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SIGNED REINFORCEMENT NUMBERS OF CERTAIN GRAPHS

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Abstract

Let $G$ be a graph with vertex set $V(G)$. A function $f : V(G) \rightarrow \{-1, 1\}$ is a signed dominating function of $G$ if, for each vertex of $G$, the sum of the values of its neighbors and itself is positive. The signed domination number of a graph $G$, denoted $\gamma_s(G)$, is the minimum value of $\sum_{v \in V(G)} f(v)$ over all the signed dominating functions $f$ of $G$. The signed reinforcement number of $G$, denoted $R_s(G)$, is defined to be the minimum cardinality $|S|$ of a set $S$ of edges such that $\gamma_s(G + S) < \gamma_s(G)$. In this paper, we initialize the study of signed reinforcement number and determine the exact values of $R_s(G)$ for several classes of graphs.

Keywords: Signed domination, signed reinforcement number.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of $v \in V(G)$ are defined by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$ and $d_G(v) = |N_G(v)|$, respectively. For $S \subseteq V(G)$, $N_G(S)$ is defined to be the union of the open neighborhoods $N_G(v)$ for all $v \in S$ and $N_G[S] = N_G(S) \cup S$. Let $\Delta(G)$ denote the maximum degree of a graph $G$. A vertex of degree one in $G$ is called a leaf; a support vertex of $G$ is a vertex adjacent with a leaf of $G$. Let $L(G)$ and $S(G)$ denote the set of leaves of $G$ and the set of support vertices of $G$, respectively. For two sets $A, B \subseteq V(G)$, let $E(A, B) = \{e = xy \mid x \in A, y \in B\}$ and $e(A, B) = |E(A, B)|$.

Let $G = (V, E)$ be a graph and $f : V \rightarrow R$ is a real-valued function on $V$. The weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$. For $S \subseteq V$, define $f(S) = \sum_{v \in S} f(v)$. Then $\omega(f) = f(V)$.

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For any $v \in V$, let $f[v] = f(N[v])$ for notation convenience. A function $f : V \to \{-1, 1\}$ is called a signed dominating function (abbreviated by SDF) if $f[v] \geq 1$ for all $v \in V$. The signed domination number of $G$ is $\gamma_s(G) = \min\{\omega(f) \mid f$ is a SDF of $G\}$. A $\gamma_s(G)$-function is a signed dominating function of $G$ of weight $\gamma_s(G)$. Signed domination was first introduced by Dunbar et al. in [4] and further studied in [1, 3, 6, 7, 9, 14, 10, 11, 12, 13, 15].

The reinforcement number of a graph $G$ is a measurement of the stability of the domination in $G$. The reinforcement number of a graph $G$ is the smallest number of edges which must be added to $G$ to decrease the domination number of $G$ (the classic domination number of a graph $G$ is the minimum cardinality of a subset $D$ of $V(G)$ such that for each $v \in V(G), N[v] \cap D \neq \emptyset$). The definition was first introduced by Kok and Mynhardt [8]. During the past twenty years, the reinforcement number associated with domination parameters were studied in literatures, for example, Ghoshal et al.[5] defined and studied the reinforcement number associated with the strong domination number; Gayla et al.[2] studied the reinforcement number associated with the fractional domination number.

In this paper, we define the signed reinforcement number of a graph $G$, denoted $R_s(G)$, to be the minimum cardinality of a set $S$ of edges in the complement graph $G^c$ of $G$ such that $\gamma_s(G+S) < \gamma_s(G)$. A minimum edge set $S \subseteq E(G^c)$ with $\gamma_s(G+S) < \gamma_s(G)$ is called a signed reinforcement set of $G$. Note that the signed reinforcement set of a graph $G$ maybe doesn’t exist, for example, for $K_n$, the complete graph on $n$ vertices or $C_4$, the cycle on 4 vertices. So if the signed reinforcement set of a graph $G$ doesn’t exist, we define $R_s(G) = 0$.

The paper is organized as follows. Section 2 gives some lemmas about signed domination numbers and signed reinforcement numbers. Sections 3 and 4 determine the exact values of the signed reinforcement numbers of paths, cycles and wheels. Section 5 gives a sharp bound of the signed reinforcement number of trees.

2. Lemmas

In this section, we will give some useful lemmas about signed dominating functions of a graph $G$. Let $K_n$, $P_n$ and $C_n$ denote a complete graph, a path and a cycle on $n$ vertices, respectively. The following lemmas are given in [4] and the proof of them can be found in [4].

Lemma 2.1. [4] A signed dominating function $f$ on a graph $G$ is minimal if and only if for every vertex $v \in V$ with $f(v) = 1$, there exists a vertex $u \in N[v]$ with $f[u] \in \{1, 2\}$.

Lemma 2.2. [4] If $f$ is a signed dominating function of a graph $G$, then $f(v) = 1$ for any $v \in L(G) \cup S(G)$.

Lemma 2.3. [4] Let $G$ be a graph on $n$ vertices. Then $\gamma_s(G) = n$ if and only if $V(G) = L(G) \cup S(G)$.

Lemma 2.4. [4] If $G$ has more than three vertices and maximum degree $\Delta \leq 3$, then $\gamma_s(G) \geq \frac{n}{3}$.
The following lemma gives a lower bound for the signed domination number of a graph $G$ with precisely one vertex with maximum degree four.

**Lemma 2.5.** Let $G$ be a graph with order $n$ and maximum degree four. If $G$ has precisely one vertex with maximum degree four, then $\gamma_s(G) \geq \frac{n-2}{3}$.

**Proof.** Let $f$ be a $\gamma_s$-function and let $P$ and $M$ be the reverse images of $+1$ and $-1$ under $f$. Then $|P| + |M| = n$ and $\gamma_s(G) = |P| - |M|.$

If $M = \emptyset$, then $\gamma_s(G) = n > \frac{n-2}{3}$.

If $M \neq \emptyset$, we evaluate the number, $e(M, P)$, of edges between $P$ and $M$ in $G$.

For any $v \in M$, to guarantee $f[v] \geq 1$, there exist at least two edges from $v$ to $P$, which means that $e(M, P) \geq 2|M|$.

On the other hand, for each $v \in P$, to guarantee $f[v] \geq 1$, $|N(v) \cap M| \leq |N(v) \cap P|$. Hence there are at most $\lfloor \frac{d(v)}{2} \rfloor$ edges from $v$ to $M$. Since $G$ has precisely one vertex with maximum degree four, $e(P, M) \leq |P| - 1 + 2 = |P| + 1$.

Hence, $2|M| \leq e(M, P) \leq |P| + 1$. Combine with $|P| + |M| = n$, we have $|M| \leq \frac{n+1}{3}$ and $|P| \geq \frac{2n-1}{3}$. So, $\gamma_s(G) = |P| - |M| \geq \frac{2n-1}{3} - \frac{n+1}{3} = \frac{n-2}{3}$.

The signed domination numbers of paths, cycles and stars were given in [4].

**Lemma 2.6.** [4]

1. $\gamma_s(K_{1,n-1}) = n$, $n \geq 2$;
2. $\gamma_s(P_n) = n - 2\lfloor \frac{n-2}{3} \rfloor$, $n \geq 2$;
3. $\gamma_s(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$, $n \geq 3$.

**Lemma 2.7.** Let $G$ be a connected graph with $|V(G)| \geq 3$. If $\gamma_s(G) = |V(G)|$, then $R_s(G) = 1$.

**Proof.** Since $\gamma_s(G) = |V(G)|$, by Lemma 2.3, $V(G) = L(G) \cup S(G)$. Let $f$ be a $\gamma_s$-function. Since $\gamma_s(G) = |V(G)|$, $f \equiv 1$. Since $|V(G)| \geq 3$, $|L(G)| \geq 2$. Let $u, v \in L(G)$ and $w$ be the support vertex of $u$. Since $f \equiv 1$, $f[w] \geq 3$. Then if we replace the value $1$ by $-1$ on $u$ and adding the edge $uv$ to $G$, then the reduced function is a SDF of $G + uv$ with weight $|V(G)| - 2 < \gamma_s(G)$. So $R_s(G) = 1$.

**Lemma 2.8.** For any graph $G$, if $\gamma_s(G + A) < \gamma_s(G)$ for some set $A \subseteq E(G^c)$, then $\gamma_s(G + A) \leq \gamma_s(G) - 2$.

**Proof.** Let $f$ and $g$ be minimum signed dominating functions of $G + A$ and $G$, respectively, and let $f^{-1}(a)$ and $g^{-1}(a)$ denote the reversed imaginary of $a$ under $f$ and $g$. Since $\gamma_s(G + A) < \gamma_s(G)$, $|f^{-1}(1)| \leq |g^{-1}(1)| - 1$ (equivalently, $|f^{-1}(-1)| \geq |g^{-1}(-1)| + 1$). Hence $\gamma_s(G + A) = |f^{-1}(1)| - |f^{-1}(-1)| \leq |g^{-1}(1)| - |g^{-1}(-1)| - 2 = \gamma_s(G) - 2$. 

3. The signed reinforcement numbers of stars, paths and cycles

Since \( V(K_{1,n-1}) = L(K_{1,n-1}) \cup S(K_{1,n-1}) \), by Lemmas 2.3 and 2.7, \( R_s(K_{1,n-1}) = 1 \) if \( n \geq 3 \). Hence, we have the following observation.

**Observation 3.1.** Let \( n \geq 3 \). Then \( R_s(K_{1,n-1}) = 1 \).

**Theorem 3.2.** For \( n \geq 3 \),

\[
R_s(P_n) = \begin{cases} 
2, & n \equiv 2 \pmod{3} \\
1, & \text{otherwise}
\end{cases}
\]

**Proof.** Denote \( V(P_n) = \{v_1, v_2, \cdots, v_n\} \).

If \( n = 3k \) or \( 3k+1 \) for some integer \( k \geq 1 \), then

\[
\gamma_s(P_{3k} + v_1v_{3k}) = \gamma_s(C_{3k}) = k < k + 2 = \gamma_s(P_{3k})
\]

and

\[
\gamma_s(P_{3k+1} + v_1v_{3k+1}) = \gamma_s(C_{3k+1}) = k + 1 < k + 3 = \gamma_s(P_{3k+1}).
\]

This implies that \( R_s(P_n) = 1 \) if \( n \neq 2 \pmod{3} \).

If \( n = 3k + 2 \) for some integer \( k \geq 1 \), let \( G \) be the graph obtained from \( P_{3k+2} \) by adding two edges \( v_1v_3, v_3v_{3k+2} \). Now, define a function \( f \) as follows:

\[
f(v_i) = \begin{cases} 
-1, & i \equiv 1 \pmod{3} \\
1, & \text{otherwise}
\end{cases}
\]

It is an easy task to check that \( f[v_i] = 1 \) for every \( i \in [1, 3k + 2] \). So \( f \) is a SDF of \( G \). Hence, \( \gamma_s(G) \leq f(V(G)) = k < k + 2 = \gamma_s(P_{3k+2}) \). Therefore, \( R_s(P_{3k+2}) \leq 2 \).

Now we show that \( R_s(P_{3k+2}) = 2 \). If there exists some edge \( e \notin E(P_{3k+2}) \) such that \( \gamma_s(P_{3k+2} + e) < \gamma_s(P_{3k+2}) \), then, by Lemma 2.8, \( \gamma_s(P_{3k+2} + e) \leq \gamma_s(P_{3k+2}) - 2 = k \). Since \( \Delta(P_{3k+2} + e) \leq 3 \), by Lemma 2.4, \( \gamma_s(P_{3k+2} + e) \geq \left\lceil \frac{3k+2}{3} \right\rceil = k + 1 > k \geq \gamma_s(P_{3k+2} + e) \), a contradiction.

**Lemma 3.3.** Let \( n \geq 3 \) and \( n \equiv 0 \) or \( 1 \pmod{3} \). Then

\[
R_s(C_n) = \begin{cases} 
0, & n = 3, 4 \\
3, & n \geq 6.
\end{cases}
\]

**Proof.** Denote \( V(C_n) = \{v_0, v_1, \cdots, v_{n-1}\} \) and \( E(C_n) = \{v_iv_{i+1} \mid i = 0, 1, \cdots, n-1\} \), where the "+" is under modulo \( n \). If \( n = 3 \), then \( C_n = K_3 \) and \( R_s(C_n) = R_s(K_3) = 0 \).

If \( n = 4 \), we can check that \( \gamma_s(C_4) = 2 = \gamma_s(C_4 + v_1v_3) = \gamma_s(C_4 + v_0v_2) = \gamma_s(C_4 + \{v_1v_3, v_0v_2\}) \) and hence \( R_s(C_4) = 0 \).
If \( n \geq 6 \), let \( G \) be the graph obtained from \( C_n \) by adding three edges \( v_1v_3, v_1v_5 \) and \( v_3v_5 \) and define a function \( f : V(G) \to \{-1, 1\} \) by

\[
f(v_i) = \begin{cases} 
-1, & i = 2, 4 \text{ or } 3j \text{ for } j \in [2, \lfloor \frac{n}{4} \rfloor] \\
1, & \text{otherwise.}
\end{cases}
\]

It is an easy task to check that \( f[v_i] \geq 1 \) for any \( i \in [1, n] \). So \( f \) is a SDF of \( G \) and hence \( \gamma_s(G) \leq f(V(G)) = n - 2(\lfloor \frac{n}{4} \rfloor + 1) = n - 2(\lfloor \frac{n}{4} \rfloor) - 2 < n - 2(\lfloor \frac{n}{3} \rfloor) = \gamma_s(C_n) \) (Lemma 2.6 (3)). So we have \( R_s(C_n) \leq 3 \).

Next we will show that \( R_s(C_n) \geq 3 \) and so the result follows. Suppose to the contrary that there exist two edges \( e_1, e_2 \not\in E(C_n) \) such that \( \gamma_s(C_n + \{e_1, e_2\}) < n - 2(\lfloor \frac{n}{3} \rfloor) = \gamma_s(C_n) \). By Lemma 2.8, \( \gamma_s(C_n + \{e_1, e_2\}) \leq \gamma_s(C_n) - 2 \).

If \( e_1, e_2 \) are independent, then \( \Delta(C_n + \{e_1, e_2\}) \leq 3 \). By Lemma 2.4, \( \gamma_s(C_n + \{e_1, e_2\}) \geq \lfloor \frac{n-2}{3} \rfloor = n - 2(\lfloor \frac{n}{3} \rfloor) = \gamma_s(C_n) \), a contradiction.

If \( e_1, e_2 \) have a common end, then \( C_n + \{e_1, e_2\} \) has precisely one vertex with maximum degree four. By Lemma 2.5, \( \gamma_s(C_n + \{e_1, e_2\}) \geq \lfloor \frac{n-2}{3} \rfloor \geq n - 2(\lfloor \frac{n}{3} \rfloor) - 1 = \gamma_s(C_n) - 1 \), a contradiction too. \( \square \)

**Lemma 3.4.** If \( n \equiv 2 \pmod{3} \) and \( n \geq 5 \), then \( R_s(C_n) = 2 \).

**Proof.** Let \( V(C_n) \) and \( E(C_n) \) be defined the same as in the former proof and let \( G \) be the graph obtained by adding two edges \( v_1v_3 \) and \( v_3v_5 \). Suppose \( n = 3k + 2 \) (\( k \geq 1 \)). Define a function \( f : V(G) \to \{-1, 1\} \) by

\[
f(v_i) = \begin{cases} 
-1, & i = 2 \text{ or } 3j + 1 \text{ for } j \in [1, k] \\
1, & \text{otherwise.}
\end{cases}
\]

It is an easy task to check that \( f[v_i] \geq 1 \) for any \( i \in [1, n] \). So \( f \) is a SDF of \( G \) and \( \gamma_s(G) \leq f(V(G)) = k < k + 2 = \gamma_s(C_n) \). Hence \( R_s(C_n) \leq 2 \).

If we can show that \( R_s(C_n) \geq 2 \), then the result follows. Suppose that there exists some edge \( e \not\in E(C_n) \) such that \( \gamma_s(C_n + e) < \gamma_s(C_n) \). By Lemma 2.8, \( \gamma_s(C_n + e) \leq \gamma_s(C_n) - 2 = k \). Since \( \Delta(C_n + e) = 3 \), \( \gamma_s(C_n + e) \geq \lfloor \frac{k-2}{3} \rfloor = k + 1 \) by Lemma 2.4, a contradiction with \( \gamma_s(C_n + e) \leq k \). \( \square \)

From the above two lemmas, we have

**Theorem 3.5.** Let \( n \geq 3 \). Then

\[
R_s(C_n) = \begin{cases} 
0, & n = 3, 4 \\
3, & n \equiv 0 \text{ or } 1 \pmod{3} \text{ and } n \geq 6 \\
2, & n \equiv 2 \pmod{3}.
\end{cases}
\]
4. Wheels

A wheel is a graph obtained from a cycle by adding a new vertex such that it is adjacent with each vertex of the cycle. Let \( W_n = \{w\} \cup C_{n-1} \) denote a wheel obtained from a cycle \( C_{n-1} \) and a new vertex \( w \), called the central vertex of \( W_n \). In the following, we denote \( V(C_{n-1}) = \{v_0, v_1, \cdots, v_{n-2}\} \) and \( E(W_n) = \{wv_i, v_iv_{i+1}, i = 0, 1, \cdots, n-2\} \), where the sum is taken modulo \( n-1 \).

First we determine the signed domination number of \( W_n \).

**Lemma 4.1.** For \( n \geq 4 \), \( \gamma_s(W_n) = n - 2\lfloor \frac{n-1}{3} \rfloor \).

**Proof.** Since we can extend a SDF of \( C_{n-1} \) to be a SDF of \( W_n \) by assigning 1 to the central vertex \( w \), \( \gamma_s(W_n) \leq \gamma_s(C_{n-1}) + 1 = n - 1 - 2\lfloor \frac{n-1}{3} \rfloor + 1 = n - 2\lfloor \frac{n-1}{3} \rfloor \).

In the following, we show that \( \gamma_s(W_n) \geq n - 2\lfloor \frac{n-1}{3} \rfloor \). Let \( f \) be a minimum SDF of \( W_n \) and let \( P \) and \( M \) be the set of reverse imagines of 1 and \(-1\) under \( f \), respectively. We claim that \( f(w) = 1 \), equivalently, \( w \in P \). If \( f(w) = -1 \), to guarantee \( f(v_i) \geq 1 \) for any \( i = 0, \cdots, n-2 \), \( f(v_i) = 1 \) since \( d(v_i) = 3 \). This means that \( \gamma_s(W_n) = n - 1 > n - 2\lfloor \frac{n-1}{3} \rfloor \), a contradiction.

Since \( f[v_i] = f(w) + f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq 1 \), \( f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq 0 \). Hence at most one of three consecutive vertices on \( C_{n-1} \) is assigned \(-1\) by \( f \). This implies that \( |M| \leq \frac{n-1}{3} \). So \( \gamma_s(W_n) = n - 2|M| \geq n - 2\lfloor \frac{n-1}{3} \rfloor \). \( \square \)

**Theorem 4.2.**

1. \( R_s(W_4) = 0 \).

2. If \( n \geq 5 \),

\[
R_s(W_n) = \begin{cases} 
2, & n = 1(mod 3) \\
1, & \text{otherwise} 
\end{cases}
\]

**Proof.** (1) It follows directly from \( W_4 = K_4 \) and \( R_s(K_n) = 0 \) for any \( n \geq 2 \).

(2) If \( n = 3k \) \((k \geq 2)\) or \( n = 3k+2 \) \((k \geq 1)\), then, by Lemma 4.1, \( \gamma_s(W_n) = n - 2\lfloor \frac{n-1}{3} \rfloor = k+2 \). Now, we add an edge \( v_0v_2 \) to \( W_n \) and define a function \( g : V(W_n + v_0v_2) \to \{-1, 1\} \) as follows:

\[
g(x) = \begin{cases} 
-1, & \text{if } x = v_i \text{ and } i = 1, n - 2 \text{ or } 3j \text{ for } j \in [1, \lfloor \frac{n}{3} \rfloor - 2] \\
1, & \text{otherwise} 
\end{cases}
\]

It is an easy task to check that \( g \) is a SDF of \( W_n + v_0v_2 \). Hence \( \gamma_s(W_n + v_0v_2) \leq g(V(W_n)) = n - 2\lfloor \frac{n}{3} \rfloor = k + 2 = \gamma_s(W_{3k}) \). So \( R_s(W_n) = 1 \).

If \( n = 3k+1 \) \((k \geq 2)\), then, by Lemma 4.1, \( \gamma_s(W_{3k+1}) = 3k + 1 - 2\lfloor \frac{3k+1-1}{3} \rfloor = k + 1 \). Then we can add two edges \( v_0v_2, v_2v_4 \) to \( W_n \) and define a SDF \( g \) of \( W_n + \{v_0v_2, v_2v_4\} \) as
follows:
\[
g(x) = \begin{cases} 
-1, & \text{if } x = v_i \text{ and } i = 1, 3 \text{ or } 3i - 1 \text{ for } i \in [2, k] \\
1, & \text{otherwise}
\end{cases}
\]
Hence
\[
\gamma_s(W_{3k+1} + \{v_0v_2, v_2v_4\}) \leq g(V(W_n)) = 3k + 1 - 2(k + 1) = k - 1 < k + 1 = \gamma_s(W_{3k+1}).
\]
So \(R_s(W_{3k+1}) \leq 2\).

In the following, we will prove that \(R_s(W_{3k+1}) \geq 2\). Suppose to the contrary that there exists an edge \(e \notin E(W_n)\) such that \(\gamma_s(W_{3k+1} + e) \leq \gamma_s(W_{3k+1}) = k + 1\).

Let \(\phi\) be a minimum SDF of \(W_{3k+1} + e\) and let \(P\) and \(M\) be the reverse imaginaries of 1 and −1 under \(\phi\), respectively. Then,
\[
\begin{cases} 
|P| + |M| = 3k + 1 \\
|P| - |M| = \gamma_s(W_{3k+1} + e) \leq k
\end{cases}
\]
Since \(|M|\) and \(|P|\) are integers, the equation array implies that
\[
\begin{cases} 
|M| \geq k + 1 \\
|P| \leq 2k
\end{cases}
\]
With a same reason with \(f(w) = 1\) in the proof of Lemma 4.1, \(\phi(w) = 1\). Then \(M \subseteq V(C_{3k})\). Since \(|M| \geq k + 1\), there are three consecutive vertices \(v_{i-1}, v_i, v_{i+1}\) on \(C_{3k}\) such that two of them are in \(M\).

If the two members of \(\{v_{i-1}, v_i, v_{i+1}\} \cap M\) are consecutive on \(C_{3k}\), without loss of generality, suppose \(v_{i-1}, v_i \in M\). Then, to guarantee that \(\phi[v_{i-1}] \geq 1, \phi[v_i] \geq 1, d(v_{i-1}) \geq 4\) and \(d(v_i) \geq 4\). This is impossible since \(v_{i-1}, v_i\) can not be the two ends of the new adding edge \(e\). Hence we must have \(v_{i-1}, v_{i+1} \in M\).

To guarantee \(\phi[v_i] \geq 1, d(v_i) = 4\) and \(\phi(v_i) = 1\). This means that \(v_i\) must be an end of \(e\). Suppose \(e = v_iv_m\). Then \(\phi(v_m) = 1\). Now we compute the number of edges between \(M\) and \(P - \{w\}\) with two methods. Let \(P' = P - \{w\}\).

Since, for each \(x \in M\), \(d(x) = 3\) and \(\phi[x] = -1 + \phi(w) + \phi(N(x) \setminus \{w\}) \geq 1\), \(\phi(N(x) \setminus \{w\}) \geq 1\). So, there are two edges from \(x\) to the vertices in \(P'\), this means \(e(x, P') = 2\).
Hence \(e(M, P') = 2|M| \geq 2(k + 1)\).

Since, for each \(x \in P - \{w, v_i, v_m\}\), \(d(x) = 3\) and \(\phi[x] = 1 + 1 + \phi(N(x) \setminus \{w\}) \geq 1, \phi(N(x) \setminus \{w\}) \geq -1\). Hence there is at most one edge from \(x\) to the vertices in \(M\), which means that \(e(x, M) \leq 1\) for each \(x \in V(P - \{w, v_i, v_m\})\). For \(v_i\) and \(v_m\), there are at most \(2\) edges from \(v_i\) or \(v_m\) to vertices in \(M\). So,
\[
e(P', M) \leq |P - \{w, v_i, v_m\}| + 4 \leq 2k - 3 + 4 = 2k + 1 < 2(k + 1) \leq e(M, P'),
\]
a contradiction.
5. Trees

**Lemma 5.1.** For any tree $T$ with order $n \geq 3$, $R_s(T) \leq 3$.

*Proof.* If $\gamma_s(T) = n$, then $R_s(T) = 1 \leq 3$ by Lemma 2.7.

Now suppose $\gamma_s(G) < n$. Then $T \neq K_{1,n-1}$. Hence there exist two leaves $u_1, v_1$ such that they have two different support vertices $u_2$ and $v_2$, respectively.

If $L(T) = \{ u_1, v_1 \}$, then $T = P_n$ and so $R_s(T) \leq 2$ by Theorem 3.2.

If $L(T) \neq \{ u_1, v_1 \}$, let $w_1$ be another leaf of $T$. Then there is at least one of $u_2, v_2$ which is not adjacent with $w_1$. Without loss of generality, assume $u_2w_1 \notin E(T)$. Let $f$ be a minimum SDF of $T$. By Lemma 2.2, $f(u_i) = f(v_i) = 1, i = 1, 2$ and $f(w_1) = 1$. Let $S = \{ u_1v_1, u_2w_1 \}$ if $u_2v_2 \in E(G)$ and $S = \{ u_1v_1, u_2v_2, u_2w_1 \}$ if $u_2v_2 \notin E(G)$. We can easily modify $f$ to be a SDF $g$ of $T + S$ as follows.

$$g(x) = \begin{cases} -1, & x = u_1 \\ f(x), & x \in V(T) - \{ u_1 \} \end{cases}$$

Then $\gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2 < \gamma_s(T)$ implies that $R_s(T) \leq 3$. \[\square\]

In fact, the upper bound of the signed reinforcement number of trees given here is not sharp. In the following, we will give a sharp bound for $R_s(T)$.

**Lemma 5.2.** Let $f$ be a minimum SDF of a tree $T$. If there exists a support vertex $v$ with $f[v] \geq 3$, then $R_s(T) = 1$.

*Proof.* Let $u, w$ be two leaves of $T$ such that at least one of them is adjacent with $v$ in $T$. Then $uw$ is the desired edge to guarantee that $\gamma_s(T + uw) < \gamma_s(T)$. \[\square\]

**Lemma 5.3.** Let $f$ be a minimum SDF of a tree $T$. If there exists a support vertex $v$ with $f[v] \geq 2$, then $R_s(T) \leq 2$.

*Proof.* If $T$ is a star, then the result is clearly true. Now suppose $T$ is not a star. Then we can choose two leaves $u, v$ of $T$ such that $uv \in E(T)$ and $uv \notin E(T)$. Hence $uw, vw$ are two edges to guarantee that $\gamma_s(T + \{ uw, vw \}) < \gamma_s(T)$. So, $R_s(T) \leq 2$. \[\square\]

**Lemma 5.4.** Let $T'$ be a tree obtained from a tree $T$ ($|V(T)| \geq 3$) by adding an edge joining a leaf of $T$ with a leaf of a path $P_3$. Then $R_s(T') \leq R_s(T)$.

*Proof.* Let $v$ be a leaf of $T$ and let $u$ be its support vertex. Let $P_3 = x_1x_2x_3$ and let $T'$ be a tree obtained from $T \cup P_3$ by adding an edge $x_3v$. It is an easy task to check that $\gamma_s(T') = \gamma_s(T) + 1$. Now suppose $R_s(T) = r$ and $S$ is a set of edges with $|S| = r$ such that $\gamma_s(T + S) \leq \gamma_s(T) - 2$ (by Lemma 2.8). By Lemma 5.1, $r \leq 3$.

Let $f$ be a minimum SDF of $T + S$. Then $f(V(T)) = \gamma_s(T + S)$.
If $v$ is not incident with any edge in $S$, then $v$ is a leaf of $T+S$, too. Hence $f(v) = f(u) = 1$. We can easily extend $f$ to be a SDF, say $g$, of $T' + S$ by defining $g(x_1) = g(x_2) = 1$ and $g(x_3) = -1$ and $g(x) = f(x)$ for the other vertices. So $\gamma_s(T' + S) \leq \gamma_s(T + S) + 1 \leq \gamma_s(T) - 2 + 1 = \gamma_s(T') - 2$. This implies that $R_s(T') \leq |S| = r = R_s(T)$.

Now we suppose that $v$ is incident with some edges, denoted $vu_1, \cdots, vu_t$, in $S$.

If $f[v] \geq 2$, then we can extend $f$ to be a SDF of $T' + S$ the same as the above case and so the result is valid. So we assume that $f[v] = 1$ in the following.

Case 1. $f(v) = 1$.

If $f(u) = 1$, then $f(u_1) + \cdots + f(u_t) = -1$. Let $S' = (S - \{vu_1, \cdots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\}$. Then we can define a SDF $g$ of $T' + S'$ as follows:

$$g(x) = \begin{cases} 
-1, & x = x_3 \\
1, & x = x_1, x_2 \\
f(x), & x \in V(T)
\end{cases}$$

So $\gamma_s(T' + S') \leq g(V(T' + S')) \leq \gamma_s(T + S) + 1 \leq \gamma_s(T) - 1 = \gamma_s(T') - 2$. This implies that $R_s(T') \leq |S'| = |S| = R_s(T)$.

If $f(u) = -1$, then $f(u_1) + \cdots + f(u_t) = 1$. Let $S' = (S - \{vu_1, \cdots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\}$. We also can define a SDF $g$ of $T' + S'$ as follows:

$$g(x) = \begin{cases} 
-1, & x = x_2 \\
1, & x = x_1, x_3 \\
f(x), & x \in V(T)
\end{cases}$$

So $\gamma_s(T' + S') \leq g(V(T' + S')) \leq \gamma_s(T + S) + 1 \leq \gamma_s(T) - 1 = \gamma_s(T') - 2$ implies that $R_s(T') \leq |S'| = |S| = R_s(T)$.

Case 2. $f(v) = -1$.

If $f(u) = 1$, then $f(u_1) + \cdots + f(u_t) = 1$. Let $S' = (S - \{vu_1, \cdots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\}$. Then we can extend $f$ to be a SDF $g$ of $T' + S'$ the same as the case $f(v) = 1$ and $f(u) = -1$ and so the result is valid.

If $f(u) = -1$, then $f(u_1) + \cdots + f(u_t) \geq 3$. Since $t \leq r \leq 3$, $t = 3$ and $f(u_1) = f(u_2) = f(u_3) = 1$. Let $S' = (S - \{vu_1\}) \cup \{x_1u_1\}$ and define

$$g(x) = \begin{cases} 
-1, & x = x_2 \\
1, & x = x_1, x_3 \\
f(x), & x \in V(T)
\end{cases}$$

Then $g$ is a SDF of $T' + S'$ and $\gamma_s(T' + S') \leq g(V(T' + S')) \leq \gamma_s(T + S) + 1 \leq \gamma_s(T) - 1 = \gamma_s(T') - 2$. This also implies that $R_s(T') \leq |S'| = |S| = R_s(T)$.

Theorem 5.5. For any tree $T$ of order $n \geq 2$, $R_s(T) \leq 2$. 

Proof. We prove the result by induction on the order of $T$. Since the result is true for $T = K_2$, we assume that $n \geq 3$. If $n = 3$, then, by Theorem 3.2, $R_s(T) = 1$ and the result is true. Now assume that $n \geq 4$ and the result is true for any tree with order less than $n$.

Let $T$ be a tree with $|V(T)| = n$ and let $f : V(T) \rightarrow \{-1,1\}$ be a minimum SDF of $T$. Then $f(V(T)) = \gamma_s(T)$ and $f(v) = 1$ for any $v \in L(T) \cup S(T)$ by Lemma 2.2.

Let $P_m = v_1v_2 \cdots v_m$ be a longest path of $T$.

If $d(v_2) \geq 3$, then there are at least two leaves adjacent with $v_2$ since $P_m$ is a longest path of $T$. Since $f(v_2) = 1$, $f[v_2] \geq 3 - 1 = 2$. By Lemma 5.3, $R_s(T) \leq 2$ and so the result is true. Hence, in the following, we suppose $d(v_2) = 2$.

Case 1. $d(v_3) \geq 3$.

Case 1.1. If $v_3$ is adjacent with a leaf $x$, then $f(x) = f(v_3) = 1$. So $f[v_2] \geq 3$. By Lemma 5.2, $R_s(T) = 1$.

Case 1.2. If $v_3$ is not adjacent with any leaf of $T$, since $P_m$ is a longest path of $T$, each neighbor of $v_3$ other than $v_4$ is a support vertex of $T$. Since $d(v_2) = 2$, we can assume that each component of $T - \{v_3\}$ not containing $v_4$ is isomorphic to $K_2$. If $f(v_3) = 1$, then $f[v_2] \geq 3$. By Lemma 5.2, $R_s(T) = 1$. Now we assume $f(v_3) = -1$.

Let $y_1y_2$ be a component of $T - \{v_3\}$ other than $v_1v_2$ with $y_2v_3 \in E(T)$. Let $S = \{v_1v_3, y_1v_3\}$. Define a function $g : V(T + S) \rightarrow \{-1,1\}$ as follows:

$$g(x) = \begin{cases} -1, & x = y_1, v_1 \\ 1, & x = v_3 \\ f(x), & \text{otherwise} \end{cases}$$

It is an easy task to check that $g[x] \geq 1$ for any vertex $x \in V(T + S)$ and hence $g$ is a SDF of $T + S$. So $\gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2$ which implies that $R_s(T) \leq |S| = 2$.

Case 2. $d(v_3) = 2$.

Case 2.1. If $f(v_3) = 1$, then $f[v_2] \geq 3$ and so $R_s(T) = 1$ by Lemma 5.2.

Case 2.2. If $f(v_3) = -1$, then, to guarantee $f[v_3] \geq 1$, $f(v_4)$ must be 1.

If $d(v_4) = 2$, then, to guarantee $f[v_4] \geq 1$, $f(v_5) = 1$. Let $T' = T - \{v_1, v_2, v_3\}$. By the inductive hypothesis, $R_s(T') \leq 2$. Since $\{v_1, v_2, v_3\}$ induce a path $P_3$, by Lemma 5.4, $R_s(T) = R_s(T' + P_3) \leq R_s(T') \leq 2$.

Now assume that $d(v_4) \geq 3$.

If $v_4$ is a support vertex and $w$ is a leaf adjacent with $v_4$, then $f(w) = f(v_4) = 1$. Let $S = \{v_1v_3, vw_3\}$. We can define a SDF $g$ of $T + S$ as follows:

$$g(x) = \begin{cases} -1, & x = w, v_1 \\ 1, & x = v_3 \\ f(x), & \text{otherwise} \end{cases}$$

So $\gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2$ implying that $R_s(T) \leq 2$. 


If $v_4$ is adjacent with a support vertex $y$ such that $N(y) - \{v_4\}$ are leaves of $T$, then $f(y) = 1$. Since $f(v_4) = 1$ and the value of any leaf assigned by $f$ is 1, $f[y] \geq 3$. By Lemma 5.2, we have $R_s(T) = 1$.

By the above proofs, we can assume that: (i) each component of $T - \{v_4\}$ not containing $v_5$ is isomorphic to $P_3$ with an end adjacent with $v_4$; (ii) the value of the vertex adjacent with $v_4$ assigned by $f$ is $-1$. By this assumption, to guarantee $f[v_4] \geq 1$, there is exactly one such component, that means $d(v_4) = 2$, contradicts with the assumption $d(v_4) \geq 3$.

**Remark 5.6.** The upper bound $R_s(T) \leq 2$ is sharp since $R(P_{3k+2}) = 2$, $k \geq 1$.

**References**


