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RESEARCH ARTICLE

Bondage number of mesh networks

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Abstract The bondage number $b(G)$ of a nonempty graph G is the smallest number of edges whose removal from G results in a graph with domination number greater than that of G. Denote $P_n \times P_m$ the Cartesian product of two paths P_n and P_m. This paper determines the exact values of $b(P_n \times P_2)$, $b(P_n \times P_3)$, and $b(P_n \times P_4)$ for $n \geqslant 2$.

Keywords Bondage number, dominating set, domination number, mesh network
MSC 05C25, 05C40, 05C12, 05C69

1 Introduction

Throughout this paper, for terminologies and notations not defined here, we refer the reader to [30,31]. A graph $G = (V, E)$ is considered as an undirected and simple graph, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.

A nonempty subset $D \subseteq V(G)$ is said to be a *dominating set* of G if every vertex in G is either in D or adjacent to a vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of all dominating sets in G. A dominating set D is called minimum if $|D| = \gamma(G)$. The *bondage number* $b(G)$ of a nonempty graph G is the minimum number of edges whose removal from G results in a graph with larger domination number, that is,

$$b(G) = \min\{|B| \colon B \subseteq E(G),\ \gamma(G - B) > \gamma(G)\}.$$

A nonempty subset $B \subseteq E(G)$ is said to be a *bondage set* of G if $\gamma(G - B) > \gamma(G)$.

The concept of the bondage number was proposed by Fink et al. [8] for an undirected graph and by Carlson and Develin [4] for a digraph. However, the first result on bondage numbers was obtained by Bauer et al. [1]. There

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are many research articles on the bondage number for undirected graphs and digraphs (see, for example, [1–4,7,8,10–18,20,21,23–29,32]). In particular, Hu and Xu [14] showed that the problem determining bondage number for general graphs is NP-hard.

Apart from its own theoretical interest, the study on the bondage number was also motivated by the increasing importance in the design and analysis of interconnection networks. It is well known that the topological structure of an interconnection network can be modeled by a connected graph whose vertices represent sites of the network and whose edges represent physical communication links. A minimum dominating set in the graph corresponds to a smallest set of sites selected in the network for some particular uses, such as placing transmitters. Such a set may not work when some communication links happen fault. Since the fault is possible in real world (hacking, experimental error, terrorism, etc.), one needs to consider it. What is the minimum number of faulty links which will make all minimum dominating sets of the original network not work any more? Such a minimum number is the bondage number, which measures the robustness of a network with respect to link failures, wherever a minimum dominating set is required for some applications.

Motivated by the above relevance of bondage number, one wants to know how to compute it for a network. However, this computation is generally difficult; no efficient algorithm has been proposed yet. Therefore, it is of significance to develop a technique to determine the bondage numbers for some special graphs or networks. However, the exact value of the bondage number has been determined for only a few classes of graphs, such as complete graphs, paths, cycles, and complete t-partition graphs (see, Fink et al. [8] for the undirected cases, Huang and Xu [15] and Zhang et al. [32] for the directed cases), trees (see, Bauer et al. [1], Hartnell and Rall [11], Hartnell et al. [10], Topp and Vestergaard [29], and Teschner [27]), and de Bruijn and Kautz digraphs (see, Huang and Xu [15]).

Let P_n and C_n be a path and a cycle of order n, respectively. For the Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2, Dunbar et al. [7] determined $b(C_n \times P_2)$ for $n \geqslant 3$, Sohn et al. [24] determined $b(C_n \times C_3)$ for $n \geqslant 4$, Kang et al. [20] determined $b(C_n \times C_4)$ for $n \geqslant 4$, Huang ang Xu [18] determined $b(C_{5i} \times C_{5j})$ for any positive integers i and j, Cao et al. [2] determined $b(C_n \times C_5)$ for $n \geqslant 5$ and $n \not\equiv 3 \pmod 5$. However, $b(C_n \times C_m)$ for $m \geqslant 6$ has been not determined yet.

The mesh $P_n \times P_m$ is a very famous network, and its domination number was determined for $1 \leqslant m \leqslant 6$ [5,6,19,22]. Moreover, Gonçalves et al. have determined the domination numbers of all $n \times m$ grid graphs in [9] very recently. However, its bondage number has not been not determined yet. $P_1 \times P_m$ is isomorphic to P_m, and $b(P_m)$ is determined. In this paper, we present the exact values of $b(P_n \times P_2)$, $b(P_n \times P_3)$, and $b(P_n \times P_4)$ for $n \geqslant 2$.

The rest of the paper is organized as follows. Section 2 presents some useful results. Section 3 determines $b(P_n \times P_2)$, Section 4 determines $b(P_n \times P_3)$, and

Section 5 determines $b(P_n \times P_4)$. Some remarks are given in Section 6, in which we propose a conjecture:

$$b(P_n \times P_m) \leqslant 2, \quad m \geqslant 5.$$

2 Preliminary results

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. The *Cartesian product* of G_1 and G_2 is an undirected graph, denoted by $G_1 \times G_2$, where $V(G_1 \times G_2) = V_1 \times V_2$, two distinct vertices x_1x_2 and y_1y_2, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$, are linked by an edge in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, such an edge is called a *vertical edge*, or $x_2 = y_2$ and $x_1y_1 \in E(G_1)$, such an edge is called a *horizontal edge*. It is clear, as a graphic operation, that the Cartesian product satisfies the commutative and associative law if we identify the isomorphic graphs.

Throughout this paper, the notation P_n denotes a path with the vertex set $\{1, 2, \ldots, n\}$. The (n, m)-mesh network, denoted by $G_{n,m}$, is defined as the Cartesian product $P_n \times P_m$, with the vertex set $\{u_{i,j} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$.

The graph shown in Fig. 1 is the $(4, 3)$-mesh network $G_{4,3}$. It is clear, as a graphic operation, $G_{n,m} \cong G_{m,n}$.

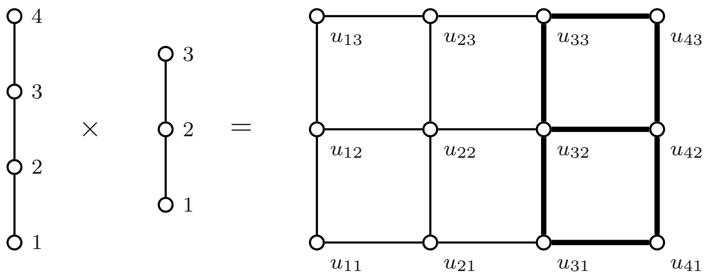

Fig. 1 $(4, 3)$-mesh network $G_{4,3} = P_4 \times P_3$

The following notation will be used in this paper. For a positive integer t with $t < n$, $G_{t,m}$ is a subgraph of $G_{n,m}$. Denote

$$H_{n-t,m} = G_{n,m} - G_{t,m},$$

that is, $H_{n-t,m}$ is a subgraph of $G_{n,m}$ induced by the set of vertices

$$\{u_{i,j} \mid t + 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}.$$

Clearly,

$$H_{n-t,m} \cong G_{n-t,m}.$$

The graph shown in Fig. 1 by heavy lines is a subgraph $H_{2,3}$ of $G_{4,3}$, where $n=4$, $t=2$ and $m=3$.

Note that both $G_{0,m}$ and $H_{n-n,m}$ are nominal graphs. For convenience of statements, we allow $G_{0,m}$ and $H_{n-n,m}$ to appear in this paper. If so, we specifically consider that their dominating sets are empty.

In Addition, let $Y_i=\{u_{i,j}: j=1,2,\ldots,m\}$ for each $i=1,2,\ldots,n$, called a set of vertical vertices of i in $G_{n,m}$.

We state some useful results on $\gamma(G_{n,m})$ to be used in this paper.

Lemma 1 [19,22] *Let P_n be a path of order $n\geqslant 1$, and C_m be a cycle of order $m\geqslant 3$. Then*

$$\gamma(G_{n,2})=\left\lceil\frac{n+1}{2}\right\rceil;\quad \gamma(G_{n,3})=n-\left\lfloor\frac{n-1}{4}\right\rfloor;$$

$$\gamma(G_{n,4})=\begin{cases} n+1, & n=1,2,3,5,6 \textit{ or } 9,\\ n, & \textit{otherwise};\end{cases}$$

$$\gamma(C_m\times C_3)=m-\left\lfloor\frac{m}{4}\right\rfloor.$$

Lemma 2 *Let D be a dominating set of $G_{n,m}$. Then*

$$\gamma(G_{i,m})\leqslant|D\cap V(G_{i+1,m})|,\quad \gamma(H_{n-i,m})\leqslant|D\cap V(H_{n-i+1,m})|$$

for $i=1,2,\ldots,n-1$ and $m\geqslant 2$.

Proof Since $H_{n-i,m}\cong G_{n-i,m}$, we only need to prove

$$\gamma(G_{i,m})\leqslant|D\cap V(G_{i+1,m})|.$$

Let

$$D'=D\cap V(G_{i+1,m}).$$

If $D'\cap Y_{i+1}=\emptyset$, then D' is a dominating set of $G_{i,m}$, and hence,

$$\gamma(G_{i,m})\leqslant|D'|.$$

Assume $D'\cap Y_{i+1}\neq\emptyset$ below. Let

$$B_i=\{j\mid u_{i+1,j}\in D'\}.$$

Then

$$D''=(D'\setminus Y_{i+1})\cup\{u_{i,j}\mid j\in B_i\}$$

is a dominating set of $G_{i,m}$ and $|D''|\leqslant|D'|$. Thus, we have

$$\gamma(G_{i,m})\leqslant|D''|\leqslant|D'|.$$

The lemma follows. $\square$

3 Bondage number of $G_{n,2}$

Theorem 1 *$b(G_{2,2}) = 3$, $b(G_{3,2}) = 2$, and $b(G_{n,2}) = 1$ if n is odd and $b(G_{n,2}) = 2$ if n is even for $n \geqslant 4$.*

Proof It is easy to verify that $b(G_{2,2}) = 3$ and $b(G_{3,2}) = 2$. In the following, consider $n \geqslant 4$. When n is odd, we consider the domination number of $G = G_{n,2} - u_{1,1}u_{1,2}$. Let D be a minimum dominating set of G. If $x \in D$ with $d_G(x) = 1$, then $D' = (D \setminus \{x\}) \cup \{y\}$ is also a dominating set, where y is the only neighbor of x. We can assume that D does not contain $u_{1,1}$ neither $u_{1,2}$ but contains $u_{2,1}$ and $u_{2,2}$. By Lemma 2,

$$|D| - 2 = |D \cap V(H_{n-2,2})| \geqslant \gamma(H_{n-3,2}).$$

Then by Lemma 1,

$$|D| \geqslant 2 + \gamma(H_{n-3,2}) = 2 + \left\lceil \frac{n-3+1}{2} \right\rceil = 1 + \gamma(G_{n,2}),$$

since n is odd, which yields $b(G_{n,2}) = 1$.

When n is even, we claim that

$$\gamma(G_{n,2}) = \gamma(G_{n,2} - e), \quad \forall\, e \in E(G_{n,2}).$$

To prove this claim, we first consider that e is a vertical edge, and let $e = u_{j,1}u_{j,2}$.

If j is even, then all the vertices $u_{i,1}$, $i \equiv 1 \pmod 4$, $u_{i,2}$, $i \equiv 3 \pmod 4$, $u_{n,1}$ if $n \equiv 0 \pmod 4$ or $u_{n,2}$ if $n \equiv 2 \pmod 4$, form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

If j is odd, then all the vertices $u_{i,1}$, $i \equiv 2 \pmod 4$, $u_{i,2}$, $i \equiv 0 \pmod 4$ and $u_{2,2}$ form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

Assume now that e is a horizontal edge. Without loss of generality, let $e = u_{j,1}u_{j+1,1}$.

If $j \equiv 2$ or $3 \pmod 4$, then all the vertices $u_{i,1}$, $i \equiv 1 \pmod 4$, $u_{i,2}$, $i \equiv 3 \pmod 4$, and $u_{n,1}$ form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

If $j \equiv 0$ or $1 \pmod 4$, then all the vertices $u_{i,2}$, $i \equiv 1 \pmod 4$, $u_{i,1}$, $i \equiv 3 \pmod 4$, and $u_{n,1}$ form a dominating set of $G_{n,2} - e$ with cardinality $\lceil \frac{n+1}{2} \rceil$.

Therefore, we have $b(G_{n,2}) \geqslant 2$. Next, we show that $b(G_{n,2}) \leqslant 2$. Let

$$e_1 = u_{2,1}u_{3,1}, \quad e_2 = u_{2,2}u_{3,2}, \quad G' = G_{n,2} - \{e_1, e_2\}.$$

Then G' consists of two connected components, one is $G_{2,2}$ and the other is $H_{n-2,2}$. By Lemma 1, we have

$$\gamma(G') = \gamma(G_{2,2}) + \gamma(H_{n-2,2}) = 2 + \left\lceil \frac{n-2+1}{2} \right\rceil = 1 + \gamma(G_{n,2}),$$

which implies $b(G_{n,2}) \leqslant 2$. Thus, $b(G_{n,2}) = 2$.

4 Bondage number of $G_{n,3}$

Proposition 1 [19] *A minimum dominating set D of $G_{n,3}$ is constructed as follows*:

$$D = \begin{cases} \{u_{i,2} \colon i \equiv 1 \pmod 4\} \cup \{u_{i,1}, u_{i,3} \colon i \equiv 3 \pmod 4\}, & n \text{ odd}, \\ \{u_{i,2} \colon i \equiv 1 \pmod 4\} \cup \{u_{i,1}, u_{i,3} \colon i \equiv 3 \pmod 4\} \cup \{u_{n,2}\}, & n \text{ even}. \end{cases}$$

Lemma 3 *For each $j = 1, 2, 3$ and $n \equiv 1, 2$ or $3 \pmod 4$, we have*

$$\gamma(G_{n,3} - u_{1,j}) \geqslant \gamma(G_{n,3}) = n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Proof It is easy to verify that the conclusion is true for $n = 1, 2, 3$. In the following, assume $n \geqslant 4$. Let $G = G_{n,3} - u_{1,j}$, and let D be a minimum dominating set of G. We only need to show

$$|D| \geqslant n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

If $(Y_1 - u_{1,j}) \cap D \neq \emptyset$, then D is a dominating set of $C_n \times C_3$. By Lemma 1,

$$|D| \geqslant \gamma(C_n \times C_3) = n - \left\lfloor \frac{n}{4} \right\rfloor = n - \left\lfloor \frac{n-1}{4} \right\rfloor$$

since $n \not\equiv 0 \pmod 4$.

If $(Y_1 - u_{1,j}) \cap D = \emptyset$, then $|Y_2 \cap D| \geqslant 2$. By Lemma 2,

$$|D \cap V(H_{n-2,3})| \geqslant \gamma(H_{n-3,3}).$$

By Lemma 1,

$$|D| \geqslant 2 + \gamma(H_{n-3,3}) = 2 + n - 3 - \left\lfloor \frac{n-3-1}{4} \right\rfloor = n - \left\lfloor \frac{n-1}{4} \right\rfloor,$$

as required. $\square$

Lemma 4 *For $n \equiv 0 \pmod 4$, we have*

$$\gamma(G_{n,3} - u_{1,1}) \geqslant \gamma(G_{n,3}) = n - \left\lfloor \frac{n-1}{4} \right\rfloor$$

Proof Let D be a minimum dominating set of $G_{n,3} - u_{1,1}$. We only need to prove

$$|D| \geqslant n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

It is easy to verify that the assertion is true for $n = 4$. In the following, we consider the case $n \geqslant 8$. We consider the following three cases, respectively.

Case 1 $u_{1,2} \in D$ or $u_{2,1} \in D$.

In this case, D is also a dominating set of $G_{n,3}$, and therefore,

$$|D| \geqslant \gamma(G_{n,3}) = n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Case 2 $u_{1,2}, u_{2,1} \notin D$ and $u_{1,3} \in D$.

In this case, $D \setminus \{u_{1,3}\}$ is a dominating set of $H_{n-1,3}$ or $H_{n-1,3} - u_{2,3}$. By Lemma 3,

$$|D \setminus \{u_{1,3}\}| \geqslant n - 1 - \left\lfloor \frac{n-1-1}{4} \right\rfloor,$$

and therefore,

$$|D| \geqslant n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Case 3 $u_{1,2}, u_{2,1} \notin D$ and $u_{1,3} \notin D$.

In this case, $u_{2,2}, u_{2,3} \in D$. We prove the conclusion by two subcases.

Subcase 3.1 $Y_3 \cap D \neq \emptyset$.

Then $D \setminus \{u_{2,2}, u_{2,3}\}$ is a dominating set of $H_{n-2,3}$ or $H_{n-2,3} - u_{3,1}$ or $H_{n-2,3} - u_{3,3}$. By Lemma 3,

$$|D \setminus \{u_{2,2}, u_{2,3}\}| \geqslant n - 2 - \left\lfloor \frac{n-2-1}{4} \right\rfloor.$$

Thus,

$$|D| \geqslant n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Subcase 3.2 $Y_3 \cap D = \emptyset$.

Then $u_{4,1} \in D$.

If $u_{4,2} \in D$ or $u_{4,3} \in D$, then $D \setminus \{u_{2,2}, u_{2,3}\}$ is a dominating set of $H_{n-2,3}$ or $H_{n-2,3} - u_{3,2}$ or $H_{n-2,3} - u_{3,3}$. By Lemma 3,

$$|D \setminus \{u_{2,2}, u_{2,3}\}| \geqslant n - 2 - \left\lfloor \frac{n-2-1}{4} \right\rfloor.$$

Thus,

$$|D| \geqslant n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

Next, assume $u_{4,2}, u_{4,3} \notin D$. Then $u_{5,3} \in D$. If $u_{5,1} \in D$ or $u_{5,2} \in D$, then $D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}\}$ is a dominating set of $H_{n-4,3}$, and hence,

$$|D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}\}| \geqslant n - 4 - \left\lfloor \frac{n-4-1}{4} \right\rfloor.$$

Thus,

$$|D| \geqslant n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

If $u_{5,1}, u_{5,2} \notin D$, then $D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}, u_{5,3}\}$ is a dominating set of $H_{n-5,3}$ or $H_{n-5,3} - u_{6,3}$. By Lemma 3,

$$|D \setminus \{u_{2,2}, u_{2,3}, u_{4,1}, u_{5,3}\}| \geqslant n - 5 - \left\lfloor \frac{n-5-1}{4} \right\rfloor.$$

Thus,

$$|D| \geqslant n - \left\lfloor \frac{n-1}{4} \right\rfloor.$$

The lemma follows. □

Corollary 1 $b(G_{n,3}) \leqslant 2$.

Proof By Lemmas 3 and 4, we have

$$\gamma(G_{n,3} - \{u_{1,1}u_{2,1}, u_{1,1}u_{1,2}\}) > \gamma(G_{n,3}).$$ □

Lemma 5 $b(G_{n,3}) = 1$ *for* $n \equiv 1$ *or* $2 \pmod 4$ *and* $n \geqslant 4$.

Proof Let D be a minimum dominating set of $G_{n,3} - u_{3,1}u_{4,1}$. We only need to prove that $|D| \geqslant 1 + \gamma(G_{n,3})$ by considering the following three cases, respectively.

Case 1 $u_{3,2}, u_{3,3} \in D$.

In this case, $|V(G_{3,3}) \cap D| = 4$. By Lemma 2,

$$|D \cap V(H_{n-3,3})| \geqslant \gamma(H_{n-4,3}).$$

By Lemma 1,

$$|D| \geqslant 4 + \gamma(H_{n-4,3}) = 4 + n - 4 - \left\lfloor \frac{n-4-1}{4} \right\rfloor = 1 + \gamma(G_{n,3}).$$

Case 2 Exactly one of $u_{3,2}$ or $u_{3,3} \in D$.

In this case, $|V(G_{3,3}) \cap D| = 3$. Then $D' = D \setminus V(G_{3,3})$ is a dominating set of $H_{n-3,3}$ or $H_{n-3,3} - u_{4,2}$ or $H_{n-3,3} - u_{4,3}$. By Lemma 3,

$$|D| = 3 + |D'| \geqslant 3 + n - 3 - \left\lfloor \frac{n-3-1}{4} \right\rfloor = n + 1 - \left\lfloor \frac{n-1}{4} \right\rfloor = 1 + \gamma(G_{n,3}).$$

Case 3 $u_{3,2}, u_{3,3} \notin D$.

In this case, $|V(G_{3,3}) \cap D| = 2$ or $|V(G_{3,3}) \cap D| = 3$.

If $|V(G_{3,3}) \cap D| = 3$, then $D \setminus V(G_{3,3})$ is a dominating set of $H_{n-3,3}$. By Lemma 1,

$$|D| \geqslant 3 + \gamma(H_{n-3,3}) = 3 + n - 3 - \left\lfloor \frac{n-3-1}{4} \right\rfloor = 1 + \gamma(G_{n,3}).$$

If $|V(G_{3,3}) \cap D| = 2$, then $V(G_{3,3}) \cap D = \{u_{1,3}, u_{2,1}\}$ and $D \setminus V(G_{3,3})$ is a dominating set of $H_{n-2,3} - u_{3,1}$. By Lemma 3 or 4,

$$|D| \geqslant 2 + n - 2 - \left\lfloor \frac{n-2-1}{4} \right\rfloor = n + 1 - \left\lfloor \frac{n-1}{4} \right\rfloor = 1 + \gamma(G_{n,3}).$$

The lemma follows. □

Lemma 6 $b(G_{n,3}) \geqslant 2$ *for* $n \equiv 0 \pmod 4$.

Proof By Proposition 1,

$$D = \{u_{i,2}\colon i \equiv 1 \pmod 4\} \cup \{u_{i,1}, u_{i,3}\colon i \equiv 3 \pmod 4\} \cup \{u_{n,2}\}$$

is a minimum dominating set, and by the symmetry of $G_{n,3}$,

$$D' = \{u_{i,2}\colon i \equiv 0 \pmod 4\} \cup \{u_{i,1}, u_{i,3}\colon i \equiv 2 \pmod 4\} \cup \{u_{1,2}\}$$

is also a minimum dominating set. It is clear that if we delete any vertical edge in $G_{n,3}$ or any horizontal edge $u_{i,1}u_{i+1,1}$ and $u_{i,3}u_{i+1,3}$ where $i \equiv 0, 1$ or $3 \pmod 4$ or any horizontal edge $u_{i,2}u_{i+1,2}$ where $i \equiv 1, 2$ or $3 \pmod 4$, D or D' is also a domination set. Next, we consider the domination number of $G_{n,3} - e$, where e is any other edge.

Let $e = u_{i,1}u_{i+1,1}$ or $e = u_{i,3}u_{i+1,3}$ where $i \equiv 2 \pmod 4$, or $e = u_{i,2}u_{i+1,2}$ where $i \equiv 0 \pmod 4$. Then

$$D'' = \{u_{i,1}, u_{i,3}\colon i \equiv 1 \pmod 4\} \cup \{u_{i,2}\colon i \equiv 3 \pmod 4\} \cup \{u_{n,2}\}$$

is a dominating set of $G - e$ with cardinality $n - \lfloor \frac{n-1}{4} \rfloor$. By Lemma 1,

$$|D''| = \gamma(G_{n,3}).$$

From the above discussion, $\gamma(G_{n,3} - e) = \gamma(G_{n,3})$ for any edge $e \in E(G_{n,3})$. Thus, $b(G_{n,3}) \geqslant 2$. □

Lemma 7 $b(G_{n,3}) \geqslant 2$ *for* $n \equiv 3 \pmod 4$.

Proof By Proposition 1,

$$D = \{u_{i,2}\colon i \equiv 1 \pmod 4\} \cup \{u_{i,1}, u_{i,3}\colon i \equiv 3 \pmod 4\}$$

is a minimum dominating set, and by the symmetry of $G_{n,3}$,

$$D' = \{u_{i,2}\colon i \equiv 3 \pmod 4\} \cup \{u_{i,1}, u_{i,3}\colon i \equiv 1 \pmod 4\}$$

is also a minimum dominating set. It is clear that if we delete any edge from $G_{n,3}$, D or D' is also a dominating set. Thus, $b(G_{n,3}) \geqslant 2$. □

Summing up the above results, we have the following theorem, immediately.

Theorem 2 *For* $n \geqslant 3$, *we have*

$$b(G_{n,3}) = \begin{cases} 1, & n \equiv 1 \text{ or } 2 \pmod 4, \\ 2, & n \equiv 0 \text{ or } 3 \pmod 4. \end{cases}$$

5 Bondage number of $G_{n,4}$

In this section, let $A = \{1, 2, 3, 5, 6, 9\}$.

Lemma 8 *Let D be a minimum dominating set of $G_{n,4}$. Then*

$$1 \leqslant |Y_1 \cap D| \leqslant 2, \quad 1 \leqslant |Y_n \cap D| \leqslant 2, \quad n \notin A.$$

Proof By Lemma 1, $|D| = n$. By the symmetry of $G_{n,4}$, we only need to prove that $1 \leqslant |Y_1 \cap D| \leqslant 2$. By contradiction. Suppose $|Y_1 \cap D| = 0$ or $|Y_1 \cap D| \geqslant 3$.

If $|Y_1 \cap D| = 0$, then $|Y_2 \cap D| = 4$. By Lemma 2,

$$|D \cap V(H_{n-2,4})| \geqslant \gamma(H_{n-3,4}).$$

By Lemma 1,

$$|D| \geqslant 4 + \gamma(H_{n-3,4}) \geqslant 4 + n - 3 = n + 1,$$

a contradiction with $|D| = n$. Thus, $|Y_1 \cap D| \geqslant 1$.

Assume now $|Y_1 \cap D| \geqslant 3$. By Lemma 2,

$$|D \cap V(H_{n-1,4})| \geqslant \gamma(H_{n-2,4}).$$

By Lemma 1,

$$|D| \geqslant 3 + \gamma(H_{n-2,4}) \geqslant 3 + n - 2 = n + 1,$$

a contradiction with $|D| = n$. Thus, $|Y_1 \cap D| \leqslant 2$. □

Lemma 9 *Let D be a minimum dominating set of $G_{n,4}$. Then $|Y_1 \cap D| = 1$ and*

$$|Y_n \cap D| = 1, \quad n \in \{4, 7, 8, 10, 11\}.$$

Proof By the symmetry of $G_{n,4}$ and Lemma 8, we only need to prove $|Y_1 \cap D| \neq 2$. Suppose, to the contrary, that there exists a minimum dominating set D of $G_{n,4}$ such that $|Y_1 \cap D| = 2$.

If $n \neq 10$ then, by Lemma 2,

$$|D \cap V(H_{n-1,4})| \geqslant \gamma(H_{n-2,4}).$$

By Lemma 1,

$$|D| \geqslant 2 + \gamma(H_{n-2,4}) \geqslant 2 + n - 1 = n + 1,$$

a contradiction with $|D| = n$.

Now, assume $n = 10$. Let $D' = D \setminus Y_1$. If $Y_2 \cap D \neq \emptyset$, then there exists a vertex $u_{2,j}$ such that $D' \cup \{u_{2,j}\}$ is a dominating set of $H_{n-1,4}$. By Lemma 1,

$$|D| = 2 + |D'| \geqslant 2 + \gamma(G_{9,4}) - 1 = 11,$$

a contradiction with $|D| = 10$. Next, we assume that $Y_2 \cap D = \emptyset$. Then $|Y_3 \cap D| \geqslant 2$. By Lemma 2,

$$|D \cap V(H_{n-3,4})| \geqslant \gamma(H_{n-4,4}).$$

By Lemma 1,

$$|D| \geqslant 4 + \gamma(H_{n-4,4}) = 4 + 7 = 11,$$

a contradiction with $|D| = 10$. The lemma follows. □

Lemma 10 *Let D be a minimum dominating set of $G_{n,4}$. Then*

$$|Y_1 \cap D| = 1, \quad |Y_n \cap D| = 1, \quad n \notin A.$$

Proof By the symmetry of $G_{n,4}$ and Lemma 8, we only need to prove $|Y_1 \cap D| \neq 2$. By Lemma 9, the statement is true for $n \in \{4, 7, 8, 10, 11\}$. We proceed by induction on $n \geqslant 12$.

Suppose that the assertion is true for any integer k with $10 \leqslant k < n$. Suppose, to the contrary, that there exists a minimum dominating set D of $G_{n,4}$ such that $|Y_1 \cap D| = 2$. If $Y_2 \cap D = \emptyset$, then $D' = D \setminus Y_1$ is a dominating set of $H_{n-2,4}$ and $|Y_3 \cap D'| \geqslant 2$. By the induction hypothesis, D' is not a minimum dominating set of $H_{n-2,4}$, and hence,

$$|D'| \geqslant \gamma(H_{n-2,4}) + 1 \geqslant n - 1$$

by Lemma 1. Then

$$|D| = 2 + |D'| \geqslant n + 1,$$

a contradiction with $|D| = n$.

If $Y_2 \cap D \neq \emptyset$, there exists a vertex $u_{2,j}$ such that

$$D'' = (D \setminus Y_1) \cup \{u_{2,j}\}$$

is a dominating set of $H_{n-1,4}$ and $|Y_2 \cap D''| \geqslant 2$. By the induction hypothesis, D'' is not a minimum dominating set of $H_{n-1,4}$, and hence,

$$|D''| \geqslant \gamma(H_{n-1,4}) + 1 \geqslant n.$$

Then

$$|D| \geqslant 2 + |D''| - 1 \geqslant n + 1,$$

a contradiction with $|D| = n$. The lemma follows. □

Theorem 3

$$b(G_{5,4}) = b(G_{9,4}) = 3, \quad b(G_{6,4}) = 2, \quad b(G_{n,4}) = 1, \quad n \notin A.$$

Proof By a careful case analysis, we can deduce that

$$b(G_{5,4}) = b(G_{9,4}) = 3, \quad b(G_{6,4}) = 2.$$

Here, we only prove $b(G_{n,4}) = 1$ for $n \notin A$. Then $n \geqslant 4$. Let D be a minimum dominating set of $G_{n,4} - u_{1,2}u_{1,3}$. By Lemma 1, we only need to show that $|D| \geqslant n + 1$. We prove the conclusion by considering the following three cases, respectively.

Case 1 $|Y_1 \cap D| = 0$.

In this case, $|Y_2 \cap D| = 4$. By Lemma 2,

$$|D \cap V(H_{n-2,4})| \geqslant \gamma(H_{n-3,4}).$$

Thus,

$$|D| \geqslant 4+\gamma(H_{n-3,4}) \geqslant n+1.$$

Case 2 $|Y_1 \cap D| \geqslant 2$.

In this case, D is a dominating set of $G_{n,4}$ with $|Y_1 \cap D| \geqslant 2$. By Lemma 10, D is not a minimum dominating set of $G_{n,4}$, and hence, $|D| \geqslant n+1$ by Lemma 1.

Case 3 $|Y_1 \cap D| = 1$.

Without loss of generality, let $u_{1,j_0} \in D$ and $j_0 \leqslant 1$. Then $u_{2,3}, u_{2,4} \in D$, and hence, $|Y_2 \cap D| \geqslant 2$.

Let $D' = D \setminus \{u_{1,j_0}\}$. If $j_0 = 2$, or $|Y_2 \cap D| \geqslant 3$, or $j_0 = 1$ and $u_{31} \in D'$, then D' is a dominating set of $H_{n-1,4}$ and let $D'' = D'$. Assume now $j_0 = 1$, $u_{3,1} \notin D$, and $Y_2 \cap D = \{u_{2,3}, u_{2,4}\}$. If $u_{3,2}$ or $u_{3,3}$ or $u_{3,4}$ belongs to D, then $D'' = (D' \setminus \{u_{2,3}\}) \cup \{u_{2,2}\}$ is a dominating set of $H_{n-1,4}$ with $|Y_2 \cap D''| \geqslant 2$.

If $n \in \{4, 7, 10\}$, then

$$|D''| \geqslant \gamma(H_{n-1,4}) = n$$

by Lemma 1. If $n \notin \{4, 7, 10\}$, then D'' is not a minimum dominating set of $H_{n-1,4}$ by Lemma 10. By Lemma 1,

$$|D''| \geqslant \gamma(H_{n-1,4}) + 1 = n.$$

Thus,

$$|D| \geqslant |D''| + 1 \geqslant n+1.$$

In the following, assume $j_0 = 1$, $u_{31} \notin D$, $Y_2 \cap D = \{u_{2,3}, u_{2,4}\}$, and $u_{3,2}, u_{3,3}, u_{3,4} \notin D$. Then $u_{4,1}, u_{4,2}$ should be in D to dominate u_{31} and u_{32}, and $D''' = D \setminus \{u_{1,1}, u_{2,3}, u_{2,4}\}$ is a dominating set of $H_{n-3,4}$ with $|Y_3 \cap D'''| \geqslant 2$.

If $n \in \{4, 8, 12\}$, then

$$|D'''| \geqslant \gamma(H_{n-3,4}) = n-2$$

by Lemma 1. If $n \notin \{4, 8, 12\}$, then D''' is not a minimum dominating set of $H_{n-3,4}$ by Lemma 10. Therefore,

$$|D'''| \geqslant \gamma(H_{n-3,4}) + 1 = n-2$$

by Lemma 1. Thus,

$$|D| \geqslant 3 + |D'''| \geqslant n+1.$$

The theorem follows. □

6 Remarks

Through determining the bondage number of $G_{n,m}$ for $2 \leqslant m \leqslant 4$, we find that if we delete the vertex $u_{1,1}$, the domination number is invariable. If m increases, the effect of $u_{1,1}$ for the domination number will be smaller and smaller in view of probability. Therefore, we expect that

$$\gamma(G_{n,m} - u_{1,1}) = \gamma(G_{n,m}), \quad m \geqslant 5,$$

and we give the following conjecture.

Conjecture 1 $b(G_{n,m}) \leqslant 2$ for $m \geqslant 5$.

In our method, determining the bondage number of a graph strongly depends on the domination number of the graph. Even the exact values of the domination number of some graphs have been determined, determining its bondage number is also very difficult. For example, the domination number of $G_{n,m}$ for $m = 5$ or 6 has been determined [5,6], we cannot determined its bondage number in our method since there are too much cases to consider. Thus, if we want to determine the bondage number of $G_{n,m}$ or to solve the Conjecture 1, we need some new method. It is what we further work on.

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