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Note on Conjectures of Bondage Numbers of Planar Graphs

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Abstract

The bondage number of a graph G is the cardinality of a smallest set of edges whose removal results in a graph with domination number larger than that of G. The bondage number measures to some extent the robustness of a network with respect to link failure. This note mainly considers some conjectures on the bondage number of a planar graph, and shows limitations of known methods and presents some new approaches to the conjectures by investigating the effects of edge deletion and contraction on the bondage number.

Mathematics Subject Classification: 05C69

Keywords: domination, bondage number, planar graphs

1 Introduction

After 30 years' development, the domination has become one of the major areas in graph theory. The reason of the rapid growth of this area may be the various applications of domination to the real-world problems. Considering possible link faults in the real-world, Fink *et al.* [3] introduced the concept of the bondage number in 1990, which measures to some extend the robustness of an interconnection network with respect to link failures. Since then, the bondage number has attracted much attention of the researchers. Recently, several authors have focused on planar graphs, obtained lots of results, and also left many conjectures and problems. In this paper, we present some approaches to them.

We mainly consider an undirected simple graph $G = (V, E)$ with vertex-set $V = V(G)$ and edge-set $E = E(G)$. For a vertex $x \in V(G)$, let $= N_G(x)$ be the set of neighbors of x. Denote the degree of x by $d_G(x) = |N_G(x)|$, the maximum and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively, and the distance between two vertices x and y by $d_G(x, y)$. The girth of G is the length of the shortest cycle in G and denoted by $g(G)$. For terminology and notation on graph theory not given here, the reader is referred to [12].

Given two vertices x and y of G, we say that x *dominates* y if $xy \in E(G)$. A subset D of $V(G)$ is called a *dominating set* if every vertex of $G - D$ is dominated by some vertex of D. The minimum cardinality of all dominating sets in G is called the *domination number* and denoted by $\gamma(G)$. A dominating set with cardinality $\gamma(G)$ is called a *γ-set* for short. The *bondage number* of a nonempty graph G, denoted by $b(G)$, is the smallest cardinality of a set of edges whose removal from G results in a graph with domination number larger than $\gamma(G)$. The bondage number was first introduced by Fink et al. [3] in 1990, as a parameter to measure the vulnerability of a network with respect to link failures. For a general graph G, determining the exact value of $b(G)$ is difficulty, since Hu and Xu [6] showed that it is NP-hard. Fink et al. [3] posed a conjecture that $b(G) \leqslant \Delta(G) + 1$ for any nonempty graph G. This conjecture was disproved by Teschner [10] three years later. In 1998, Dunbar et al. [2] proposed the same conjecture for planar graphs.

Conjecture 1.1 [2] *$b(G) \leqslant \Delta(G) + 1$ for any planar graph G.*

We have noted that $b(C_{3k+1}) = 3 = \Delta + 1$ [3] and $b(C_{4k+2} \times K_2) = 4 = \Delta + 1$ [2]. Also we can easily see that $b(K_6 - M) = 5 = \Delta + 1$ where M is a perfect matching of the complete graph K_6. These examples show that if Conjecture 1.1 is true then the upper bound is best possible for $2 \leqslant \Delta \leqslant 4$.

In 2000, Kang and Yuan [8] confirmed this conjecture for $\Delta(G) \geqslant 7$ by proving $b(G) \leqslant \min\{8, \Delta(G) + 2\}$ for any connected planar graph G. Three years later, Fischermann et al. [4] obtained further results by proving $b(G) \leqslant 6$ for any connected planar graph G with girth $g(G)$ at least 4, and also proposed some conjectures.

Conjecture 1.2 [4] *$b(G) \leqslant 7$ for any connected planar graph G.*

Conjecture 1.3 [4] *$b(G) \leqslant 5$ for any connected planar graph G with $g(G) \geqslant 4$.*

Conjecture 1.4 [4] *$b(G) \leqslant 4$ for any connected planar graph G with $g(G) \geqslant 5$.*

Carlson and Develin [1] gave short proofs for the results on planar graphs and constructed a planar graph G with $b(G) = 6$ as follows. The *corona* $G_1 \circ G_2$ is a graph formed from a copy of G_1 and $|V(G_1)|$ copies of G_2 by joining the i-th vertex of G_1 to every vertex of the i-th copy of G_2. Let H be a planar graph with $\delta(H) = 5$. Then $G = H \circ K_1$ is also a planar graph and $b(G) = 6$ by the following lemma.

Lemma 1.5 [1] *$b(H \circ K_1) = \delta(H) + 1$.*

So far, no planar graphs with $b(G) \geqslant 7$ have been known.

The rest of this paper is organized as follows. In Section 2 we consider the known ways to prove the results on planar graphs and show their limitations to the conjectures. In Section 3 we obtain some properties of minimum counterexamples which possibly exist under the assumption that the conjectures do not hold. Finally we conclude this paper in Section 4.

2 Remarks on known ways

First we give two basic upper bounds of $b(G)$. More bounds can be found in [3, 9].

Lemma 2.1 [3, 11] *$b(G) \leqslant d_G(x) + d_G(y) - 1$ for any two distinct vertices x and y with $d_G(x, y) \leqslant 2$ in G.*

Lemma 2.2 [5] *$b(G) \leqslant d_G(x) + d_G(y) - 1 - |N_G(x) \cap N_G(y)|$ for any two adjacent vertices x and y in G.*

In [1, 4, 8], the authors obtained various results for planar graphs by Lemma 2.1, Lemma 2.2 and Euler's Formula. Their main results can be stated the following theorems.

Theorem 2.3 [1, 8] *$b(G) \leqslant \min\{8, \Delta(G) + 2\}$ for any connected planar graph G.*

Theorem 2.4 [4] *For any connected planar graph G,*

$$b(G) \leqslant \begin{cases} 6, & \text{if } g(G) \geqslant 4; \\ 5, & \text{if } g(G) \geqslant 5; \\ 4, & \text{if } g(G) \geqslant 6; \\ 3, & \text{if } g(G) \geqslant 8. \end{cases}$$

To prove these results, a basic way is to find two vertices x and y in G satisfying the conditions in Lemma 2.1 or Lemma 2.2 such that the value bounded $b(G)$, $d_G(x)+d_G(y)$ or $d_G(x)+d_G(y)-|N_G(x)\cap N_G(y)|$ is as small as possible. Precisely, let

$$B(G)=\min_{x,y\in V(G)}\left\{\begin{array}{l}\{d_G(x)+d_G(y)-1:1\leqslant d_G(x,y)\leqslant 2\}\bigcup\\ \{d_G(x)+d_G(y)-|N_G(x)\cap N_G(y)|-1:d_G(x,y)=1\}\end{array}\right\}.$$

Then by Lemma 2.1 and Lemma 2.2, we have

$$b(G)\leqslant B(G).$$

The proofs given in [1, 4, 8] indeed imply the following stronger results.

Theorem 2.3′ $B(G)\leqslant \min\{8,\Delta(G)+2\}$ *for any connected planar graph* G.

Theorem 2.4′ *For any connected planar graph* G,

$$B(G)\leqslant\begin{cases}6, & \text{if } g(G)\geqslant 4;\\ 5, & \text{if } g(G)\geqslant 5;\\ 4, & \text{if } g(G)\geqslant 6;\\ 3, & \text{if } g(G)\geqslant 8.\end{cases}$$

Now, a natural question is whether Conjecture 1.1 ∼ 1.4 can be proved by Lemma 2.1 and Lemma 2.2. In other words, are the following stronger conjectures valid?

Conjecture 1.1′ $B(G)\leqslant\Delta(G)+1$ *for any planar graph* G.

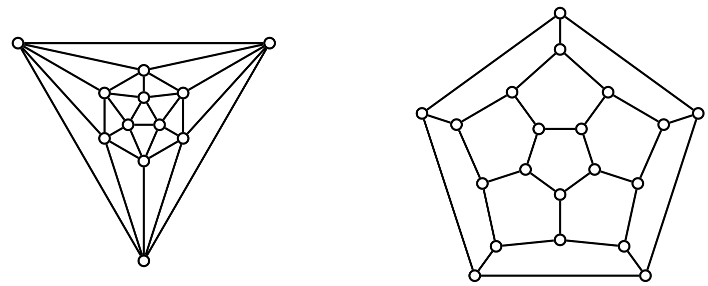
Conjecture 1.2′ $B(G)\leqslant 7$ *for any connected planar graph* G.

Conjecture 1.3′ $B(G)\leqslant 5$ *for any connected planar graph* G *with* $g(G)\geqslant 4$.

Conjecture 1.4′ $B(G)\leqslant 4$ *for any connected planar graph* G *with* $g(G)\geqslant 5$.

It is clear that Conjecture 1.1′ ∼ Conjecture 1.4′ imply Conjecture 1.1 ∼ Conjecture 1.4, respectively, by $b(G)\leqslant B(G)$. This fact seems to mean that the conjectures can be proved by Lemma 2.1 and Lemma 2.2. However, Conjecture 1.1′ ∼ Conjecture 1.4′ are disproved by the following theorems.

The construction of our proofs uses the operation of *subdividing an edge* xy, i.e, replacing the edge xy by a 2-path xvy through a new vertex v, which is called the *subdividing vertex* or *s-vertex* for short. We say *subdividing* the edge xy *twice* if xy is replaced by a 3-path xv_1v_2y.

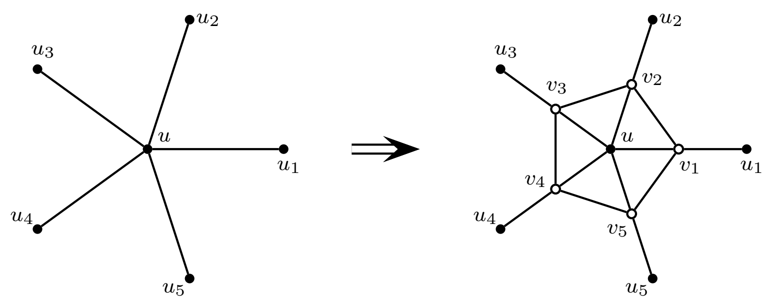


(a) icosahedron (b) dodecahedron

Figure 1: The two regular planar graphs

Theorem 2.5 *There is a planar graph G with $B(G) = 8$.*

Proof. Let H be the icosahedron, shown in Figure 1 (a), and let H' be a graph obtained by subdividing each edge of H. Then for each vertex u in H' there are five subdividing vertices adjacent to u. Link these five vertices to form a cycle and keep the planarity, see Figure 2. Let such a resulting graph be G.



a vertex u and $N_H(u)$ in H a vertex $u, N_G(u)$ and s-vertices in G

Figure 2: A graph G constructed in Theorem 2.5

Assume $V(H) = \{u_1, \ldots, u_n\}$. Let $S = \{v_1, \ldots, v_m\}$ be the set of subdividing vertices in H'. Then $V(G) = V(H) \cup S$. It is easy to observe that, $d_G(u_i) = 5$, $d_G(v_j) = 6$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Furthermore, $d_G(u_i, u_j) = 2$ $(i \neq j)$ and $|N_G(x) \cap N_G(y)| \leqslant 2$ for every edge xy of G. Thus $B(G) = 5 + 6 - 1 - 2 = 8$. ∎

Theorem 2.6 *There is a planar graph G with $B(G) = 6$.*

Proof. Let H be a 3-cube, and let H' be a graph obtained by subdividing each edge in H twice. Link the subdividing vertices properly such that the resulting graph is a planar graph with girth 4. Let such a resulting graph be G, see Figure 3.

Note that all subdividing vertices have degree 4 and $d_G(u, v) \geqslant 3$ for any two vertices u and v in H. Thus $B(G) = 4 + 3 - 1 = 6$. ∎

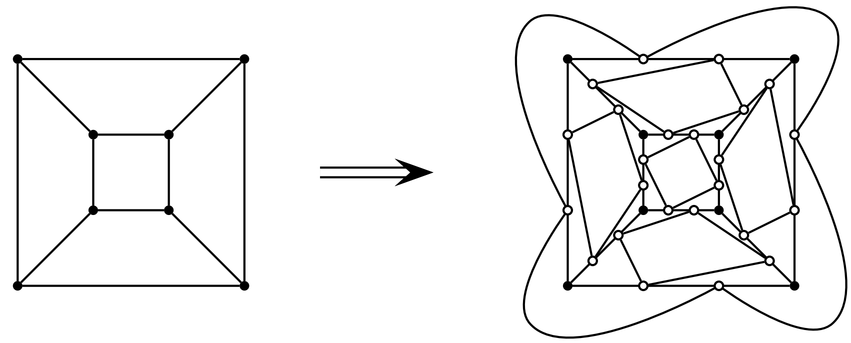


Figure 3: A graph G constructed in the proof of Theorem 2.6

Theorem 2.7 *There is a planar graph G with $B(G) = 5$.*

Proof. The dodecahedron G, shown in Figure 1 (b), is a planar graph with $g(G) = 5$ and $B(G) = 5$. ∎

The theorems $2.5 \sim 2.7$ disprove the conjectures $1.2' \sim 1.4'$, respectively. Conjecture $1.1'$ is invalid, either, by anyone of the theorems $2.5 \sim 2.7$. As a result, Lemma 2.1 and Lemma 2.2 are not enough to prove the conjectures $1.1 \sim 1.4$ (if they are right). Therefore, a new method is need to prove these conjectures.

Remark A careful argument for the dominations in examples provided in the proofs the theorems $2.5 \sim 2.7$ shows that their bondage numbers are not large enough to disprove the conjectures $1.1 \sim 1.4$. However, all the known upper bounds (cf. [2, 9]) can not prove that these examples satisfy the conjectures $1.1 \sim 1.4$. Hence these conjectures can not be proved only by the known upper bounds.

3 Minimum Counterexamples

As mentioned in Section 2, the known upper bounds are not enough to prove anyone of the conjectures $1.1 \sim 1.4$. Thus we present some new approach to them in this section. Note that, if one of these conjectures is invalid, then there exists a counterexample G which is minimum with respect to $|V(G)| + |E(G)|$. We call such a graph G a *minimum counterexample*. Let G_1, G_2, G_3 and G_4 be the possible existing minimum counterexamples to the conjectures $1.1 \sim 1.4$, respectively. It follows from Theorem 2.3 and Theorem 2.4 that

$$b(G_1) = \Delta(G) + 2, \; b(G_2) = 8, \; b(G_3) = 6, \; b(G_4) = 5.$$

By Theorem 2.4, it is easy to see that $g(G_2) = 3, g(G_3) = 4$ and $g(G_4) = 5$. In order to obtain further properties of these minimum counterexamples, we consider how the bondage number changes under some operation of a graph G which decreases $|V(G)| + |E(G)|$ and preserves the planarity. A simple operation satisfying this requirement is the edge deletion.

Lemma 3.1 *Let e be an edge of G. Then $b(G-e) \geqslant b(G)-1$. In addition, $b(G-e) \leqslant b(G)$ if $\gamma(G-e) = \gamma(G)$.*

Proof. Let $E' \subseteq E(G-e)$ with $|E'| = b(G-e)$. Then $\gamma(G-e-E') > \gamma(G-e)$, and so $b(G) \leqslant |\{e\} \cup E'| = b(G-e)+1$.

Assume $\gamma(G-e) = \gamma(G)$ and $E'' \subseteq E(G)$ with $|E''| = b(G)$ such that $\gamma(G-E'') > \gamma(G)$.

If $e \notin E''$ then $\gamma(G-e-E'') \geqslant \gamma(G-E'') > \gamma(G)$.

If $e \in E''$ then $\gamma(G-e-E' \setminus \{e\}) = \gamma(G-E'') > \gamma(G)$.

Thus $b(G-e) \leqslant |E''| = b(G)$. ∎

Theorem 3.2 *$b(G_i - e) = b(G_i) - 1$ for any edge e in G_i, $i = 1, 2, 3, 4$.*

Proof. By Lemma 3.1, $b(G_i - e) \geqslant b(G_i) - 1$ for any edge $e \in E(G)$. Note that $G_i - e$ is a planar graph with $g(G_i - e) \geqslant g(G_i)$ and $\Delta(G_i - e) \leqslant \Delta(G_i)$. Thus, if $b(G_i - e) \geqslant b(G_i)$, then $G_i - e$ is also a counterexample, a contradiction to the minimum of G_i. Hence $b(G_i - e) = b(G_i) - 1$ for any edge $e \in E(G)$. ∎

Next we consider the effect of the edge contraction on the bondage number. Given a graph G, the *contraction* of G by the edge $e = xy$, denoted by G/xy, is the graph obtained from $G - e$ by replacing x and y with a new vertex v_{xy} (*contracted vertex*) which is adjacent to all vertices in $N_{G-e}(x) \cup N_{G-e}(y)$. It is easy to observe that $|V(G/xy)| + |E(G/xy|) < |V(G)| + |E(G)|$ and G/xy is also planar if G if planar.

Lemma 3.3 *$\gamma(G) - 1 \leqslant \gamma(G/xy) \leqslant \gamma(G)$ for any edge xy of G.*

Proof. Let D be a γ-set of G. If neither x nor y belongs to D, then D is a dominating set in G/xy. If $D \cap \{x, y\} \neq \emptyset$, then $(D \setminus \{x, y\}) \cup \{v_{xy}\}$ is a dominating set in G/xy, since v_{xy} dominates all neighbors of x and y. It follows that $\gamma(G/xy) \leqslant |D| = \gamma(G)$.

Conversely, let D' be a γ-set in G/xy. If $v_{xy} \in D'$, then $D = D' \setminus \{v_{xy}\} \cup \{x, y\}$ is a dominating set of G. If $v_{xy} \notin D'$, then D' contains a vertex u such that $uv_{xy} \in E(G/xy)$. By the definition of edge contraction, $ux \in E(G)$ or $uy \in E(G)$, which implies that $D' \cup \{y\}$ or $D' \cup \{x\}$ is a dominating set of G. Thus $\gamma(G) \leqslant |D'| + 1 = \gamma(G/xy) + 1$. ∎

Theorem 3.4 *$b(G/xy) \geqslant b(G)$ if $\gamma(G/xy) = \gamma(G)$ and $N_G(x) \cap N_G(y) = \emptyset$ for some edge xy of G.*

Proof. Let $E' \subseteq E(G/xy)$ with $|E'| = b(G/xy)$. Then $\gamma(G/xy - E') > \gamma(G/xy)$. Suppose that $N_G(x) \cup N_G(y) = \{z_1, \ldots, z_t\}$. Then the hypothesis $N_G(x) \cap N_G(y) = \emptyset$ yields the following claim immediately.

Claim *For any $i \in \{1, \ldots, t\}$, there exists exact one of $\{x, y\}$, denoted by v_i, such that $v_i z_i \in E(G)$.*

Suppose that $v_{xy}z_i \in E'$ if $i \leqslant k$ and $v_{xy}z_i \notin E'$ if $i \geqslant k+1$ ($k \in \{0, 1, \ldots, t\}$), without loss of generality. Let

$$E'' = (E' \setminus \{v_{xy}z_1, \ldots, v_{xy}z_k\}) \cup \{v_1z_1, \ldots, v_kz_k\}.$$

Then $|E''| = |E'|$ and $xy \notin E''$. We show that $(G - E'')/xy = G/xy - E'$. Then by Lemma 3.3,

$$\gamma(G - E'') \geqslant \gamma((G - E'')/xy) = \gamma(G/xy - E') > \gamma(G/xy) = \gamma(G).$$

Hence $b(G) \leqslant |E''| = |E'| = b(G/xy)$.

Let $G_1 = (G - E'')/xy$ and $G_2 = G/xy - E'$. Note that $V(G_1) = V(G_2) = V(G/xy)$. We need only to show $E(G_1) = E(G_2)$. To this aim, consider two vertices $u, v \in V(G/xy)$. If $u, v \neq v_{xy}$, then

$$uv \in E(G_1) \Leftrightarrow uv \in E(G - E'') \Leftrightarrow uv \in E(G - E') \Leftrightarrow uv \in E(G_2).$$

Now consider $v = v_{xy}$ and $u \neq v$. By the above claim and the definition of E'', we have $uv_{xy} \in E'$ if and only if $uv_i \in E''$. Thus

$$\begin{aligned} uv_{xy} \in E(G_1) &\Leftrightarrow uv_i \in E(G - E'') \Leftrightarrow uv_i \in E(G), uv_i \notin E'' \\ &\Leftrightarrow uv_{xy} \in E(G/xy), uv_{xy} \notin E' \Leftrightarrow uv_{xy} \in E(G_2). \end{aligned}$$

Therefore $E(G_1) = E(G_2)$. The proof of the theorem is complete. ∎

Theorem 3.4 is best possible. To illustrate this fact, we need some simple examples.

Example 3.5 [3] $b(C_n) = \begin{cases} 3 & \text{if } n \equiv 1 (mod\ 3), \\ 2 & \text{otherwise.} \end{cases}$ *for a cycle C_n of order $n \geqslant 3$,*

Example 3.6 [3] $b(K_n) = \lceil \frac{n}{2} \rceil$ *for a complete graph K_n of order $n \geqslant 2$.*

The above examples show that the conditions of Theorem 3.4 are necessary. Clearly $\gamma(C_n) = \lceil n/3 \rceil$ and $\gamma(K_n) = 1$; for any edge xy, $C_n/xy = C_{n-1}$ and $K_n/xy = K_{n-1}$. By Example 3.5, if $n \equiv 1 \pmod 3$, then $\gamma(C_n/xy) < \gamma(G)$ and $b(C_n/xy) = 2 < 3 = b(C_n)$. Thus the result in Theorem 3.4 is generally invalid without the hypothesis $\gamma(G/xy) = \gamma(G)$. Furthermore, the condition

$N_G(x) \cap N_G(y) = \emptyset$ can not be omitted even if $\gamma(G/xy) = \gamma(G)$ is provided, since for odd n, $b(K_n/xy) = \lceil \frac{n-1}{2} \rceil < \lceil \frac{n}{2} \rceil = b(K_n)$, by Example 3.6.

On the other hand, the above examples also show that the equality in $b(G/xy) \geqslant b(G)$ may hold ($b(C_n/xy) = b(C_n) = 2$ if $n \equiv 0 \pmod 3$, $b(K_n/xy) = b(K_n)$ if n is even). Thus the bound on $b(G)$ given in Theorem 3.4 is tight. However, $b(G/xy)$ can be arbitrarily larger than $b(G)$ when all the conditions are satisfied. Given a graph H, let G be the graph formed from $H \circ K_1$ by adding a new vertex x and joining it to an vertex y of degree one in $H \circ K_1$. Then $G/xy = H \circ K_1$, $\gamma(G) = \gamma(G/xy)$ and $N_G(x) \cap N_G(y) = \emptyset$. But $b(G) = 1$ since $\gamma(G - xy) = \gamma(G/xy) + 1$, and $b(G/xy) = \delta(H) + 1$ by Lemma 1.5. The gap between $b(G)$ and $b(G/xy)$ is $\delta(H)$.

Now we apply Theorem 3.4 to G_2.

Corollary 3.7 *$N_G(x) \cap N_G(y) \neq \emptyset$ if $\gamma(G_2/xy) = \gamma(G_2)$ for some edge xy.*

Proof. Note that G_2/xy is a simple planar graph. If $\gamma(G_2/xy) = \gamma(G_2)$ for some edge xy and $N_G(x) \cap N_G(y) = \emptyset$, then it follows from Theorem 3.4 that $b(G_2/xy) \geqslant b(G_2)$. Hence G_2/xy is a counterexample. But G_2/xy is smaller than G_2, a contradiction. ∎

Finally we consider G_1.

Lemma 3.8 *$\gamma(G/xy) \leqslant \gamma(G - x)$ for any edge xy.*

Proof. Let D be a γ-set of $G - x$. If $y \notin D$, then there exists a vertex $u \in D$ such that $uy \in E(G - x)$. Thus $uv_{xy} \in E(G/xy)$, i.e., v_{xy} is dominated by $u \in D$. Therefore D is also a dominating set of G/xy and $\gamma(G/xy) \leqslant |D| = \gamma(G)$.

Now assume $y \in D$ and let $D' = (D \setminus \{y\}) \cup \{v_{xy}\}$. If $uy \in E(G - x)$, then $uv_{xy} \in E(G/xy)$. That means, the vertices dominated by y in $G - x$ are all dominated by v_{xy} in G/xy. Thus D' is a dominating set of G/xy and $\gamma(G/xy) \leqslant |D'| = \gamma(G)$. ∎

Lemma 3.9 [2] *$\gamma(G - v) < \gamma(G)$ for all vertices $v \in V(G)$ if $b(G) > \Delta(G)$.*

Theorem 3.10 *$\gamma(G_1/xy) = \gamma(G_1) - 1$ for every edge xy.*

Proof. Let xy be an edge of G_1. Since $b(G_1) > \Delta(G_1) + 1$, then by Lemma 3.8 and Lemma 3.9, we have $\gamma(G_1/xy) \leqslant \gamma(G_1 - x) < \gamma(G_1)$. ∎

4 Conclusions

In Section 2 we defined $B(G)$ to show that the conjectures 1.1 ∼ 1.4 can not be proved only by the known upper bounds. Conversely, if one wants to find a counterexample G, he or she should first guarantee that $B(G)$ is large enough.

In Section 3 we investigate the minimum counterexamples (if exist) to these conjectures, and obtained their properties by considering edge deletion and contraction. Other operations are also worthy of investigation as long as they decrease the value of $|V(G)| + |E(G)|$ while preserving the planarity.

The results on the edge contraction may have other applications. So far, no planar graphs with large bondage number have been known to us; we can not tell the tightness of anyone of the conjectures 1.2 ∼ 1.4 even if they are proved to be true. Now we know that contracting edges from a planar graph G may lead to a planar graph with larger bondage number, since $b(G/xy)$ can be arbitrarily larger than $b(G)$. Then it is possible to find a planar graph with large bondage number from some planar graph by suitably choosing its edges to be contracted. For this purpose, we propose the following problem.

Problem 4.1 *Determine when $b(G/xy) \geqslant b(G) + k$ for some positive integer k.*

The similar consideration for edge deletions fails since $b(G - e) \leqslant b(G)$ when $b(G) \geqslant 2$, by Lemma 3.1.

In view of Theorem 3.2 and Theorem 3.10, we propose the following problems, which may be helpful to seeking counterexamples or proving their inexistence.

Problem 4.2 *Characterize the graph G with $b(G - e) = b(G) - 1$ for every edge e.*

Problem 4.3 *Characterize the graph G with $\gamma(G/e) = \gamma(G) - 1$ for every edge e.*

Lastly, we would like to make some remarks on the above problems. The graphs in Problem 4.2 and Problem 4.3 can be viewed as *bondage critical with respect to edge removal*, or *domination critical with respect to edge contraction*. Graphs in the latter case can be characterized by the property that every edge lies in the induced subgraph of some minimum dominating set (See [7].) No further characterizations are known yet.

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