1 - \delta(x_1), 1) \to (x_1 - \delta(x_1), 0) \to (x_1, 0) \to \cdots \to (e_1 - \delta(x_1), 0) \to e$.

We can see $|P_1| = |P_2| = e_1 - x_1 + 2$ and $|P_3| = |P_4| = e_1 - x_1 + 4 = e_1 - d_1 + \lceil\frac{d_1+e_1}{2}\rceil + 3$ if $d_1 - \lceil\frac{d_1+e_1}{2}\rceil + 1 \le x_1 \le e_1 - 1$. When $e_1 + 1 \le x_1 \le \lceil\frac{d_1+e_1}{2}\rceil - 1$, we have $|P_1| = |P_2| = x_1 - e_1 + 2$, $|P_3| = |P_4| = x_1 - e_1 + 4 \le \lceil\frac{d_1+e_1}{2}\rceil - e_1 + 3$.

Case 4 $x_2 = 3$, $x = (x_1, 3)$.

It is similar to Case 2 by vertex-transitivity.

Note that the length of any path of the four in the above cases is at most

$$\begin{aligned} &\max\left\{e_1 - d_1 + \left\lceil\frac{d_1+e_1}{2}\right\rceil + 3, \left\lceil\frac{d_1+e_1}{2}\right\rceil - e_1 + 3, d_1 - \left\lceil\frac{d_1+e_1}{2}\right\rceil + 4\right\} \\ &\le d_1 - \left\lceil\frac{d_1+e_1}{2}\right\rceil + 4. \end{aligned}$$

We know S can $(d,4)$-dominate any vertex $x \in V(G) - S$. So we have $\gamma_{d,4}(G) \le 2$.

Thus, we have $\gamma_{d,4}(G) = 2$ if $d_4(G) - (2e_1 - \lfloor\frac{d_1+e_1}{2}\rfloor) \le d < d_4(G)$. □

Similarly, we can discuss $\gamma_{d,4}(C_{d_1} \times C_{d_2})$ for some special values $d_2\,(< d_1)$. Next we consider $\gamma_{d,4}(C_{d_1} \times C_{d_2})$ for $d_1 = d_2 \ge 14$.

Theorem 3.3 *Let $G = C_{d_1} \times C_{d_2}$, $d_1 = d_2 \ge 14$. We have $\gamma_{d,4}(G) = 2$ if $d_4(G) - (e_1 - 2) \le d < d_4(G)$.*

Proof By Lemma 2.2, we have $d_4(G) - (e_1 - 2) = e_1 + 3$. First assume d_1 and d_2 are odd, and we only need to prove $\gamma_{d,4}(G) \le 2$ for $d = e_1 + 3$. Let $S = \{o, e\}$, where $o = (0, 0)$, $e = (e_1, e_1)$. For any vertex $x = (x_1, x_2) \in V(G) - S$, where $x_1, x_2 \in \{0, 1, \ldots, d_1 - 1\}$, we consider the following cases by vertex-transitivity and symmetry:

Case 1 $0 \le x_1 \le e_1$ and $0 \le x_2 \le e_1$.

Subcase 1a $x_1 = 1$ or 2, $x_2 = 0$.

We construct the four paths $P_i\,(1 \le i \le 4)$ as (1), (2), (3), (4). We can see $|P_1| = x_1$, $|P_2| = |P_3| = x_1 + 2$ and $|P_4| = x_1 + 8 \le 10 \le e_1 + 3$.

Subcase 1b $3 \le x_1 \le e_1 - 3$ and $x_2 = 0$.

We construct the four paths $P_i\,(1 \le i \le 4)$ as follows:

$P_1 : x = (x_1, 0) \to (x_1, 1) \to (x_1 - 1, 1) \to \cdots \to (1, 1) \to (1, 0) \to o$;

$P_2: x=(x_1,0)\rightarrow(x_1+1,0)\rightarrow(x_1+1,1)\rightarrow(x_1+1,2)\rightarrow(x_1,2)\rightarrow\cdots\rightarrow(0,2)\rightarrow(0,1)\rightarrow o$;

$P_3: x=(x_1,0)\rightarrow(x_1-1,0)\rightarrow(x_1-1,-1)\rightarrow(x_1-2,-1)\rightarrow\cdots\rightarrow(0,-1)\rightarrow o$;

$P_4: x=(x_1,0)\rightarrow(x_1,-1)\rightarrow(x_1,-2)\rightarrow(x_1-1,-2)\rightarrow\cdots\rightarrow(0,-2)\rightarrow(-1,-2)\rightarrow(-1,-1)\rightarrow(-1,0)\rightarrow o$.

We can see $|P_1|=|P_3|=x_1+2$, $|P_2|=|P_4|=x_1+6\le e_1+3$.

Subcase 1c $e_1-2\le x_1\le e_1-1$ and $x_2=0$.

We construct the three paths $P_i\,(1\le i\le 3)$ as (1), (2), (3) and the one other path P_4 as follows:

$P_4: x=(x_1,0)\rightarrow(x_1+1,0)\rightarrow\cdots\rightarrow(-1,0)\rightarrow o$.

We can see $|P_1|=x_1$, $|P_2|=|P_3|=x_1+2\le e_1+1$, $|P_4|=d_1-x_1\le d_1-e_1+2=e_1+3$.

Subcase 1d $x=(e_1,0)$.

We can easily construct four paths of length at most e_1+2 in the same way as Subcase 1c of Theorem 3.1.

Subcase 1e $1\le x_1\le -x_2+e_1-1$, where $1\le x_2\le e_1-2$.

We construct the four paths $P_i\,(1\le i\le 4)$ as follows:

$P_1: x=(x_1,x_2)\rightarrow(x_1,x_2-1)\rightarrow\cdots\rightarrow(x_1,0)\rightarrow(x_1-1,0)\rightarrow\cdots\rightarrow o$; (11)

$P_2: x=(x_1,x_2)\rightarrow(x_1-1,x_2)\rightarrow\cdots\rightarrow(0,x_2)\rightarrow(0,x_2-1)\rightarrow\cdots\rightarrow o$; (12)

$P_3: x=(x_1,x_2)\rightarrow(x_1+1,x_2)\rightarrow(x_1+1,x_2-1)\rightarrow\cdots\rightarrow(x_1+1,0)\rightarrow(x_1+1,-1)\rightarrow(x_1,-1)\rightarrow\cdots\rightarrow(0,-1)\rightarrow o$;

$P_4: x=(x_1,x_2)\rightarrow(x_1,x_2+1)\rightarrow(x_1-1,x_2+1)\rightarrow\cdots\rightarrow(0,x_2+1)\rightarrow(-1,x_2+1)\rightarrow(-1,x_2)\rightarrow\cdots(-1,0)\rightarrow o$.

We can see $|P_1=|P_2|=x_1+x_2$, $|P_3|=|P_4|=x_1+x_2+4\le e_1+3$.

Subcase 1f $-x_2+e_1-1\le x_1\le -x_2+e_1$, where $x_1\ge 1$ and $x_2\ge 1$.

We construct the two paths P_1, P_2 as (11), (12) and the other two paths P_3,P_4 as follows:

$P_3: x=(x_1,x_2)\rightarrow(x_1,x_2+1)\rightarrow\cdots\rightarrow(x_1,e_1)\rightarrow(x_1+1,e_1)\rightarrow\cdots\rightarrow e$;

$P_4: x=(x_1,x_2)\rightarrow(x_1+1,x_2)\rightarrow\cdots\rightarrow(e_1,x_2)\rightarrow(e_1,x_2+1)\rightarrow\cdots\rightarrow e$.

We can see $|P_1=|P_2|=x_1+x_2\le e_1$, $|P_3|=|P_4|=2e_1-x_1-x_2\le e_1+1$.

Case 2 $e_1+1\le x_1\le d_1-1$ and $0\le x_2\le e_1$.

Subcase 2a $e_1+1\le x_1\le d_1-1$ and $x_2=0$.

We can construct four paths between vertices x and o in the same way as Subcase 1a, Subcase 1b, Subcase 1c or Subcase 1d, respectively.

Subcase 2b $x_2+e_1+2\le x_1\le d_1-1$, where $1\le x_2\le e_1-2$.

We construct the four paths $P_i\,(1\le i\le 4)$ as follows:

$P_1: x=(x_1,x_2)\rightarrow(x_1+1,x_2)\rightarrow\cdots\rightarrow(d_1-1,x_2)\rightarrow(0,x_2)\rightarrow(0,x_2-1)\rightarrow\cdots\rightarrow(0,1)\rightarrow o$; (13)

$P_2: x=(x_1,x_2)\rightarrow(x_1,x_2-1)\rightarrow\cdots\rightarrow(x_1,0)\rightarrow(x_1+1,0)\rightarrow\cdots\rightarrow(-1,0)\rightarrow o$; (14)

$P_3: x=(x_1,x_2)\rightarrow(x_1-1,x_2)\rightarrow(x_1-1,x_2-1)\rightarrow\cdots\rightarrow(x_1-1,0)\rightarrow(x_1-1,d_1-1)\rightarrow(x_1,d_1-1)\rightarrow\cdots\rightarrow(d_1-1,d_1-1)\rightarrow(0,d_1-1)\rightarrow o$;

$P_4: x=(x_1,x_2)\rightarrow(x_1,x_2+1)\rightarrow(x_1+1,x_2+1)\rightarrow\cdots\rightarrow(d_1-1,x_2+1)\rightarrow(0,x_2+1)\rightarrow(1,x_2+1)\rightarrow(1,x_2)\rightarrow\cdots\rightarrow(1,0)\rightarrow o$.

We can see $|P_3| = |P_2| = d_1 - x_1 + x_2$, $|P_3| = |P_4| = d_1 - x_1 + x_2 + 4 \leq d_1 - e_1 + 2 = e_1 + 3$.

Subcase 2c $x_2 + e_1 \leq x_1 \leq x_2 + e_1 + 1$, where $x_1 \leq d_1 - 1$ and $x_2 \geq 1$.

We construct the two paths $P_i\,(1 \leq i \leq 2)$ as (13), (14) and the other two paths $P_i\,(3 \leq i \leq 4)$ as follows:

$P_1 : x = (x_1, x_2) \to (x_1 - 1, x_2) \to \cdots \to (e_1, x_2) \to (e_1, x_2 + 1) \to \cdots \to e$;

$P_2 : x = (x_1, x_2) \to (x_1, x_2 + 1) \to \cdots \to (x_1, e_1) \to (x_1 - 1, e_1) \to \cdots \to e$;

We can see $|P_1| = |P_2| = d_1 - x_1 + x_2 \leq d_1 - e_1 = e_1 + 1$, $|P_3| = |P_4| = x_1 - x_2 \leq e_1 + 1$.

Case 3 $0 \leq x_1 \leq e_1$ and $e_1 + 1 \leq x_2 \leq d_1 - 1$.

It is similar to Case 2 by vertex-transitivity and symmetry.

Case 4 $e_1 + 1 \leq x_1 \leq d_1 - 1$ and $e_1 + 1 \leq x_2 \leq d_1 - 1$.

Subcase 4a $e_1 + 1 \leq x_1 \leq -x_2 + d_1 + e_1 - 2$, where $e_1 + 1 \leq x_2 \leq d_1 - 3$.

We construct the four paths $P_i\,(1 \leq i \leq 4)$ as follows:

$P_1 : x = (x_1, x_2) \to (x_1 - 1, x_2) \to \cdots \to (e_1, x_2) \to (e_1, x_2 - 1)$
$\to \cdots \to (e_1, e_1 + 1) \to e$; (15)

$P_2 : x = (x_1, x_2) \to (x_1, x_2 - 1) \to \cdots \to (x_1, e_1) \to (x_1 - 1, e_1)$
$\to \cdots \to (e_1 + 1, e_1) \to e$; (16)

$P_3 : x = (x_1, x_2) \to (x_1 + 1, x_2) \to (x_1 + 1, x_2 - 1) \to \cdots \to (x_1 + 1, e_1 - 1) \to (x_1, e_1 - 1) \to (e_1, e_1 - 1) \to e$;

$P_4 : x = (x_1, x_2) \to (x_1, x_2 + 1) \to (x_1 - 1, x_2 + 1) \to \cdots \to (e_1 - 1, x_2 + 1) \to (e_1 - 1, x_2) \to \cdots \to (e_1 - 1, e_1) \to e$.

We can see $|P_1| = |P_2| = x_1 + x_2 - 2e_1$, $|P_3| = |P_4| = x_1 + x_2 - 2e_1 + 4 \leq e_1 + 3$.

Subcase 4b $-x_2 + d_1 + e_1 - 1 \leq x_1 \leq -x_2 + d_1 + e_1$, where $x_1 \geq e_1 + 1$ and $x_2 \geq e_1 + 1$.

We construct the two paths $P_i\,(1 \leq i \leq 2)$ as (15), (16) and the other two paths $P_i\,(3 \leq i \leq 4)$ as follows:

$P_3 : x = (x_1, x_2) \to (x_1 + 1, x_2) \to \cdots \to (d_1 - 1, x_2) \to (0, x_2) \to (0, x_2 + 1) \to \cdots \to (0, d_1 - 1) \to o$;

$P_4 : x = (x_1, x_2) \to (x_1, x_2 + 1) \to \cdots \to (x_1, d_1 - 1) \to (x_1, 0) \to (x_1 + 1, 0) \to \cdots \to (d_1 - 1, 0) \to o$.

We can see $|P_1| = |P_2| = x_1 + x_2 - 2e_1 \leq d_1 - e_1 = e_1 + 1$, $|P_3| = |P_4| = 2d_1 - x_1 - x_2 \leq d_1 - e_1 + 1 = e_1 + 2$.

From the above cases, we know there exist four internally disjoint (x, S)-paths of length at most $e_1 + 3$ in G. So S can $(d, 4)$-dominate any vertex x in $G - S$, S is a $(d, 4)$-dominating set of G.

Next we assume d_1 and d_2 are even, and so we only need to consider the case of $0 \leq x_1 \leq e_1$ and $0 \leq x_2 \leq e_1$ in the same way as Case 1 by vertex-transitivity and symmetry. We easily see S can also $(d, 4)$-dominate any vertex in $V(G) - S$.

So we have $\gamma_{d,4}(G) \leq 2$. □

4 Conclusion and Problems

For the undirected toroidal mesh $C(d_1, d_2, \ldots, d_n)$, we prove that $\gamma_{d,4}(C(d_1, 3)) = 2$ if and only if $d_4(G) - e_1 \leq d < d_4(G)$ for $d_1 \geq 5$, $\gamma_{d,4}(C(d_1, 4)) = 2$ if $d_4(G) - (2e_1 - \lfloor \frac{d_1 + e_1}{2} \rfloor) \leq d < d_4(G)$

for $d_1 \geq 24$, and $\gamma_{d,4}(C(d_1, d_2)) = 2$ if $d_4(G) - (e_1 - 2) \leq d < d_4(G)$ for $d_1 = d_2 \geq 14$. To determine the values of $\gamma_{d,4}(C(d_1, 4))$ for $d < d_4(G) - (2e_1 - \lfloor \frac{d_1+e_1}{2} \rfloor)$, $\gamma_{d,4}(C(d_1, d_2))$ for $d < d_4(G) - (e_1 - 2)$ and $\gamma_{d,2n}(C(d_1, d_2, \ldots, d_n))$ for $d < \mathrm{diam}(C(d_1, d_2, \ldots, d_n))$ are worth studying further.

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