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On the bounds of feedback numbers of (n,k)-star graphs ☆

Jian Wang [a], Xirong Xu [a], Dejun Zhu [a], Liqing Gao [a], Jun-Ming Xu [b,*]

[a] *School of Computer Science and Technology, Dalian University of Technology, Dalian, 116024, China*
[b] *School of Mathematical Sciences, University of Science and Technology of China, Wentsun Wu Key Laboratory of CAS, Hefei, Anhui, 230026, China*

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ABSTRACT

The feedback number of a graph G is the minimum number of vertices whose removal from G results in an acyclic subgraph. We use $f(n,k)$ to denote the feedback number of the (n,k)-star graph $S_{n,k}$ and $p(n,k)$ the number of k-permutations of an n-element set. This paper proves that

$$p(n,k)-2(k-1)!\binom{n}{k-1}\leqslant f(n,k)\leqslant p(n,k)-2(k-1)!\sum_{i=1}^{\theta}\binom{n-2i+1}{k-i},$$

where $\theta=\min\{k-1,n-k+1\}$.

1. Introduction

Let $G=(V,E)$ be a graph without loops and multiple edges, with vertex-set $V(G)$ and edge-set $E(G)$. A subset $F\subset V(G)$ is called *a feedback set* if the subgraph $G-F$ is acyclic, that is, if $G-F$ is a forest. The minimum cardinality of a feedback set is called *the feedback number* of G.

Determining the feedback number is quite difficult even for some well-known graphs, such as the hypercube [3]. In fact, the problem determining feedback number for a graph was proved to be NP-complete by Karp in 1972 (see the 7th of 21 problems in [4]). However, some upper bounds of feedback numbers for some well-known graphs have been established (see, for example, [1], the recent article [8] and references cited therein). In particular, Wang et al. [5] gave an upper bound of the feedback number for the n-star graph S_n, which has $n!$ vertices. There is a large gap between $n!$ and $(n+1)!$ if S_n is extended to S_{n+1}. To compensate for this shortcoming, Chiang and Chen [2] proposed the (n,k)-star graph $S_{n,k}$, where $S_{n,n-1}=S_n$ and $S_{n,1}=K_n$, the complete graph on n vertices. Let $f(n,k)$ denote the feedback number of $S_{n,k}$. This paper proves that

$$p(n,k)-2(k-1)!\binom{n}{k-1}\leqslant f(n,k)\leqslant p(n,k)-2(k-1)!\sum_{i=1}^{\theta}\binom{n-2i+1}{k-i},$$

where $\theta=\min\{k-1,n-k+1\}$ and $p(n,k)$ is the number of k-permutations of an n-element set.

The proof of the result is in Section 3. In Section 2, we give the definition of the (n,k)-star graph $S_{n,k}$, several lemmas and construct a feedback set of $S_{n,k}$.

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* Corresponding author.
E-mail address: xujm@ustc.edu.cn (J.-M. Xu).

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2. Some definitions and lemmas

Throughout this paper, we follow Xu [6] for graph-theoretical terminology and notation not defined here. Let $G=(V,E)$ be a graph. For two vertices x and y, $xy \in E(G)$ implies that x and y are adjacent, that is, there is an edge connecting x and y in G. For a subset $S \subset V(G)$, $N_G(S)$ denotes the set of neighbors of S, namely, $N_G(S)=\{x \in V(G-S)\colon\ xy \in E(G), y \in S\}$, and $G[S]$ denotes the subgraph induced by S. The set S is called *cycle-free* or *acyclic* if $G[S]$ is acyclic, that is, $G[S]$ has no cycles. Let I_n be the set $\{1,2,3,\ldots,n\}$, and let $P(n,k)$ be the set of k-permutations on I_n for $1 \leqslant k < n$, that is, $P(n,k)=\{x_1x_2\ldots x_k \mid x_i \in I_n,\ x_i \neq x_j,\ 1 \leqslant i \neq j \leqslant k\}$. Let $p(n,k)=|P(n,k)|$. Then $p(n,k)=n!/(n-k)!$.

Definition 2.1. The (n,k)-star graph, denoted by $S_{n,k}$, is specified by two integers n and k where $1 \leqslant k < n$. The vertex-set of $S_{n,k}$ is $P(n,k)$. The adjacency is defined as follows: a vertex $x_1x_2\ldots x_i\ldots x_k$ is adjacent to a vertex

(1) $x_ix_2\cdots x_{i-1}x_1x_{i+1}\cdots x_k$, where $2 \leqslant i \leqslant k$ (swap x_1 with x_i).
(2) $\alpha x_2x_3\cdots x_k$, where $\alpha \in I_n-\{x_i \mid 1 \leqslant i \leqslant k\}$ (replace x_1 by α).

Fig. 1 shows a $(4,2)$-star graph $S_{4,2}$.

The edges of type (1) are referred to as *i-edges* ($2 \leqslant i \leqslant k$), and the edges of type (2) are referred to as *1-edges*. The vertices of type (1) are referred to as *swap-adjacent vertices*, and the vertices of type (2) are referred to as *unswap-adjacent vertices*. We also call i-edge as *swap-edge* ($2 \leqslant i \leqslant k$), and call 1-edge as *unswap-edge*. Clearly, every vertex in $S_{n,k}$ has $k-1$ swap-adjacent vertices and $n-k$ unswap-adjacent vertices. Usually, if $x=x_1x_2\ldots x_k$ is a vertex in $S_{n,k}$, we call x_i the *i-th bit* for each $i=1,2,\ldots,k$.

It has been shown by Chiang and Chen [2] that $S_{n,k}$ is an $(n-1)$-regular $(n-1)$-connected vertex-transitive graph. In order to simplify our arguments, we will partition $P(n,k)$ into some subsets according an equivalence relation on $P(n,k)$.

Definition 2.2. A relation R on the set $P(n,k)$ is defined as follows. For any two elements $x=x_1x_2\ldots x_k$ and $y=y_1y_2\ldots y_k$ in $P(n,k)$, we have

$$xRy \quad \Leftrightarrow \quad x_i=y_i \quad \text{for each } i=2,3,\ldots,k.$$

It is a simple exercise to verify that the relation R defined in Definition 2.2 is an equivalence relation on $P(n,k)$.

For each $u \in P(n,k)$, the equivalence class of u is the set

$$[u]=\{x \mid xRu,\ x \in P(n,k)\}.$$

For example, see Fig. 1, if $u=14$ is a vertex in $S_{4,2}$, then $[u]=\{14,24,34\}$. So, all equivalence classes form a partition of $P(n,k)$, denoted by

$$\mathscr{P}(n,k)=\{[u] \mid u \in P(n,k)\}.$$

For example, see Fig. 1,

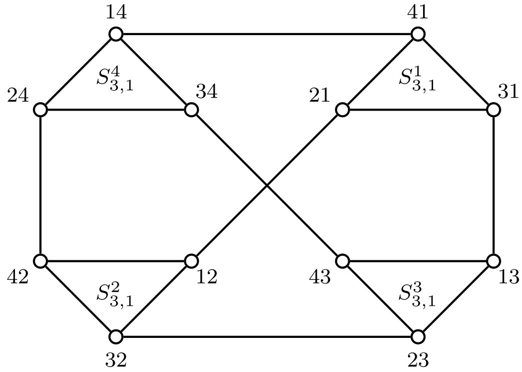

Fig. 1. $(4,2)$-star graph $S_{4,2}$.

$$\mathscr{P}(4,2) = \{\{21,31,41\},\{12,32,42\},\{13,23,43\},\{14,24,34\}\}.$$

Clearly, $|\mathscr{P}(n,k)|=p(n,k-1)$. It is also clear that, for any two distinct elements x and y in $P(n,k)$, if they are in different equivalence classes, say $x \in [u]$ and $y \in [v]$, then there is a swap-edge between x and y. On the other hand, if they are in the same equivalence class, say $x,y \in [u]$, then there is an unswap-edge between x and y, which implies that the subgraph of $S_{n,k}$ induced by $[u]$ is isomorphic to a complete graph K_{n-k+1}. Since $S_{n,1}$ is isomorphic to K_n, whose feedback number is $n-2$, we can assume $k \geqslant 2$ in the following discussion.

Definition 2.3. Define a mapping

$$\sigma : P(n,k-1) \to \mathscr{P}(n,k)$$

subject to, for any $\alpha=x_2x_3\ldots x_k \in P(n,k-1)$,

$$\sigma(\alpha)=\{x\alpha \mid x \in I_n \setminus \{x_2,x_3,\ldots,x_k\}\}.$$

Lemma 2.1. *The mapping σ is a bijection from $P(n,k-1)$ to $\mathscr{P}(n,k)$.*

Proof. On the one hand, for any two distinct elements $\alpha=x_2x_3\ldots x_k$ and $\beta=y_2y_3\ldots y_k$ in $P(n,k-1)$, we have $\sigma(\alpha)=\{x\alpha \mid x \in I_n \setminus \{x_2,x_3,\ldots,x_k\}\}$ and $\sigma(\beta)=\{y\beta \mid y \in I_n \setminus \{y_2,y_3,\ldots,y_k\}\}$. Clearly, $\sigma(\alpha) \cap \sigma(\beta)=\emptyset$, which means σ is an injection.

On the other hand, for any $[u] \in \mathscr{P}(n,k)$, say $u=x_1x_2\ldots x_k$, let us set $\alpha=x_2x_3\ldots x_k$. Then $\alpha \in P(n,k-1)$ and $\sigma(\alpha)=\{x\alpha \mid x \in I_n \setminus \{x_2,x_3,\ldots,x_k\}\}=[u]$. Then σ is a surjection.

It follows that σ is a bijection from $P(n,k-1)$ to $\mathscr{P}(n,k)$. The lemma follows. $\square$

By Lemma 2.1, $\mathscr{P}(n,k)$ can be viewed as

$$\mathscr{P}(n,k)=\{\sigma(\alpha) \mid \alpha \in P(n,k-1)\},$$

which is a partition of vertex-set $V(S_{n,k})$, that is, $V(S_{n,k})=\bigcup_{\alpha \in P(n,k-1)} \sigma(\alpha)$, and $\sigma(\alpha_i) \cap \sigma(\alpha_j)=\emptyset$ for any two distinct $\alpha_i,\alpha_j \in P(n,k-1)$.

Since σ is a bijection, $P(n,k)$ can be decomposed into $p(n,k-1)$ subsets by $\sigma(\alpha)$, each of them induces a subgraph of $S_{n,k}$, denoted by $S^{\alpha}_{n,k-1}$, which is isomorphic to K_{n-k+1}, for each $\alpha \in P(n,k-1)$, see Fig. 1 for $S_{4,2}$.

A subset $S \subseteq V(S_{n,k})$ is called an *acyclic set* of $S_{n,k}$ if the subgraph of $S_{n,k}$ induced by S contains no cycle. A feedback set F of $S_{n,k}$ is called to be *minimum* if $|F| = f(n,k)$. It is clear that S is an acyclic set of $S_{n,k}$ if and only if $V(S_{n,k} - S)$ is a feedback set of $S_{n,k}$. Let $\mathscr{S}_{n,k}$ denote the set of all acyclic sets of $S_{n,k}$. Then

$$f(n,k) = \left|V(S_{n,k})\right| - \max\left\{|S| \,\middle|\, S \in \mathscr{S}_{n,k}\right\}.$$

Let $S \in \mathscr{S}_{n,k}$. Since the feedback number of K_{n-k+1} is $n-k-1$, an acyclic subgraph of K_{n-k+1} has at most two vertices. In other words, for each $\alpha \in P(n,k-1)$, at most two vertices in $S^{\alpha}_{n,k-1}$ ($\cong K_{n-k+1}$) are contained in S. This fact implies $|S| \leqslant 2p(n,k-1)$, and so $|V(S_{n,k})| - |S| \geqslant |V(S_{n,k})| - 2p(n,k-1)$. Thus, we obtain a lower bound of $f(n,k)$ immediately

$$\begin{aligned} f(n,k) &= \left|V(S_{n,k})\right| - \max\left\{|S| \,\middle|\, S \in \mathscr{S}_{n,k}\right\} \\ &\geqslant \left|V(S_{n,k})\right| - 2p(n,k-1). \end{aligned}$$

We state this lower bound as the following lemma.

Lemma 2.2. $f(n,k) \geqslant |V(S_{n,k})| - 2p(n,k-1)$.

In the following discussion, our aim is to give an upper bound of $f(n,k)$. Thus, to obtain an upper bound of $f(n,k)$, a usual way is to construct an acyclic set of $S_{n,k}$.

We attempt to construct an acyclic set of $S_{n,k}$ based on a simple observation. The set of vertices in $P(n,k)$ whose first position is exactly one less or more than other $k-1$ positions is certainly an acyclic set. For example, in $S_{4,2}$, the set $S = \{12, 23, 34, 21, 32, 43\}$ is an acyclic set, see Fig. 1. Such a chosen acyclic set S is maybe small. Thus, we add some vertices to S from $P(n,k-1)$ chosen according to the above way.

To state our way in detail, we need some notations.

For each $\alpha = x_1 \cdots x_{k-1} \in P(n,k-1)$, since $x_i \in I_n$ for each $i = 1,2,\ldots,k$, we can assign it a sequence $\gamma_1(\alpha)\gamma_2(\alpha)\cdots\gamma_{k-1}(\alpha)$ satisfying $\gamma_1(\alpha) < \gamma_2(\alpha) < \cdots < \gamma_{k-1}(\alpha)$. For example, if $\alpha = 527 \in P(n,3)$, then $\gamma_1(\alpha) = 2$, $\gamma_2(\alpha) = 5$ and $\gamma_3(\alpha) = 7$.

Let $\theta = \min\{k-1, n-k+1\}$. By the hypothesis of $2 \leqslant k \leqslant n-1$, we have $\theta \geqslant 1$. For each $m \in I_\theta = \{1,2,\ldots,\theta\}$, let

$$\begin{aligned} &X_1 = \left\{\alpha \in P(n,k-1) \,\middle|\, \gamma_1(\alpha) > 1\right\}, \\ &Y_{k-1} = \left\{\alpha \in P(n,k-1) \,\middle|\, \gamma_{k-1}(\alpha) < n\right\}, \\ &X_m \\ &\quad = \left\{\alpha \in P(n,k-1) \,\middle|\, \begin{array}{l} \gamma_i(\alpha) = 2i-1 \text{ for each } i \in I_{m-1} \\ \gamma_m(\alpha) > 2m-1 \end{array}\right\} \\ &\quad \text{for } m > 1, \\ &Y_{k-m} \\ &\quad = \left\{\alpha \in P(n,k-1) \,\middle|\, \begin{array}{l} \gamma_{k-i}(\alpha) = n-2i+2,\ i \in I_{m-1} \\ \gamma_{k-m}(\alpha) < n-2m+2 \end{array}\right\} \\ &\quad \text{for } m > 1. \end{aligned} \tag{2.1}$$

By the definition of X_m, if $n \geqslant 2k-2$, then $\theta = k-1$ and $n \geqslant \gamma_\theta(\alpha) > 2(k-1)-1 = 2k-3$; if $n \leqslant 2k-3$, then $\theta = n-k+1 \leqslant k-2$, since $2\theta - 1 < \gamma_\theta(\alpha) < \gamma_{\theta+1}(\alpha) < \cdots < \gamma_{k-1}(\alpha) \leqslant n$, we have $n \geqslant 2\theta - 1 + (k-\theta) = \theta + k - 1$, that is, $\gamma_\theta(\alpha) = 2\theta$, $\gamma_{\theta+i}(\alpha) = 2\theta + i$ for $i = 1,\ldots,k-1-\theta$ and $\gamma_{k-1}(\alpha) = n$. Thus, X_m is well-defined for any $m \in I_\theta$ when $\theta + k - 1 \leqslant n \leqslant 2k-3$.

Similarly, Y_{k-m} is also well-defined for any $m \in I_\theta$.

For example, in $S_{6,4}$, for $\alpha \in P(6,3)$, we have $\theta = 3$, and

$$\begin{cases} X_1 = \left\{\alpha \in P(6,3) \,\middle|\, \gamma_1(\alpha) > 1\right\}, \\ Y_3 = \left\{\alpha \in P(6,3) \,\middle|\, \gamma_3(\alpha) < 6\right\}, \\ X_2 = \left\{\alpha \in P(6,3) \,\middle|\, \gamma_1(\alpha) = 1,\ \gamma_2(\alpha) > 3\right\}, \\ Y_2 = \left\{\alpha \in P(6,3) \,\middle|\, \gamma_3(\alpha) = 6,\ \gamma_2(\alpha) < 4\right\}, \\ X_3 = \left\{\alpha \in P(6,3) \,\middle|\, \gamma_1(\alpha) = 1,\ \gamma_2(\alpha) = 3,\ \gamma_3(\alpha) > 5\right\}, \\ Y_1 = \left\{\alpha \in P(6,3) \,\middle|\, \gamma_3(\alpha) = 6,\ \gamma_2(\alpha) = 4,\ \gamma_1(\alpha) < 2\right\}. \end{cases}$$

In other words, the set X_m defined in (2.1) is such a set of $(k-1)$-elements in $P(n,k-1)$ which contains $(m-1)$ elements $1,3,\ldots,2m-3$ but m elements $2,4,\ldots,2m-2,2m-1$ in I_n. The former have $p(k-1,m-1)$ choices and the latter have $p(n-2m+1,k-m)$ choices from $P(n,k-1)$. Similarly, the set Y_{k-m} defined in (2.1) is such a set of $(k-1)$-elements in $P(n,k-1)$ which contains $(m-1)$ elements $n,n-2,\ldots,n-2m$ but m elements $n-1,n-3,\ldots,n-2m+3,n-2m+2$ in I_n. The former have $p(k-1,m-1)$ choices and the latter have $p(n-2m+1,k-m)$ choices from $P(n,k-1)$. Thus, for any $m \in I_\theta$,

$$\begin{aligned} |X_m| &= |Y_{k-m}| \\ &= p(k-1,m-1)p(n-2m+1,k-m). \end{aligned} \tag{2.2}$$

Let

$$\begin{aligned} X_0 &= P(n,k-1) - \bigcup_{m=1}^{\theta} X_m, \\ Y_k &= P(n,k-1) - \bigcup_{m=1}^{\theta} Y_{k-m}. \end{aligned} \tag{2.3}$$

It is easy to see that both $\{X_0, X_1, X_2, \ldots, X_\theta\}$ and $\{Y_k, Y_{k-1}, Y_{k-2}, \ldots, Y_{k-\theta}\}$ are partitions of $P(n,k-1)$. Furthermore, we have the following conclusion.

Lemma 2.3. *$\{X_i \cap Y_{k-j} \mid i,j \in \{0,1,\ldots,\theta\}\}$ is a partition of $P(n,k-1)$.*

Proof. We first show that, for any $i,j,s,t \in \{0,1,\ldots,\theta\}$, either $(X_i \cap Y_{k-j}) \cap (X_s \cap Y_{k-t}) = \varnothing$ or $X_i \cap Y_{k-j} = X_s \cap Y_{k-t}$.

Noting that both $\{X_0, X_1, X_2, \ldots, X_\theta\}$ and $\{Y_k, Y_{k-1}, Y_{k-2}, \ldots, Y_{k-\theta}\}$ are partitions of $P(n,k-1)$, we prove this conclusion according to the relationship among i,j,s,t.

If $s = i$ and $t = j$, then $X_i \cap Y_{k-j} = X_s \cap Y_{k-t}$.

If $s = i$ and $t \neq j$, then

$$\begin{aligned} (X_i \cap Y_{k-j}) \cap (X_s \cap Y_{k-t}) &= (X_i \cap X_s) \cap (Y_{k-j} \cap Y_{k-t}) \\ &= X_i \cap \varnothing = \varnothing. \end{aligned}$$

If $s \neq i$ and $t = j$, then

$$(X_i \cap Y_{k-j}) \cap (X_s \cap Y_{k-t}) = (X_i \cap X_s) \cap (Y_{k-j} \cap Y_{k-t}) = \varnothing \cap Y_{k-j} = \varnothing.$$

If $s \neq i$ and $t \neq j$, then

$$(X_i \cap Y_{k-j}) \cap (X_s \cap Y_{k-t}) = (X_i \cap X_s) \cap (Y_{k-j} \cap Y_{k-t}) = \varnothing \cap \varnothing = \varnothing.$$

It is clear that

$$\bigcup_{i=0}^{\theta}\bigcup_{j=0}^{\theta}(X_i \cap Y_{k-j}) = \bigcup_{i=0}^{\theta}\left(X_i \cap \bigcup_{j=0}^{\theta} Y_{k-j}\right) = \bigcup_{i=0}^{\theta} X_i = P(n,k-1).$$

Thus, the lemma follows. □

For each $m \in I_\theta$, let

$$S_m = \{x\alpha \in P(n,k) \mid \alpha \in X_m, x = \gamma_m(\alpha) - 1\},$$
$$T_{k-m} = \{x\alpha \in P(n,k) \mid \alpha \in Y_{k-m}, x = \gamma_{k-m}(\alpha) + 1\}. \quad (2.4)$$

Clearly, $S_i \cap S_j = \varnothing$, $T_i \cap T_j = \varnothing$ and $S_i \cap T_j = \varnothing$ for any $i, j \in I_\theta$. Thus,

$$|S_m| = |X_m| = |Y_{k-m}| = |T_{k-m}| \quad \text{for each } m \in I_\theta. \quad (2.5)$$

Let

$$S = S_1 \cup S_2 \cup \cdots \cup S_\theta,$$
$$T = T_{k-1} \cup T_{k-2} \cup \cdots \cup T_{k-\theta}. \quad (2.6)$$

From the definitions of S_m and T_{k-m}, we can easily find the following proposition.

Proposition 2.1. *For any $\alpha \in P(n,k-1)$,*
$\sigma(\alpha) \cap S = \varnothing$ if $\alpha \in X_0$ and $|\sigma(\alpha) \cap S| = 1$ if $\alpha \in X_i$ for any $i \in I_\theta$;
$\sigma(\alpha) \cap T = \varnothing$ if $\alpha \in Y_k$ and $|\sigma(\alpha) \cap T| = 1$ if $\alpha \in Y_{k-j}$ for any $j \in I_\theta$.

From Proposition 2.1, we have the following proposition immediately.

Proposition 2.2. *For any $\alpha \in P(n,k-1)$,*
$|\sigma(\alpha) \cap (S \cup T)| = 2$ if $\alpha \in X_i \cap Y_{k-j}$ for any $i, j \in I_\theta$;
$|\sigma(\alpha) \cap (S \cup T)| = 1$ if $\alpha \in X_i \cap Y_k$ for any $i \in I_\theta$;
$|\sigma(\alpha) \cap (S \cup T)| = 1$ if $\alpha \in X_0 \cap Y_{k-j}$ for any $j \in I_\theta$;
$|\sigma(\alpha) \cap (S \cup T)| = 0$ if $\alpha \in X_0 \cap Y_k$.

Lemma 2.4. *Let $\alpha \in P(n,k-1)$.*
If $\alpha \in X_0$, then either

$$\gamma_i(\alpha) = 2i - 1 \quad \text{for each } i = 1, 2, \ldots, k-1 \quad (2.7)$$

or there exists some $\ell \in I_n$ such that

$$\gamma_i(\alpha) = 2i - 1 \quad \text{for each } i = 1, 2, \ldots, \ell,$$
$$\gamma_{\ell+1}(\alpha) = 2\ell. \quad (2.8)$$

If $\alpha \in Y_k$, then either

$$\gamma_{k-j}(\alpha) = n - 2j + 2 \quad \text{for each } j = 1, 2, \ldots, k-1 \quad (2.9)$$

or there exists some $\ell \in I_n$ such that

$$\gamma_{k-j}(\alpha) = n - 2j + 2 \quad \text{for each } j = 1, 2, \ldots, \ell,$$
$$\gamma_{k-\ell-1}(\alpha) = n - 2\ell + 1. \quad (2.10)$$

Proof. If $\alpha \in X_0$, then $\alpha \notin X_1$, which implies $\gamma_1(\alpha) = 1$, and $\alpha \notin X_2$, which implies $\gamma_2(\alpha) \leqslant 3$.

If $\gamma_2(\alpha) = 2$, we get $\ell = 1$. If $\gamma_2(\alpha) = 3$ and $\alpha \notin X_3$, then $\gamma_3(\alpha) \leqslant 5$.

If $\gamma_3(\alpha) = 4$, we get $\ell = 2$. Else, if $\gamma_3(\alpha) = 5$ and $\alpha \notin X_4$, then $\gamma_4(\alpha) \leqslant 7$.

In general, we have $\gamma_i(\alpha) = 2i - 1$ for each $i = 1, 2, \ldots, k-1$, or we can find an ℓ such that $\gamma_i(\alpha) = 2i - 1$ for each $i = 1, 2, \ldots, \ell$ and $\gamma_{\ell+1}(\alpha) = 2\ell$.

In the same argument, we can prove that the conclusion is true for $\alpha \in Y_k$. □

Lemma 2.5. *The set $S \cup T$ is an acyclic set in $S_{n,k}$.*

Proof. Let H be the subgraph of $S_{n,k}$ induced by $S \cup T$ and let u be any vertex in H. In order to prove the lemma, we only need to prove that u is not in a cycle in H. Since $u \in S \cup T$, by (2.6), there exist i and j in I_θ such that $u \in S_i \cup T_{k-j}$. We can, without loss of generality, assume $u \in S_i$. We want to prove that u is not in a cycle in H.

By the definition of S_i in (2.4), there exists $\alpha \in X_i$ such that $u = (\gamma_i(\alpha) - 1)\alpha$, where $\gamma_i(\alpha) > 2i - 1$. By Lemma 2.3, we consider two cases depending on $\alpha \in X_i \cap Y_k$ or $\alpha \in X_i \cap Y_{k-j}$.

Case 1. $\alpha \in X_i \cap Y_k$.

Since $\alpha \in X_i$, by Proposition 2.2, we have $\sigma(\alpha) \cap (S \cup T) = \{u\}$. Thus, $\sigma(\alpha) \cap S_i = \{u\}$, $\sigma(\alpha) \cap S_j = \varnothing$ and $\sigma(\alpha) \cap T_{k-j} = \varnothing$ for any $j \neq i$. So, the neighbors of u in H are all swap-adjacent vertices. By Lemma 2.4, $\alpha \in Y_k$ satisfies (2.9) or (2.10).

If $\alpha \in Y_k$ satisfies (2.9) then, when $j = k-1$, $\gamma_1(\alpha) = n - 2k + 4$. Since $\alpha \in X_i$, we consider two cases depending on $i = 1$ or $i > 1$ by (2.1).

(a) If $i > 1$, then $\gamma_1(\alpha) = 1$ by (2.1). Thus, we have $n - 2k + 4 = 1$, that is, $n = 2k - 3$, and so $\theta = \min\{k-1, n-k+1\} = k-2$. This fact implies that α consists of all $k-1$ odd integers in I_{2k-3}. However, $\gamma_i(\alpha) \neq 2i - 1$ in $\alpha \in X_i$ and $1 \leqslant i \leqslant \theta = k-2$, a contradiction.

(b) If $i = 1$, then $\gamma_1(\alpha) > 1$ by (2.1). Thus, $n - 2k + 4 = \gamma_1(\alpha) > 1$, and $u = (\gamma_1(\alpha) - 1)\alpha = (n - 2k + 3)\alpha$. Since $\gamma_{k-j}(\alpha) = n - 2j + 2$ $(j = 1, 2, \ldots, k-1)$ by (2.1), it is clear that u is an isolated vertex in $G[S \cup T]$.

If $\alpha \in Y_k$ satisfies (2.10), then the degree of u in H is at most one. Moreover, if the degree of u is one then its neighbor is a swap-adjacent vertex obtained from u by replacing the first bit of u with $\gamma_{k-1-\ell}(\alpha)$. Thus, u is not in a cycle in H.

Case 2. $\alpha \in X_i \cap Y_{k-j}$.

In this case, $|\sigma(\alpha) \cap (S \cup T)| = 2$ by Proposition 2.2. Then u has the only unswap-adjacent vertex, say, v in

$\sigma(\alpha) \cap T_{k-j}$. Since $u = (\gamma_i(\alpha) - 1)\alpha$, $v = (\gamma_{k-j}(\alpha) + 1)\alpha$ by (2.4). Thus, $k - j \geqslant i$ since $\alpha \in X_i \cap Y_{k-j} \neq \varnothing$. So, we have that

$$\begin{cases} \gamma_1(\alpha) = 1, \\ \dots \\ \gamma_{i-1}(\alpha) = 2i - 1, \\ \gamma_i(\alpha) > 2i - 1, \\ \dots \\ \gamma_{k-j}(\alpha) < n - 2j + 2, \\ \gamma_{k-j+1}(\alpha) = n - 2j + 4, \\ \dots \\ \gamma_{k-1}(\alpha) = n. \end{cases}$$

We want to prove that u is not in a cycle in H. To understand our proof, let us see an example first. For example, in $S_{12,7}$, $u = 56\,378(12)1$, then $\alpha = 6378(12)1 \in X_3 \cap Y_5$, where $k = 7$, $i = 3$, $j = 2$, and

$$\begin{cases} \gamma_1(\alpha) = 1, \\ \gamma_2(\alpha) = 3, \\ \gamma_3(\alpha) = 6 > 5, \\ \gamma_4(\alpha) = 7, \\ \gamma_5(\alpha) = 8 < 10, \\ \gamma_6(\alpha) = 12. \end{cases}$$

The vertex u has the only unswap-adjacent vertex $v = 96\,378(12)1$ and the only swap-adjacent vertex $v_1^- = 86\,375(12)1$ in H, where v_1^- is obtained from u by swapping the first bit (5) with 8. The vertex v_1^- has no unswap-adjacent vertex and has the only swap-adjacent vertex $u_1^+ = 69\,378(12)1$ in H obtained from v by swapping the first bit (9) with 6. The vertex u_1^+ has the only unswap-adjacent vertex $v_1^+ = (10)9378(12)1$ in H. The vertex v_1^+ has the only swap-adjacent vertex $u_2^+ = 793(10)8(12)1$ in H obtained from v_1^+ by swapping the first bit (10) with 7. The vertex u_2^+ has no unswap-adjacent vertex. Thus, the subgraph of H induced by $\{v_1^-, u, v, u_1^+, v_1^+, u_2^+\}$ is a path $(v_1^-, u, v, u_1^+, v_1^+, u_2^+)$ in H whose edges are alternately in swap-edges and unswap-edges starting and ending with swap-edges.

In general, there are two possible cases: either $\gamma_{k-j}(\alpha) = \gamma_{k-j-1}(\alpha) + 1$ or $\gamma_i(\alpha) = \gamma_{i+1}(\alpha) - 1$.

If $\gamma_{k-j}(\alpha) = \gamma_{k-j-1}(\alpha) + 1$, then u has the only swap-adjacent vertex, say v_1^-, obtained from u by swapping the first bit with $\gamma_{k-j}(\alpha)$, and denoted by $v_1^- = \gamma_{k-j}(\alpha)\alpha_1^-$. If $\gamma_i(\alpha_1^-) > 2i - 1$, then v_1^- has the only unswap-adjacent vertex $u_1^- = (\gamma_i(\alpha_1^-) - 1)\alpha_1^-$; else v_1^- has no unswap-adjacent vertex. Similarly, if $\gamma_{k-j}(\alpha_1^-) = \gamma_{k-j-1}(\alpha_1^-) + 1$, then u_1^- has the only swap-adjacent vertex, say v_2^-, obtained from u_1^- by swapping the first bit with $\gamma_{k-j}(\alpha_1^-)$, and denoted by $v_2^- = \gamma_{k-j}(\alpha_1^-)\alpha_2^-$.

Continue this process. Since $\gamma_i(\alpha_\ell^-) = \gamma_i(\alpha_{\ell-1}^-) - 1$ (where $\alpha_0^- = \alpha$), this process will stop in finite steps. Until either $\gamma_{k-j}(\alpha_\ell^-) > \gamma_{k-j-1}(\alpha_\ell^-) + 1$, which implies that u_ℓ^- has no swap-adjacent vertex, or $\gamma_i(\alpha_\ell^-) = 2i - 1$, which implies that v_ℓ^- has no unswap-adjacent vertex. In other words, this process must stop when it meets with a vertex of degree one. So, we can get the vertex-sequence $(u, v_1^-, u_1^-, v_2^-, u_2^-, \dots, v_\ell^-, u_\ell^-)$, whose induced subgraph of H is not a cycle, but a path. Thus, u is not in a cycle in H.

Similarly, if $\gamma_i(\alpha) = \gamma_{i+1}(\alpha) - 1$, then v has the only swap-adjacent vertex, say u_1^+, obtained from v by swapping the first bit with $\gamma_i(\alpha)$, and denoted by $u_1^+ = \gamma_i(\alpha)\alpha_1^+$. If $\gamma_{k-j}(\alpha_1^+) < n - 2j + 2$, then u_1^+ has the only unswap-adjacent vertex v_1^+, which is $(\gamma_{k-j}(\alpha_1^+) + 1)\alpha_1^+$. Similarly, if $\gamma_i(\alpha_1^+) = \gamma_{i+1}(\alpha_1^+) - 1$, then v_1^+ has the only swap-adjacent vertex, say u_2^+, obtained from v_1^+ by swapping the first bit with $\gamma_i(\alpha_1^+)$, and denoted by $u_2^+ = \gamma_i(\alpha_1^+)\alpha_2^+$.

Continue this process. Since $\gamma_{k-j}(\alpha_\ell^+) = \gamma_{k-j}(\alpha_{\ell-1}^+) + 1$ (where $\alpha_0^+ = \alpha$), this process will stop in finite steps. Until either $\gamma_i(\alpha_\ell^+) < \gamma_{i+1}(\alpha_\ell^+) - 1$, which implies that v_ℓ^+ has no swap-adjacent vertex, or $\gamma_{k-j}(\alpha_\ell^+) = n - 2j + 2$, which implies that u_ℓ^+ has no unswap-adjacent vertex. In other words, this process must stop when it meets with a vertex of degree one. So, we can get the vertex-sequence $(v, u_1^+, v_1^+, u_2^+, v_2^+, \dots, u_\ell^+, v_\ell^+)$, whose induced subgraph of H is not a cycle, but a path. Thus, v is not in a cycle in H.

Thus, we proved that u is not in a cycle in H, and the lemma follows. □

By Lemma 2.5, we immediately have the following conclusion.

Lemma 2.6. *The set $V(S_{n,k}) - (S \cup T)$ is a feedback set of $S_{n,k}$.*

3. Proofs of main results

Theorem 3.1. *For each k with $2 \leqslant k < n$ and $\theta = \min\{k-1, n-k+1\}$,*

$$p(n,k) - 2(k-1)!\binom{n}{k-1} \leqslant f(n,k) \leqslant p(n,k) - 2(k-1)!\sum_{i=0}^{\theta-1}\binom{n-2i-1}{k-i-1}.$$

Proof. Since $S \cap T = \varnothing$, by (2.5) and (2.2), we have that

$$|S \cup T| = 2\sum_{i=1}^{\theta} p(k-1, i-1)p(n-2i+1, k-i) = 2\sum_{i=1}^{\theta} \frac{(k-1)!}{(k-i)!}\frac{(n-2i+1)!}{(n-k-i+1)!} = 2(k-1)!\sum_{i=1}^{\theta}\binom{n-2i+1}{k-i}.$$

Thus, by Lemma 2.6, we have

$$f(n,k) \leqslant \left|V(S_{n,k})\right| - |S \cup T|$$
$$= p(n,k) - 2(k-1)! \sum_{i=1}^{\theta} \binom{n-2i+1}{k-i}.$$

Combining this with Lemma 2.2, the theorem holds. □

Remarks. The lower bound given in Theorem 3.1 can be reachable in the following senses. When $k = 2$ and $k = 3$, these lower bounds are $n(n-3)$ and $n(n-1)(n-4)$, respectively. Very recently, Xu et al. [7] have showed that $f(n,2) = n(n-3)$ and $f(n,3) = n(n-1)(n-4)$.

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