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Generalized Measures of Edge Fault Tolerance in (n,k)-star Graphs

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Abstract: This paper considers a kind of generalized measure $\lambda_s^{(h)}$ of fault tolerance in the (n,k)-star graph $S_{n,k}$ for $2\leqslant k\leqslant n-1$ and $0\leqslant h\leqslant n-k$, and determines $\lambda_s^{(h)}(S_{n,k})=\min\{(n-h-1)(h+1),(n-k+1)(k-1)\}$, which implies that at least $\min\{(n-k+1)(k-1),(n-h-1)(h+1)\}$ edges of $S_{n,k}$ have to remove to get a disconnected graph that contains no vertices of degree less than h. This result shows that the (n,k)-star graph is robust when it is used to model the topological structure of a large-scale parallel processing system.

Keywords: Combinatorics, fault-tolerant analysis, (n,k)-star graphs, edge-connectivity, h-super edge-connectivity

I. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network. The connectivity $\lambda(G)$ of a graph G is an important measurement for fault-tolerance of the network, and the larger $\lambda(G)$ is, the more reliable the network is.

A subset of vertices B of a connected graph G is called an *edge-cut* if $G-B$ is disconnected. The *edge connectivity* $\lambda(G)$ of G is defined as the minimum cardinality over all edge-cuts of G. Because λ has many shortcomings, one proposes the concept of the h-super edge connectivity of G, which can measure fault tolerance of an interconnection network more accurately than the classical connectivity λ.

A subset of vertices B of a connected graph G is called an *h-super edge-cut,* or *h-edge-cut* for short, if $G-B$ is disconnected and has the minimum degree at least h. The *h-super edge-connectivity* of G, denoted by $\lambda_s^{(h)}(G)$, is defined as the minimum cardinality over all h-edge-cuts of G. It is clear that, if $\lambda_s^{(h)}(G)$ exists, then

$$\lambda(G)=\lambda_s^{(0)}(G)\leqslant\lambda_s^{(1)}(G)\leqslant\lambda_s^{(2)}(G)\leqslant\cdots\leqslant\lambda_s^{(h-1)}(G)\leqslant\lambda_s^{(h)}(G).$$

For any graph G and integer h, determining $\lambda_s^{(h)}(G)$is quite difficult. In fact, the existence of $\lambda_s^{(h)}(G)$ is an open problem so far when $h\geqslant 1$. Some results have been obtained on $\lambda_s^{(h)}(G)$ for particular classes of graphs and small h's (see Section 16.7 in [5]).

This paper is concerned about $\lambda_s^{(h)}$ for the (n,k)-star graph $S_{n,k}$. For the h-super connectivity, several authors have done some work. For $k=n-1$, $S_{n,n-1}$is isomorphic to a star graph S_n. Akers and Krishnamurthy [1] determined $\lambda(S_n)=n-1$ for $n\geqslant 2$ and $\lambda_s^{(1)}(S_n)=2n-4$ for $n\geqslant 3$. In this paper, we show the following result.

Theorem: *If $2\leqslant k\leqslant n-1$ and $0\leqslant h\leqslant n-k$, then*

$$\lambda_s^{(h)}(S_{n,k}) = \begin{cases} (n-h-1)(h+1) & for\ h \leqslant k-2\ and\ h \leqslant \frac{n}{2}-1, \\ (n-k+1)(k-1) & otherwise. \end{cases}$$

This result implies that at least $\min\{(n-k+1)(k-1),(n-h-1)(h+1)\}$ edges of $S_{n,k}$ have to remove to get a disconnected graph that contains no vertices of degree less than h. The proof of this result is in Section 3. In Section 2, we recall the structure of $S_{n,k}$ and some lemmas used in our proofs.

II. Definitions and Lemmas

For integers n and k with $1 \leqslant k \leqslant n-1$, let $I_n = \{1,2,\ldots,n\}$ and $P(n,k)=\{p_1p_2\ldots p_k:\ p_i \in I_n, p_i \neq p_j,\ 1 \leqslant i \neq j \leqslant k\}$, the set of k-permutations on I_n. Clearly, $|P(n,k)| = n!/(n-k)!$.

Definition 2.1: *The (n,k)-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$. The adjacency is defined as follows: a vertex $p=p_1p_2\ldots p_i\ldots p_k$ is adjacent to a vertex*

(a) $p_ip_2\cdots p_{i-1}p_1p_{i+1}\cdots p_k$*, where $2 \leqslant i \leqslant k$ (swap p_1 with p_i).*

(b) $\alpha p_2p_3\cdots p_k$*, where $\alpha \in I_n \setminus \{p_i:\ 1 \leqslant i \leqslant k\}$ (replace p_1 by α).*

The vertices of type (a) are referred to as *swap-neighbors* of p and the edges between them are referred to as *swap-edge* or *i-edges*. The vertices of type (b) are referred to as *unswap-neighbors* of p and the edges between them are referred to as *unswap-edges*. Clearly, every vertex in $S_{n,k}$ has $k-1$ swap-neighbors and $n-k$ unswap-neighbors. Usually, if $x=p_1p_2\ldots p_k$ is a vertex in $S_{n,k}$, we call p_i the *i-th bit* for each $i \in I_k$.

The (n,k)-star graph $S_{n,k}$ is proposed by Chiang and Chen [4]. Some nice properties of $S_{n,k}$ are compiled by Cheng and Lipman (see Theorem 1 in [2]).

Lemma 2.2: *$S_{n,k}$ is $(n-1)$-regular $(n-1)$-connected.*

Lemma 2.3: *For any $\alpha = p_1p_2\cdots p_{k-1} \in P(n,k-1)$ $(k \geqslant 2)$, let $V_\alpha = \{p\alpha:\ p \in I_n \setminus \{p_i:\ i \in I_{k-1}\}\}$. Then the subgraph of $S_{n,k}$ induced by V_α is a complete graph of order $n-k+1$, denoted by K^α_{n-k+1}.*

Let $S^{t:i}_{n-1,k-1}$ denote the subgraph of $S_{n,k}$ induced by vertices with the t-th bit i for $2 \leqslant t \leqslant k$. The following lemma is a slight modification of the result of Chiang and Chen [4].

Lemma 2.4: *For a fixed integer t with $2 \leqslant t \leqslant k$, $S_{n,k}$ can be decomposed into n subgraphs $S^{t:i}_{n-1,k-1}$, which is isomorphic to $S_{n-1,k-1}$, for each $i \in I_n$. Moreover, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{t:i}_{n-1,k-1}$ and $S^{t:j}_{n-1,k-1}$ for any $i,j \in I_n$ with $i \neq j$.*

Since $S_{n,1} \cong K_n$, we only consider the case of $k \geqslant 2$ in the following discussion.

Lemma 2.5: *If $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$, then*

$$\lambda_s^{(h)}(S_{n,k}) \leqslant \begin{cases} (n-h-1)(h+1) & for\ \ h \leqslant \frac{n}{2}-1, \\ (n-k+1)(k-1) & otherwise. \end{cases}$$

Proof: By our hypothesis of $2 \leqslant k \leqslant n-1$, for any $\alpha \in P(n,k-1)$, we can choose a subset $X \subseteq V(K^\alpha_{n-k+1})$ such that $|X| = h+1$. Then the subgraph of K^α_{n-k+1} induced by X is a complete graph K_{h+1}. Let B be the set of incident edges with and not within X. Since $S_{n,k}$ is $(n-1)$-regular and K_{h+1} is h-regular, we have that

$$|B| = (n-h-1)(h+1).$$

Clearly, B is an edge-cut of $S_{n,k}$. Let x be any vertex in $S_{n,k}-X$, and $d(x)$ denote the number of edges incident with x in $S_{n,k}-X$. In order to prove that B is an h-edge-cut, we only need to show $d(x) \geqslant h$. Note that X is contained in $S^i_{n-1,k-1}$ and edges between $S^i_{n-1,k-1}$ and $S^j_{n-1,k-1}$ are independent for any $i,j \in I_n$ with $i \neq j$ by

Lemma 2.4. If x is in $S^i_{n-1,k-1} - K^{\alpha}_{n-k+1}$ or is in $S^j_{n-1,k-1}$ with $i \neq j$, then $d(x) \geqslant n-2 \geqslant n-k \geqslant h$. For $x \in V(K^{\alpha}_{n-k+1} - X)$, if exists, then $d(x) = n-1-|X| = n-h-2 \geqslant h$ for $h \leqslant \frac{n}{2}-1$. Therefore, B is an h-edge-cut of $S_{n,k}$, and so

$$\lambda_s^{(h)}(S_{n,k}) \leqslant |B| = (n-h-1)(h+1) \quad \text{for } h \leqslant \frac{n}{2} - 1.$$

For $h \geqslant \frac{n}{2}$, we choose $X = V(K^{\alpha}_{n-k+1})$. Then $|B| = (n-k+1)(k-1)$. For any x in $S^i_{n-1,k-1} - X$ or $S^j_{n-1,k-1}$ with $i \neq j$, we have $d(x) \geqslant n-2 \geqslant n-k \geqslant h$. Thus, B is an h-edge-cut of $S_{n,k}$, and so

$$\lambda_s^{(h)}(S_{n,k}) \leqslant |B| = (n-k+1)(k-1) \quad \text{for } h \geqslant \frac{n}{2}.$$

The lemma follows.

Corollary 2.6: $\lambda_s^{(h)}(S_{n,2}) = n-1$ *for* $0 \leqslant h \leqslant n-2$.

Proof: On the one hand, $\lambda_s^{(h)}(S_{n,2}) \leqslant n-1$ by Lemma 2.5 when $k=2$. On the other hand, $\lambda_s^{(h)}(S_{n,2}) \geq \lambda(S_{n,2}) = n-1$ by Lemma 2.2.

The following lemma shows the relations between $(n-h-1)(h+1)$ and $(n-k+1)(k-1)$.

Lemma 2.7: *For* $2 \leqslant k \leqslant n-1, 0 \leqslant h \leqslant n-k$*, let*

$$\psi(h,k) = \min\{(n-h-1)(h+1), (n-k+1)(k-1)\}.$$

If $h \leqslant \frac{n}{2} - 1$*, then*

$$\psi(h,k) = \begin{cases} (n-h-1)(h+1) & \text{if } 0 \leqslant h \leqslant k-2; \\ (n-k+1)(k-1) & \text{if } h \geqslant k-1. \end{cases}$$

Proof: Let $f(x) = (n-x)x$, then $\psi(h,k) = \min\{f(h+1), f(k-1)\}$. It can be easily checked that $f(x)$ is a convex function on the interval $[0,n]$, the maximum value is reached at $x = \frac{n}{2}$. Thus, $f(x)$ is an increasing function on the interval $[0, \frac{n}{2}]$.

If $0 \leqslant h \leqslant k-2$, then $h+1 \leqslant k-1$. Since $h \leqslant n-k$, $h+1 \leqslant n-k+1$ and $\min\{k-1, n-k+1\} \leqslant \frac{n}{2}$. Thus, when $h \leqslant \frac{n}{2}-1$, $f(h+1) \leqslant f(k-1) = f(n-k+1)$, and so $\psi(h,k) = f(h+1) = (n-h-1)(h+1)$.

If $h \geqslant k-1$, then $k-1 < h+1 \leqslant \frac{n}{2}$, $f(k-1) < f(h+1)$, so $\psi(n,k) = f(k-1) = (n-k+1)(k-1)$.

The lemma follows.

To state and prove our main results, we need some notations. Let B be a minimum h-edge-cut of $S_{n,k}$. Clearly, $S_{n,k} - B$ has exactly two connected components. Let X and Y be two vertex-set of two connected components of $S_{n,k} - B$. For a fixed $t \in I_k \setminus \{1\}$ and any $i \in I_n$, let

$$\begin{aligned} &X_i = X \cap V(S^{t:i}_{n-1,k-1}), \\ &Y_i = Y \cap V(S^{t:i}_{n-1,k-1}), \\ &B_i = B \cap E(S^{t:i}_{n-1,k-1}) \text{ and} \\ &B_{ij} = B \cap E(S^{t:i}_{n-1,k-1}, S^{t:j}_{n-1,k-1}), \end{aligned}$$

and let

$$\begin{aligned} &J = \{i \in I_n : \ X_i \neq \emptyset\}, \\ &J' = \{i \in J : \ Y_i \neq \emptyset\} \ \text{ and} \\ &T = \{i \in I_n : \ Y_i \neq \emptyset\}. \end{aligned}$$

Lemma 2.8: *Let* B *be a minimum* h*-edge-cut of* $S_{n,k}$ *and* X *be the vertex-set of a connected component of* $S_{n,k} - B$*. If* $3 \leqslant k \leqslant n-1$ *and* $1 \leqslant h \leqslant n-k$ *then, for any* $t \in I_k \setminus \{1\}$*,*

(a) B_i *is an* $(h-1)$*-edge-cut of* $S^{t:i}_{n-1,k-1}$ *for any* $i \in J'$*,*

(b) $\lambda_s^{(h)}(S_{n,k}) \geqslant |J'|\ \lambda_s^{(h-1)}(S_{n-1,k-1})$.

Proof. (a) By the definition of J', B_i is an edge-cut of $S^{t:i}_{n-1,k-1}$for any $i \in J'$. For any vertex x in $S^{t:i}_{n-1,k-1} - B_i$, since x has degree at least h in $S_{n,k} - S$ and has exactly one neighbor outsider $S^{t:i}_{n-1,k-1}$, x has degree at least $h-1$ in $S^{t:i}_{n,k} - B_i$. This fact shows that B_i is an $(h-1)$-edge-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J'$.

(b) By the assertion (a), we have $|B_i| \geqslant \lambda_s^{(h-1)}(S_{n-1,k-1})$, and so

$$\lambda_s^{(h)}(S_{n,k}) = |B| \geqslant \sum_{i \in J'} |B_i| \geqslant |J'|\lambda_s^{(h-1)}(S_{n-1,k-1}).$$

The lemma follows.

III. Proof of Theorem

By Lemma 2.5 and Lemma 2.7, we only need to prove that, for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$,

$$\lambda_s^{(h)}(S_{n,k}) \geqslant \begin{cases} (n-h-1)(h+1) & \text{for } h \leqslant k-2 \text{ and } h \leqslant \frac{n}{2}-1, \\ (n-k+1)(k-1) & \text{otherwise.} \end{cases}$$

Let $\omega(h,k) = \max\{(n-h-1)(h+1), (n-k+1)(k-1)\}$.

We proceed by induction on $k \geqslant 2$ and $h \geqslant 0$. The inequality is true for $k=2$ and any h with $0 \leqslant h \leqslant n-2$ by Corollary 2.6. The inequality is also true for $h=0$ and any k with $2 \leqslant k \leqslant n-1$ since $\lambda_s^{(0)}(S_{n,k}) = \lambda(S_{n,k}) = n-1$. Assume the induction hypothesis for $k-1$ with $k \geqslant 3$ and for $h-1$ with $h \geqslant 1$, that is,

$$\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant \begin{cases} (n-h)h & \text{for } h \leqslant k-3 \text{ and } h \leqslant \frac{n-1}{2}, \\ (n-k+2)(k-2) & \text{otherwise.} \end{cases}$$

Let B be a minimum h-edge-cut of $S_{n,k}$ and X be the vertex-set of a minimum connected component of $S_{n,k} - B$. By Lemma 2.5, we have

$$|B| \leqslant \omega(h,k). \tag{1}$$

Use notations defined in Section II. Choose $t \in I_k \setminus \{1\}$ such that $|J|$ is as large as possible. For each $i \in I_n$, we write $S^i_{n-1,k-1}$ for $S^{t:i}_{n-1,k-1}$ for short.

We first show $|J| = 1$. Suppose to the contrary $|J| \geqslant 2$. We will deduce contradictions by considering three cases depending on $|J'| = 0$, $|J'| = 1$ or $|J'| \geqslant 2$.

Case 1. $|J'| = 0$.

In this case, $X_i \neq \emptyset$ and $Y_i = \emptyset$ for each $i \in J$, that is, $J \cap T = \emptyset$. By $|J| \geqslant 2$ and the minimality of X, $|T| \geqslant 2$. Assume $\{i_1, i_2\} \subseteq J$ and $\{i_3, i_4\} \in T$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{i_1}_{n-1,k-1}$ (resp. $S^{i_2}_{n-1,k-1}$) and $S^{i_3}_{n-1,k-1}$ (resp. $S^{i_4}_{n-1,k-1}$), all of which are contained in B. Since $J \cap T = \emptyset$, we have that

$$|B| \geqslant 4\ \tfrac{(n-2)!}{(n-k)!}.$$

For $k=3$,

$$|B| \geqslant 4\ \tfrac{(n-2)!}{(n-k)!} \geqslant 4(n-2) > 2(n-2)$$

Combining Lemma 2.5 with Lemma 2.7 yields $|B| \leqslant \lambda_s^{(h)}(S_{n,3}) \leqslant 2(n-2)$, a contradiction. For $k \geqslant 4$, it is easy to check that

$$\begin{aligned}|B| &\geqslant 4\,\frac{(n-2)!}{(n-k)!} \geqslant 4(n-2)(n-3) = (2n-4)(2n-6)\\ &> \max\{(n-h-1)(h+1),(n-k+1)(k-1)\}\\ &= \omega(h,k),\end{aligned}$$

which contradicts the inequality (1).

Case 2. $|J'| = 1$.

Without loss of generality, assume $J' = \{1\}$. By Lemma 2.8 (a), B_1 is an $(h-1)$-edge-cut of $S^1_{n-1,k-1}$. By $|J| \geqslant 2$, there exists an $i \in J - J'$ such that $X_i = V(S^i_{n-1,k-1})$. By the minimality of X, there exists some $j \in T - J'$ such that $Y_j = V(S^j_{n-1,k-1})$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^i_{n-1,k-1}$ and $S^j_{n-1,k-1}$, thus $|B_{ij}| = \frac{(n-2)!}{(n-k)!} \geqslant n-2$. We consider the following two cases.

If $\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant (n-h)h$, then

$$\begin{aligned}|B| &\geqslant |B_1| + |B_{ij}|\\ &\geqslant (n-h)h + (n-2)\\ &> (n-h-1)h + (n-h-1)\\ &= (n-h-1)(h+1),\end{aligned}$$

If $\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant (n-k+2)(k-2)$, then

$$\begin{aligned}|B| &\geqslant |B_1| + |B_{ij}|\\ &\geqslant (n-k+2)(k-2) + (n-2)\\ &> (n-k+1)(k-2) + (n-k+1)\\ &= (n-k+1)(k-1).\end{aligned}$$

Therefore, we have $|B| > \omega(h,k)$, which contradicts the inequality (1).

Case 3. $|J'| \geqslant 2$.

By Lemma 2.8 (b), if $\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant (n-h)h$ then

$$\begin{aligned}|B| &\geqslant |J'|\lambda_s^{(h-1)}(S_{n-1,k-1})\\ &\geqslant 2(n-h)h \geqslant (n-h)h + (n-h)\\ &> (n-h-1)(h+1),\end{aligned}$$

if $\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant (n-k+2)(k-2)$ then

$$\begin{aligned}|B| &\geqslant |J'|\lambda_s^{(h-1)}(S_{n-1,k-1})\\ &\geqslant 2(n-k+2)(k-2)\\ &\geqslant (n-k+2)(k-2) + (n-k+2)\\ &> (n-k+1)(k-1).\end{aligned}$$

Therefore, we have $|B| > \omega(h,k)$, which contradicts the inequality (1).

Thus, we have $|J| = 1$. By the choice of t, the i-th bits of all vertices in X are same for each $i = 2, 3, \ldots, k$, and so X is a complete graph. Thus, we have that

$$\lambda_s^{(h)}(S_{n,k}) \quad = |B| = (n-|X|)|X|$$

Since $h+1 \leqslant |X| \leqslant n-k+1$ and $f(x) = (n-x)\,x$ is a convex function on the interval $[0,n]$, we have that

$$\lambda_s^{(h)}(S_{n,k}) \quad = |B| = (n-|X|)|X| \geqslant \psi(h,k),$$

where $\psi(h,k)$ is defined in Lemma 2.7.

If $h \leqslant \frac{n}{2} - 1$, using Lemma 2.7, we have

$$\lambda_s^{(h)}(S_{n,k}) \geqslant \psi(h,k) = \begin{cases} (n-h-1)(h+1) & \text{if } 0 \leqslant h \leqslant k-2; \\ (n-k+1)(k-1) & \text{if } h \geqslant k-1. \end{cases}$$

If $h \geqslant \frac{n}{2}$, then $X = V(K_{n-k+1})$. Otherwise, there exists some $x \in V(K_{n-k+1} - X)$ such that

$$h \leqslant d(x) = n-1-|X| \leqslant n-h-2,$$

which implies $h \leqslant \frac{n}{2} - 1$, a contradiction. Therefore, we have $|X| = n-k+1$, and

$$\lambda_s^{(h)}(S_{n,k}) = |B| = (n-|X|)|X| = (n-k+1)(k-1) \text{ for } h \geqslant \tfrac{n}{2}.$$

By the induction principle, the theorem follows.

As we have known, when $k = n-1$, $S_{n,n-1}$ is isomorphic to the star graph S_n. Akers and Krishnamurthy [1] determined $\lambda(S_n)$ and $\lambda_s^{(1)}(S_n)$, which can be obtained from our result by setting $k = n-1$ and $h = 0, 1$, respectively.

Corollary: $\lambda(S_n) = n-1$ for $n \geqslant 2$ and $\lambda_s^{(1)}(S_n) = 2n-4$ for $n \geqslant 3$.

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