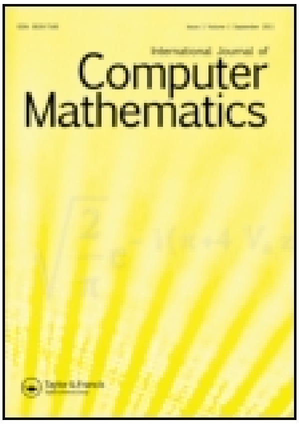

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The design of large dependable multiprocessor systems requires quick and precise mechanisms for detecting the faulty nodes. The system-level fault diagnosis is the process of identifying faulty processors in a system through testing. This paper shows that the largest connected component of the survival graph contains almost all remaining vertices in the hierarchical hypercube HHC_n when the number of faulty vertices is up to two or three times of the traditional connectivity. Based on this fault resiliency, we establish that the conditional diagnosability of HHC_n ($n = 2^m + m$, $m \geq 2$) under the comparison model is $3m - 2$, which is about three times of the traditional diagnosability.

Keywords: fault tolerance; comparison diagnosis; diagnosability; hierarchical hypercubes

2010 AMS Subject Classifications: 05C90; 05C40

1. Introduction

Processors of a multiprocessor system are connected according to a given interconnection network. Fault tolerance is especially important for interconnection networks, since failures of network components are inevitable when the size of network grows largely. To be reliable, the rest of the network should stay connected when component faults occur. Obviously, this can only be guaranteed if the number of faults is smaller than the minimum degree of the network. When the number of faults is larger than the minimum degree, some extensions of connectivity are necessary, since the graph may become disconnected. Some generalizations of connectivity were introduced and examined for various classes of graphs in [6], including super connectedness and tightly super connectedness, where only one singleton can appear in the remaining network. As the number of faults of the graph increases, it is desirable that when a few processors separated from the rest, the largest component of the surviving network stays connected and the network will continue to be able to function. Many interconnection networks have been examined in this aspect, when the number of faults is roughly twice the minimum degree [4,13]. One can even go

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further and ask what happens when more vertices are deleted. This has been examined for the hypercube in [28–30] and for certain Cayley graphs generated by transpositions in [5], and it has been shown that the surviving network has a large component containing almost all vertices.

The process of identifying faulty processors in a system by analysing the outcomes of available inter-processor tests is called system-level diagnosis. Preparata *et al.* [21] established a foundation of system diagnosis and an original diagnostic model, called the Preparata Metze Chien (PMC) model. Its target is to identify the exact set of all faulty vertices before their repair or replacement. All tests are performed between two adjacent processors, and it was assumed that a test result is reliable (respectively, unreliable) if the processor that initiates the test is fault-free (respectively, faulty). The comparison-based diagnosis models, first proposed by Malek [17] and Chwa and Hakimi [8], have been considered to be a practical approach for fault diagnosis in the multiprocessor systems. In these models, the same job is assigned to a pair of processors in the system and their outputs are compared by a central observer. This central observer performs diagnosis using the outcomes of these comparisons. Maeng and Malek [16] extended Malek's comparison approach to allow the comparisons carried out by the processors themselves. Sengupta and Dahbura [22] developed this comparison approach such that the comparisons have no central unit involved.

Lin *et al.* [15] introduced the conditional diagnosis under the comparison model. By evaluating the size of connected components, they obtained that the conditional diagnosability of the star graph S_n under the comparison model is $3n - 7$, which is about three times larger than that of the classical diagnosability of star graphs. Using the same method, Hsu *et al.* [11] proved that the conditional diagnosability of the hypercube Q_n is $3n - 5$. This idea was attributed to Lai *et al.* [14] who were the first to use a conditional diagnosis strategy. They obtained that the conditional diagnosability of the hypercube Q_n is $4n - 7$ under the PMC model. Furthermore, Hsu *et al.* [11] exposed the difference between these two conditional diagnosis models. Recently, Zhou [33] investigated the conditional diagnosability of the crossed cube CQ_n through the fault tolerance of CQ_n. Zhou and Xiao [34] obtained the conditional diagnosability of the alternating group network. By a similar processing style to [11,33], this paper will focus on the conditional diagnosability of hierarchical hypercube (HHC), which is a more complicated variant of hypercube.

The hypercube network suffers from a practical limitation: as n increases, it becomes more difficult to design and fabricate the nodes of Q_n because of the large fan-out. To remove the limitation, the cube-connected cycles (CCC for short) network was designed as a substitute for the hypercube network. The node degree of a CCC network is restricted to 3. However, this restriction degrades the performance of a CCC network at the same time. For example, a CCC network has a greater diameter than a hypercube network having the same number of nodes. Taking both the practical limitation and the performance into account, the HHC network was proposed as a compromise between the hypercube network and the CCC network. An HHC network, which has a two-level structure, takes hypercubes as basic modules and connects them in a hypercube manner. An HHC network has a logarithmic diameter, which is the same as a hypercube network. Since the topology of one HHC network is closely related to the topology of a hypercube network, it inherits some favourable properties from the latter.

This paper contains four sections in addition to the introduction. Section 2 introduces some definitions, notations and the structure of the hierarchical hypercube HHC_n. Section 3 is devoted to the fault resiliency of HHC_n and derives the extra connectivities. Section 4 concentrates on the conditional diagnosability of HHC_n. Section 5 contains some concluding remarks.

2. Hierarchical hypercubes

An interconnection network is conveniently represented by an undirected graph. The vertices (edges) of the graph represent the nodes (links) of the network. Throughout this paper, the terms vertex and node, edge and link, and graph and network are used interchangeably. For notation and terminology not defined here, we follow [25]. Specifically, we use a graph $G = G(V, E)$ to represent an interconnection network, where a vertex $u \in V$ represents a processor and an edge $(u, v) \in E$ represents a link between vertices u and v. If at least one end-vertex of an edge is faulty, the edge is said to be faulty; otherwise, the edge is said to be fault-free.

Let S be a subset of $V(G)$. The subgraph of G induced by S, denoted by $G[S]$, is the graph with the vertex set S and the edge set $\{(u, v) \mid (u, v) \in E(G), u, v \in S\}$. For any vertex u and one subgraph H in G, $N_H(u)$ denotes the set of all neighbours of u in H, that is, $N_H(u) = \{v \in H \mid (u, v) \in E\}$. Let S be a subgraph of G or a subset of $V(G)$, and let $N_H(S) = (\bigcup_{u \in S} N_H(u)) \setminus S$. We also denote $N[S] = N(S) \cup S$. For brevity, $N[u] = N(u) \cup \{u\}$, $N(\{u, v\})$ and $N[\{u, v\}]$ are written as $N(u, v)$ and $N[u, v]$, respectively. The union $G = G_1 \cup G_2$ of graphs G_1 and G_2 is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. We use $d(u, v)$ to denote the distance between u and v, the length of a shortest path between u and v in G. The diameter of G is defined as the maximum distance between any two vertices in G.

For any subset $F \subset V$, the notation $G - F$ denotes a graph obtained by removing all vertices in F from G and deleting those edges with at least one end-vertex in F, simultaneously. If $G - F$ is disconnected, F is called a *separating set*. A separating set F is called a *k-separating set* if $|F| = k$. The maximal connected subgraphs of $G - F$ are called *components*. The *connectivity* $\kappa(G)$ of G is defined as the minimum k for which G has a k-separating set; otherwise, $\kappa(G)$ is defined as $n - 1$ if $G = K_n$. A graph G is called to be *k-connected* if $\kappa(G) \geq k$. A k-separating set is called to be *minimum* if $k = \kappa(G)$.

Network reliability is one of the major factors in designing the topology of an interconnection network. Because of its elegant topological properties and the ability to emulate a wide variety of other frequently used networks, the hypercube has been one of the most popular interconnection networks for parallel computer/communication systems. However, when dealing with the parallel computers of very large scale, the port limitation due to the technology greatly forbid the use of hypercube networks.

An n-dimensional cube-connected cycle (CCC$_n$ for short) can be obtained by replacing each node of n-dimensional hypercube Q_n with a cycle of n nodes so that they are connected to the n neighbours of the original node in Q_n. Actually, an HHC network is a modification of a CCC in which the cycle is replaced with a hypercube [18–20]. An n-dimensional hierarchical cube (HHC$_n$ for short) can be constructed as follows: start with a Q_{2^m} network and then replace each node of it with a hypercube Q_m. Since there are a total of $2^{2^m} \times 2^m = 2^{2^m+m}$ nodes, each node in the HHC$_n$ network can be uniquely represented by a binary sequence $b_{n-1}b_{n-2}\cdots b_0$, where $n = 2^m + m$. In Figure 1, an example with $m = 2$ is shown. For convenience, $b_{n-1}b_{n-2}\cdots b_0$ is expressed as a two-tuple (X, Y), where $X = b_{n-1}b_{n-2}\cdots b_m$ tells which Q_m the node is located in and $Y = b_{m-1}b_{m-2}\cdots b_0$ gives the address of the node in the located Q_m network. Let $Y^l = b_{m-1}b_{m-2}\cdots b_{l+1}\overline{b_l}b_{l-1}\cdots b_0$ (or $X^{m+l} = b_{n-1}b_{n-2}\cdots b_{m+l+1}\overline{b_{m+l}}b_{m+l-1}\cdots b_0$) denote the binary sequence obtained by complementing b_l (or b_{m+l}) of X (or Y). The network HHC$_n$ can be defined in terms of graph as follows.

Definition 2.1 [18–20] *The node set of an n-dimensional HHC* (HHC$_n$ *for short*) *network is* $\{(X, Y) \mid X = b_{n-1}b_{n-2}\cdots b_m, Y = b_{m-1}b_{m-2}\cdots b_0$ *and* $b_i \in \{0, 1\}$ *for all* $0 \leq i \leq n - 1$, *where* $n = 2^{2^m+m}$ *and* $m \geq 1\}$. *Node adjacency of* HHC$_n$ *is defined as follows*: (X, Y) *is adjacent to*

(1) (X, Y^l) *for all* $0 \leq l \leq m - 1$ *and*
(2) $(X^{m+\mathrm{dec}(Y)}, Y)$, *where* $\mathrm{dec}(Y)$ *is the decimal value of* Y.

The spanning subgraph $\{(X, Y) \mid X = b_{n-1}b_{n-2}\cdots b_m, Y = b_{m-1}b_{m-2}\cdots b_0$ and $b_i \in \{0, 1\}$ for all $0 \leq i \leq n - 1\}$ of HHC$_n$ with $\mathrm{dec}(X) = j$ is denoted as HHC$_n^j$.

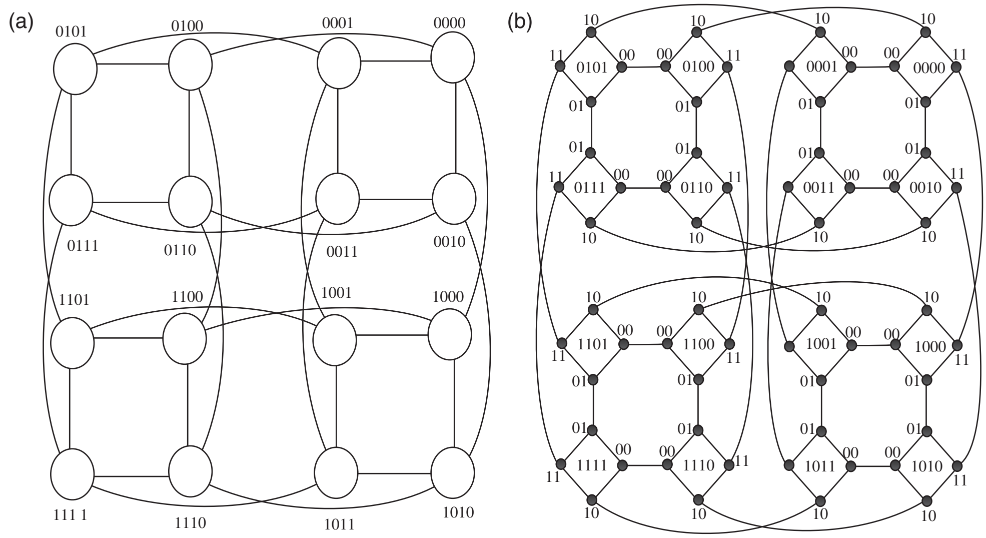

Figure 1. (a) Hypercube Q_{2^2} and (b) hierarchical hypercube HHC_6.

Edges defined by (1) are referred to as *internal edges*, and those defined by (2) are referred to as *external edges*. Internal edges are within Q_m networks and each of external edges connects two Q_m networks. Note that the HHC_n network is $(m+1)$-regular, symmetric and has a diameter of 2^{m+1}. In subsequent discussion, whenever a node v of an HHC_n network is mentioned, we use v_X and v_Y to denote the X part and Y part of v, respectively. For any two distinct vertices u and v in HHC_n

$$|N(u) \cap N(v)| \begin{cases} = 0, & \text{if } d(u,v) \geq 3, \\ \leq 2, & \text{if } d(u,v) = 2, \\ = 0, & \text{if } d(u,v) = 1. \end{cases} \tag{1}$$

To simplify the description of the HHC_n structure, integers and their binary encoding are used interchangeably in this paper, and it assumes that $n = 2^m + m$ and $m \geq 1$. The set of edges E is the union of two set E_{int} and E_{ext}, which are the sets of internal and external edges, respectively, as the following equation illustrates:

$$E_{\text{int}} = \{((i,j),(i,j + f_k(j) \times 2^k)) \mid 0 \leq i < 2^{n-m}, 0 \leq j < 2^m, 0 \leq k < m\},$$
$$E_{\text{ext}} = \{((i,j),(i + f_j(i) \times 2^j, j)) \mid 0 \leq i < 2^{n-m}, 0 \leq j < 2^m\},$$

where ($\mathrm{bit}_k(x)$ denotes the kth coordinate of the binary representation of x)

$$f_k(x) = \begin{cases} 1, & \text{if } \mathrm{bit}_k(x) = 0, \\ -1, & \text{if } \mathrm{bit}_k(x) = 1. \end{cases} \tag{2}$$

Thus, every vertex (i,j) of HHC_n is connected to the following:

(1) m vertices in the same sub-hypercube through internal edges. These are the vertices whose addresses are found by changing only one bit of the j part of the address.
(2) exactly one vertex in a neighbour sub-hypercube through an external vertex corresponding to change of the jth bit of the i part of the address.

Wu *et al.* [23] gave a construction of node disjoint paths of any two vertices in HHC with the estimation of length on these paths. Wu *et al.* [24] showed that the HHC is bipancyclic. Some fault tolerant problems are still open for HHCs.

3. Fault tolerance of the HHC

The connectivity $\kappa(G)$ of a graph G is an important parameter to measure the fault tolerance of the network, while it has an obvious deficiency in that it tacitly assumes that all elements in any subset of G can fail potentially at the same time. To compensate for this shortcoming, it is natural to generalize the classical connectivity by introducing some conditions or restrictions on the separating set S and/or the components of $G - S$.

The connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal results in a disconnected or a trivial (one vertex) graph. A k-regular k-connected graph is *super k-connected* if any one of its minimum separating sets is a set of the neighbours of some vertex. If, in addition, the deletion of a minimum separating set results in a graph with two components (one of which has only one vertex), then the graph is *tightly super k-connected*. For example, the complete bipartite graph $K_{n,n}$ is n-super connected but not tightly n-super connected. The notions of super connectedness and tightly super connectedness were first introduced in [1,6], respectively.

Esfahanian [9] first introduced the concepts of the restricted separating set and the restricted connectivity of a graph G. A set S of vertices is a *restricted separating set* if $G - S$ is disconnected and $N(x)$ is not completely contained in S for any vertex x of G. The *restricted connectivity* of G, denoted by $\kappa_{\mathrm{r}}(G)$, is the minimum cardinality of a restricted separating set. Considering it is difficult to examine whether a separating set is restricted, Xu *et al.* [26] formally proposed the super connectivity, one weaker concept than the restricted connectivity. A separating set S of G is *super* if $G - S$ contains no isolated vertices. The *super connectivity* of G, denoted by $\kappa_{\mathrm{s}}(G)$, is the minimum cardinality of a super separating set. Clearly, $\kappa(G) \leq \kappa_{\mathrm{s}}(G) \leq \kappa_{\mathrm{r}}(G)$ if $\kappa_{\mathrm{r}}(G)$ exists [26].

Fábrega and Fiol [10] generalized the concept of super connectivity to *h-extra connectivity* for an undirected graph. Let G be a connected undirected graph and h be an integer with $0 \leq h \leq \delta(G)$. A subset $S \subset V(G)$ is called an *h-extra separating set* if $G - S$ is disconnected and every connected component contains at least $h + 1$ vertices. The *h-extra connectivity* $\kappa_o^{(h)}(G)$ is defined as

$$\kappa_o^{(h)}(G) = \min\{|S| \,|S \text{ is an } h\text{-extra separating set of } G\}.$$

It follows from definitions that the h-extra connectivity can provide a more accurate measurement than the connectivity or super connectivity for fault tolerance of a large-scale interconnection network.

Usually, if the surviving graph $G - S$ contains a large connected component C when $G - S$ is not connected, the component C may be used as the functional subsystem, without incurring severe performance degradation. Thus, in evaluating a distributed system, it is indispensable to estimate the size of the maximal connected components of the underlying graph when the structure begins to lose processors.

Yang *et al.* [28–30] proved that the hypercube Q_n with f faulty processors has a component of size at least $2^n - f - 1$ if $f \leq 2n - 3$, and size at least $2^n - f - 2$ if $f \leq 3n - 6$. Recently, Yang and Meng [27] determined the extra connectivity of hypercubes, Hsu *et al.* [12] went further to establish the component connectivity of hypercubes. Yang *et al.* [31], Cheng and Lipman [2] and Cheng and Lipták [3] evaluated the size of surviving graph $S_n - F$ of star graph S_n, where F is a subset of $V(S_n)$ with $|F| \leq 2n - 4$. Cheng and Lipták [5] extended the results on S_n with linearly many faults. Yuan *et al.* [32] generalized the results on (n, k)-star graphs. Cheng *et al.* [7]

presented a similar result for the two-tree-generated networks with linearly many faults. In this section, we detail on the fault resilience of the hierarchical hypercube HHC_n. When $m = 1$, HHC_n with $n = 2^m + m$ is a trivial cycle of length 8. Therefore, we consider that $m \geq 2$ for the time being.

LEMMA 3.1 [18–20,23,24] *For $n = 2^m + m$ and $m \geq 2$, HHC_n has the following combinatorial properties*:

(1) HHC_n *has* 2^{2^m+m} *vertices with regular degree* $m + 1$.
(2) HHC_n *has the vertex connectivity of* $m + 1$ *and the edge connectivity* $m + 1$.
(3) HHC_n *has the diameter* 2^{m+1}.
(4) HHC_n *is a bipartite graph and it is bipancyclic.*

LEMMA 3.2 [11,28–30] *Let F be a set of faulty vertices in the hypercube Q_m with $|F| \leq 2m - 3$ and $m \geq 2$. If $Q_m - F$ is disconnected, then $Q_m - F$ has two connected components and one of which is an isolated vertex.*

Throughout this paper, the notation F denotes the set of faulty vertices in HHC_n. A subgraph H of HHC_n is called to be *fault-free* if $V(H) \cap F = \emptyset$. We denote

$$F_i = \mathrm{HHC}_n^i \cap F \quad \text{and} \quad f_i = |F_i| \quad \text{for } i \in [0, 2^{2^m} - 1], \tag{3}$$

$$I = \{i \,|\, f_i = |F_i| \geq m \quad \text{for } \ i \in [0, 2^{2^m} - 1]\}, \quad J = [0, 2^{2^m} - 1] - I. \tag{4}$$

Furthermore, we define

$$\mathrm{HHC}_n^J - F_J = \bigcup_{j \in J} (\mathrm{HHC}_n^j - F_j). \tag{5}$$

LEMMA 3.3 *Let F be a set of faulty vertices in* HHC_n *with $|F| \leq 3m - 3$ and $n = 2^m + m, m \geq 2$. Then,* $\mathrm{HHC}_n^J - F_J$ *is connected.*

Proof Furthermore, we define $J_1 = \{j \in J \mid \ < f_j = |F_j| \leq m - 1\}$ and $J_0 = J \setminus J_1$. By contracting each HHC_n^j $(j \in [0, 2^{2^m} - 1])$ as a new vertex, we obtain a new graph, say G, which is isomorphic to Q_{2^m} and has connectivity 2^m. The vertex of G corresponding to HHC_n^j $(j \in J_0)$ can be seen as a fault-free vertex, and the vertex of G corresponding to HHC_n^i $(i \in I \cup J_1)$ can be seen as a faulty vertex. Since $2^m > 3m - 3 \geq |F|$ for $m \geq 2$, G is connected, so $\mathrm{HHC}_n - \mathrm{HHC}_n^{J_1 \cup I} = \mathrm{HHC}_n^{J_0}$ is connected.

For any $j \in J_1$, HHC_n^j is isomorphic to the hypercube Q_m, whose connectivity is m. Thus, $\mathrm{HHC}_n^j - F_j$ is connected. To prove that $\mathrm{HHC}_n^J - F_J$ is connected, we need to consider three cases as follows.

Case 1. $|I| = 0$

For each $j \in J_1$, there exist exactly 2^m matching edges between HHC_n^j and $\mathrm{HHC}_n - \mathrm{HHC}_n^j$. Since $2^m > 3m - 3$ for $m \geq 2$, $\mathrm{HHC}_n^j - F_j$ is connected to $\mathrm{HHC}_n^{J_0}$. By the arbitrariness of $j \in J_1$, $\mathrm{HHC}_n^{J_1} - F_{J_1}$ is connected to $\mathrm{HHC}_n^{J_0}$. Hence, $\mathrm{HHC}_n^J - F_J$ is connected.

Case 2. $|I| = 1$.

For each $j \in J_1$, there exist exactly 2^m matching edges between HHC_n^j and $\mathrm{HHC}_n - \mathrm{HHC}_n^j$, at most one of which is in HHC_n^I. Since $2^m - 1 > 3m - 3 - m = 2m - 3$ for $m \geq 2$, $\mathrm{HHC}_n^j - F_j$ is connected to $\mathrm{HHC}_n^{J_0}$. By the arbitrariness of $j \in J_1$, $\mathrm{HHC}_n^{J_1} - F_{J_1}$ is connected to $\mathrm{HHC}_n^{J_0}$. Hence, $\mathrm{HHC}_n^J - F_J$ is connected.

Case 3. $|I| = 2$.

For each $j \in J_1$, there exist exactly 2^m matching edges between HHC_n^j and $\mathrm{HHC}_n - \mathrm{HHC}_n^j$, at most two of which is in HHC_n^I. Since $2^m - 2 > 3m - 3 - 2m = m - 3$ for $m \geq 2$, $\mathrm{HHC}_n^j - F_j$ is connected to $\mathrm{HHC}_n^{J_0}$. By the arbitrariness of $j \in J_1$, $\mathrm{HHC}_n^{J_1} - F_{J_1}$ is connected to $\mathrm{HHC}_n^{J_0}$. Hence, $\mathrm{HHC}_n^{J} - F_J$ is connected. ■

THEOREM 3.4 *For $n = 2^m + m$ and $m \geq 2$, HHC_n is tightly super $m+1$-connected.*

Proof Let F be a minimum separating set in HHC_n. Then, using the notations defined in Equations (3) and (4), we have that

$$|F| = \sum_{i \in [0, 2^{2^m}-1]} f_i = \kappa(\mathrm{HHC}_n) = m + 1.$$

By the definition of tightly super connectivity, we need to show that $\mathrm{HHC}_n - F$ has exactly two components, one of them is a single vertex. If $f_i \leq m-1$ for any $i \in [0, 2^{2^m}-1]$, $\mathrm{HHC}_n - F$ is still connected by Lemma 3.3. Now, we consider two cases.

Case 1. There exists some $i_0 \in [0, 2^{2^m}-1]$ such that $f_{i_0} = m + 1$.

In this case, by Lemma 3.1, $f_i = 0$ for any $i \in [0, 2^{2^m}-1]$ and $i \neq i_0$, HHC_n^i is connected. $\mathrm{HHC}_n - \mathrm{HHC}_n^{i_0}$ is still connected by Lemma 3.3. Every vertex of $\mathrm{HHC}_n^{i_0} - F_{i_0}$ has exactly one fault-free neighbour vertex in $\mathrm{HHC}_n - \mathrm{HHC}_n^{i_0}$, so $\mathrm{HHC}_n - F$ is still connected, a contradiction.

Case 2. $f_{i_0} = m$ for some $i_0 \in [0, 2^{2^m}-1]$.

By the hypothesis, there exists some $i_1 \in [0, 2^{2^m}-1]$ with $i_1 \neq i_0$ such that $f_{i_1} = 1$. Since HHC_n^i is isomorphic to the m-dimensional hypercube Q_m which is m-connected, HHC_n^i is still connected for any $i \in [0, 2^{2^m}-1]$ with $i \neq i_0$. As $\mathrm{HHC}_n^{i_0}$ is tightly super m-connected, $\mathrm{HHC}_n^{i_0} - F_{i_0}$ has at most one vertex isolated from $\mathrm{HHC}_n - (V(\mathrm{HHC}_n^{i_0}) \cup (F - F_{i_0}))$. Since $f_{i_1} = 1$, $\mathrm{HHC}_n - F$ has exactly two connected components, one of which is an isolated vertex. ■

LEMMA 3.5 *Let F be a separating set of HHC_n with $|F| \leq 3m-3$ and $m \geq 2$. If there is some $i_0 \in [0, 2^{2^m}-1]$ such that $|F| - f_{i_0} \leq 1$, then $\mathrm{HHC}_n - F$ has exactly two components, one of which is a single vertex.*

Proof We use the notations defined in Equations (3) and (4) in the following. By the hypothesis, for any $i \in [0, 2^{2^m}-1] - \{i_0\}$,

$$f_i \leq |F| - f_{i_0} \leq 1.$$

Since $\mathrm{HHC}_n - F$ is disconnected, and $\mathrm{HHC}_n - (\mathrm{HHC}_n^{i_0} \cup F)$ is connected by Lemma 3.3, there is a component of $\mathrm{HHC}_n - F$ that contains no vertices in $\mathrm{HHC}_n^J - F_J$. Let H be the union of such components of $\mathrm{HHC}_n - F$. Thus, $N_{\mathrm{HHC}_n - \mathrm{HHC}_n^{i_0}}(H) \subseteq F \setminus F_{i_0}$, and we have that

$$|V(H)| \leq |F| - f_{i_0} \leq 1$$

which yields $|V(H)| \leq 1$, that is to say, H is a single vertex, say u. By the choice of H, other components of $\mathrm{HHC}_n^{i_0} - F_{i_0}$ must be connected to $\mathrm{HHC}_n^J - F_J$. Since $\mathrm{HHC}_n^J - F_J$ is connected by Lemma 3.3, $\mathrm{HHC}_n - (F \cup \{u\})$ is connected. It follows that $\mathrm{HHC}_n - F$ has exactly two components, one of which is a single vertex. The lemma follows. ■

LEMMA 3.6 *For $n = 2^m + m$ and $m \geq 2$, let F be a separating set of HHC_n with $|F| \leq 3m-3$ and H be the union of connected components of $\mathrm{HHC}_n - F$ which are in $\mathrm{HHC}_n^i - F_i$ for some $i \in [0, 2^{2^m}-1]$. If $N_{\mathrm{HHC}_n^i}(H) \subseteq F_i$, then $|V(H)| \leq 2$.*

Proof Let $h = |V(H)|$. We would like to prove $h \leq 2$. Suppose to the contrary that $h \geq 3$. Take a subset $T \subseteq V(H)$ with $|T| = 3$. Let $T' = V(H - T)$. By the hypothesis, $N_{\mathrm{HHC}_n^i}(T) \setminus T' \subseteq F_i$. Note that HHC_n^i is isomorphic to hypercube Q_m. We denote $T = \{x, y, z\}$ and discuss as follows.

If $H[T]$ has no edges, then every pair of vertices in T has at most two common neighbours by Equation (2), and hence

$$|N_{\mathrm{HHC}_n^i}(T)| \geq 3m - 5.$$

If $H[T]$ has only one edge, say $e = (x, y)$, then x and y have no common neighbours, z and x (respectively, y) have at most two common neighbours by Equation (2), but two cases cannot occur meanwhile as there are no cycles of odd length. It follows that

$$|N_{\mathrm{HHC}_n^i}(T)| \geq 3m - 4.$$

Similarly, by Equation (2), we can obtain that if $H[T]$ has two edges, then

$$|N_{\mathrm{HHC}_n^i}(T)| \geq 3m - 5.$$

Summing all cases, we have that

$$\begin{aligned} f_i &\geq |N_{\mathrm{HHC}_n^i}(T) \setminus T'| \\ &\geq |N_{\mathrm{HHC}_n^i}(T)| - (h-3) \\ &\geq 3m - 5 - (h-3) \\ &= 3m - 2 - h, \end{aligned}$$

that is,

$$f_i \geq 3m - 2 - h. \tag{6}$$

Since $N_{\mathrm{HHC}_n - \mathrm{HHC}_n^i}(H) \subseteq F - F_i$, $|F| - f_i \geq h$, from which we have that

$$f_i \leq |F| - h \leq 3m - 3 - h,$$

that is,

$$f_i \leq 3m - 3 - h. \tag{7}$$

Combining Equation (6) with Equation (7) deduces a contradiction. Thus, $h \leq 2$. ■

THEOREM 3.7 *The 1-extra connectivity of* HHC_n ($n = 2^m + m$, $m \geq 2$) *is* $\kappa_0^{(1)}(\mathrm{HHC}_n) = 2m$.

Proof We choose an edge (u, v) in some subgraph HHC_n^i. Obviously, $|N(u, v)| = 2m$, $\mathrm{HHC}_n - N[u, v]$ is still connected by Lemma 3.3. Each connected component of $\mathrm{HHC}_n - N(u, v)$ has order at least 2. Thus, we have $\kappa_0^{(1)}(\mathrm{HHC}_n) \leq 2m$.

Now, we show that $\kappa_0^{(1)}(\mathrm{HHC}_n) > 2m - 1$. Let F be a set of faulty vertices in HHC_n with $|F| \leq 2m - 1$ such that $\mathrm{HHC}_n - F$ is disconnected.

If $|I| \geq 2$, then $|F| \geq 2m$, a contradiction. Now, we set $I = \{i_0\}$.

Let H be the union of all components of $\mathrm{HHC}_n - F$ that contain no vertex in $\mathrm{HHC}_n^J - F_J$. Thus, H is in $\mathrm{HHC}_n^{i_0}$. By the choice of H, other components of $\mathrm{HHC}_n^{i_0} - F_{i_0}$ must be connected to $\mathrm{HHC}_n^J - F_J$. Since $\mathrm{HHC}_n^J - F_J$ is connected, $\mathrm{HHC}_n - (F \cup V(H))$ is connected. Thus, to complete the proof, it suffices to show that $|V(H)| = 1$. By Lemma 3.6, we only need to show that $|V(H)| = 2$ is not possible. Suppose to the contrary that $H = \{u, v\}$. Obviously, $N(u, v) \subseteq F$.

If u is not adjacent to v, then $d(u, v) \geq 2$, $|N(u) \cap N(v)| \leq 2$. By Equation (2), we have

$$\begin{aligned} |F| &\geq |N(u) \cup N(v)| \\ &\geq |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ &\geq 2(m+1) - |N(u) \cap N(v)| \end{aligned}$$

$$\geq 2(m+1)-2$$
$$> |F|,$$

a contradiction.

If (u,v) is an edge of HHC_n, then $|N(u)\cap N(v)|=0$. By Equation (2), we have

$$\begin{aligned}|F| &\geq |N(u,v)|\\ &\geq |N(u)|+|N(v)|-|\{u,v\}|\\ &\geq 2(m+1)-2\\ &> |F|,\end{aligned}$$

a contradiction. ■

We now discuss the fault tolerance of HHC_n with more faulty vertices up to $3m-3$ when $m\geq 2$.

LEMMA 3.8 *Let F be a set of faulty vertices in* HHC_n $(n=2^m+m, m\geq 2)$ *with* $|F|\leq 3m-3$. *If* $\mathrm{HHC}_n - F$ *is disconnected, then it either has two components, one of which is an isolated vertex or an isolated edge, or has three components, two of which are isolated vertices.*

Proof Since $\mathrm{HHC}_n - F$ is disconnected, F is a separating set of HHC_n.

If there exists some $i\in[0,2^{2^m}-1]$ such that $f_i\geq 3m-4$, and hence

$$|F|-f_i\leq 1,$$

by Lemma 3.5, $\mathrm{HHC}_n - F$ has exactly two components, one of which is a single vertex. Now, we consider that $f_i\leq 3m-5$ for any $i\in[0,2^{2^m}-1]$.

Let H be the union of all components of $\mathrm{HHC}_n - F$ that contain no vertex in $\mathrm{HHC}_n^J - F_J$, and let $h=|V(H)|$. Since $\mathrm{HHC}_n^J - F_J$ is connected, H is in HHC_n^I. By the choice of H, other components of $\mathrm{HHC}_n^I - F_I$ must be connected to $\mathrm{HHC}_n^J - F_J$. Since $\mathrm{HHC}_n^J - F_J$ is connected, $\mathrm{HHC}_n - (F\cup V(H))$ is connected. To complete the proof, it suffices to show that $h\leq 2$.

If $|I|\geq 3$, then $|F|\geq 3m > 3m-3\geq |F|$, a contradiction. Now, we set $1\leq |I|\leq 2$ in the following.

If $|I|=1$, then $h\leq 2$ by Lemma 3.6. Now, we suppose that $I=\{i_1,i_2\}$, and let h_1 and h_2 be the numbers of vertices of H that lie in $\mathrm{HHC}_n^{i_1}$ and $\mathrm{HHC}_n^{i_2}$, respectively.

Obviously, $f_i\leq 2m-3$ for any $i\in I$, otherwise, $|F|\geq 3m-2$, which is a contradiction. We have $h_1\leq 1$ and $h_2\leq 1$ by Lemma 3.2. Thus, $h=h_1+h_2\leq 2$. ■

Lemma 3.8 tells that $\kappa_0^{(2)}(\mathrm{HHC}_n) > 3m-3$. Now, we choose a path, $P=(u,v,w)$, of length 3 in some subgraph HHC_n^i. Obviously, $|N(u,v,w)|=3m-2$, $\mathrm{HHC}_n - N[u,v,w]$ is still connected and has order at least 3. Thus, we have the following.

THEOREM 3.9 *The 2-extra connectivity of hierarchical hypercube* HHC_n $(n=2^m+m$ *and* $m\geq 2)$ *is* $\kappa_0^{(2)}(\mathrm{HHC}_n)=3m-2$.

4. Diagnosability of HHCs

The comparison diagnosis strategy of a graph $G=(V,E)$ can be modelled as a multi-graph $M=(V,C)$, where C is a set of labelled edges. If the processors u and v can be compared by the processor w, provided that u and v are both adjacent to w, there exists a labelled edge (u, v) in C, denoted by $(u, v)_w$. We call w the comparator of u and v. Since different comparators can compare the same pair of processors, M is a multi-graph. Denote the comparison result as $\sigma((u, v)_w)$ such that $\sigma((u, v)_w) = 0$ if the outputs of u and v agree, and $\sigma((u, v)_w) = 1$ if the outputs disagree. If the comparator w is fault-free and $\sigma((u, v)_w) = 0$, the processors u and v are fault-free; while $\sigma((u, v)_w) = 1$, at least one of the three processors u, v and w is faulty. The collection of the comparison results defined as a function $\sigma : C \to \{0, 1\}$ is called the *syndrome* of the diagnosis. If the comparator w is faulty, the comparison result is unreliable. A faulty comparator can lead to unreliable results, so a set of faulty vertices may produce different syndromes. A subset $F \subsetneq V$ is said to be *compatible* with a syndrome σ if σ can arise from the circumstance that all vertices in F are faulty and all vertices in $V - F$ are fault-free. A system G is said to be *diagnosable* if, for every syndrome σ, there is a unique $F \subset V$ that is compatible with σ. A system is said to be t-diagnosable if the system is diagnosable as long as the number of faulty vertices does not exceed t. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, written as $t(G)$. Let $\sigma_F = \{\sigma \mid \sigma \text{ is compatible with } F\}$. Two distinct subsets F_1 and F_2 of $V(G)$ are said to be *indistinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_2} \neq \phi$, and distinguishable otherwise [11,15,22]. There are several different ways to verify whether a system is t-diagnosable under the comparison approach. The following lemma obtained by Sengupta and Dahbura [22] gives necessary and sufficient conditions to ensure distinguishability.

LEMMA 4.1 [22] *Let G be a graph and F_1 and F_2 be two distinct subsets of vertices in G. The pair (F_1, F_2) is distinguishable if and only if at least one of the following conditions is satisfied.*

(1) *There are two distinct vertices u and $w \in V(G - F_1 \cup F_2)$ and a vertex $v \in F_1 \Delta F_2$ such that $(u, v)_w \in C$, where $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$.*
(2) *There are two distinct vertices u and $v \in F_1 \setminus F_2$ (or $F_2 \setminus F_1$) and a vertex $w \in V(G - F_1 \cup F_2)$ such that $(u, v)_w \in C$.*

Lin *et al.* [15] introduced the so-called conditional diagnosability of a system under the situation that no set of faulty vertices can contain all neighbours of any vertex in the system. A fault-set $F \subset V(G)$ is called a conditional fault set if $N(v)$ is not a subset of F for every vertex v in $V(G)$. A system $G(V, E)$ is said to be conditionally t-diagnosable if F_1 and F_2 are distinguishable for each pair (F_1, F_2) of distinct conditional fault sets in G with $|F_1| \leq t$ and $|F_2| \leq t$. The *conditional diagnosability* of G, denoted by $t_c(G)$, is defined as the maximum value of t for which G is conditionally t-diagnosable. Clearly, $t_c(G) \geq t(G)$. This section will focus on the conditional diagnosability of HHCs.

LEMMA 4.2 *Let F_1 and F_2 be any two distinct conditional fault sets of $|F_1| \leq 3m - 2$ and $|F_2| \leq 3m - 2$. Denote by H the maximum component of* $\mathrm{HHC}_n - F_1 \cap F_2$ *with $n = 2^m + m$, $m \geq 2$. Then, for every vertex $u \in F_1 \Delta F_2$, $u \in H$.*

Proof Without loss of generality, we assume that $u \in F_1 - F_2$. Since F_2 is a conditional fault set, there is a vertex $v \in (\mathrm{HHC}_n - F_2) - \{u\}$ such that $(u, v) \in E(\mathrm{HHC}_n)$. Suppose that u is not a vertex of H. Then, v is not in H, hence u and v are in one small component of $\mathrm{HHC}_n - F_1 \cap F_2$. Since F_1 and F_2 are distinct, we have

$$|F_1 \cap F_2| \leq 3m - 3.$$

Hence, $\mathrm{HHC}_n[u, v]$ forms a component K_2 in $\mathrm{HHC}_n - F_1 \cap F_2$ by Lemma 3.8, that is, the vertex u is the unique neighbour of v in $\mathrm{HHC}_n - F_1 \cap F_2$. This is a contradiction since F_1 is a conditional fault set, but all the neighbours of v are faulty in F_1. ∎

LEMMA 4.3 [15] *Let G be a graph with $\delta(G) \geq 2$, and let F_1 and F_2 be any two distinct conditional fault sets of G with $F_1 \subset F_2$. Then, (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.*

THEOREM 4.4 $t_c(\mathrm{HHC}_n) = 3m - 2$ *for* $n = 2^m + m$ *and* $m \geq 2$.

Proof We first prove that $t_c(\mathrm{HHC}_n) \leq 3m - 2$ for $m \geq 2$. In fact, when $n \geq 3$, we select four vertices $x, y, z, u \in V(\mathrm{HHC}_n)$, such that (x, y, z, u) is a cycle of length 4. Set $A = N[x, y, z]$, $F_1 = A - \{y, z\}$ and $F_2 = A - \{x, y\}$. We obtain

$$|F_1| = |F_2| = 3m - 1 \quad \text{and} \quad |F_1 - F_2| = |F_2 - F_1| = 1.$$

It is easy to check that F_1 and F_2 are two conditional fault sets and are indistinguishable. Thus, we have

$$t_c(\mathrm{HHC}_n) \leq 3m - 2.$$

Now, we prove that $t_c(\mathrm{HHC}_n) \geq 3m - 2$ for $m \geq 2$.

Let F_1 and F_2 be any two distinct conditional fault sets of HHC_n. If $|F_1| \leq 3m - 2$, $|F_2| \leq 3m - 2$ for $m \geq 2$. It suffices to prove that (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.

By Lemma 4.3, (F_1, F_2) is a distinguishable conditional pair if $F_1 \subset F_2$ or $F_2 \subset F_1$. Now, we assume that $|F_1 - F_2| \geq 1$ and $|F_2 - F_1| \geq 1$. Let $S = F_1 \cap F_2$. Then, we have $|S| \leq 3m - 3$ for $m \geq 2$. Let H be the largest connected component of $\mathrm{HHC}_n - F_1 \cap F_2$. By Lemma 4.2, every vertex in $F_1 \Delta F_2$ is in H.

We claim that H has a vertex u outside of $F_1 \cup F_2$ that has no neighbour in S. Since every vertex has degree $m + 1$, the vertices in S can have at most $(m + 1)|S|$ neighbours in H. There are at most $|F_1| + |F_2| - |S|$ vertices in $F_1 \cup F_2$ and at most two vertices of $\mathrm{HHC}_n - S$ may not belong to H by Lemma 3.8. Thus, we have

$$\begin{aligned} &2^{2^m+m} - (m + 1)|S| - (|F_1| + |F_2| - |S|) - 2 \\ &\geq 2^{2^m+m} - (m + 2) \times (3m - 3) - 2 \\ &\geq 4. \end{aligned}$$

Thus, there must be some vertex of H outside of $F_1 \cup F_2$, which has no neighbours in S. Let u be such a vertex.

If u has no neighbour in $F_1 \cup F_2$, we can find a path of length at least 2 within H to a vertex v in $F_1 \Delta F_2$. We may assume that v is the first vertex of $F_1 \Delta F_2$ on this path, and let q and w be the two vertices on this path immediately before v (we may have $u = q$), hence q and w are not in $F_1 \cup F_2$. The existence of the edges (q, w) and (w, v) ensures that (F_1, F_2) is a distinguishable conditional pair of HHC_n by Lemma 4.1. Now, we assume that u has a neighbour in $F_1 \Delta F_2$. Since the degree of u is at least 3, and u has no neighbour in S, there are three possibilities:

(1) u has two neighbours in $F_1 \setminus F_2$,
(2) u has two neighbours in $F_2 \setminus F_1$ or
(3) u has at least one neighbour outside $F_2 \cup F_1$.

In each sub-case above, by Lemma 4.1, (F_1, F_2) is a distinguishable conditional pair of HHC_n under the comparison diagnosis model, hence the proof is complete. ∎

5. Conclusion

The paper derives the fault resiliency of HHCs and then uses the fault resiliency to evaluate fault diagnosability of HHCs under the comparison model. The traditional diagnosability of HHC_n with $n = 2^m + m$ and $m \geq 2$ under the comparison model is only $m + 1$, while the conditional diagnosability of HHC_n is $3m - 2$, which is about three times that of the traditional diagnosability under the comparison model. The fault resiliency of an HHC may also reveal its conditional connectivity of high order. This method can also be applied to other complex network structures.

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