CONDITIONAL FAULT DIAGNOSABILITY OF DUAL-CUBES

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The growing size of the multiprocessor system increases its vulnerability to component failures. It is crucial to locate and replace the faulty processors to maintain a system's high reliability. The fault diagnosis is the process of identifying faulty processors in a system through testing. This paper shows that the largest connected component of the survival graph contains almost all of the remaining vertices in the dual-cube DC_n when the number of faulty vertices is up to twice or three times of the traditional connectivity. Based on this fault resiliency, this paper determines that the conditional diagnosability of DC_n ($n \geq 3$) under the comparison model is $3n - 2$, which is about three times of the traditional diagnosability.

Keywords: Fault tolerance; comparison diagnosis; conditional diagnosability; dual-cubes.

1. Introduction

Processors of a multiprocessor system are connected according to a given interconnection network. Fault-tolerance is especially important for interconnection networks, since failures of interconnection network components are inevitable. To be reliable, the rest of the network should retain connection when component faults

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occur. Obviously, this can only be guaranteed if the number of faults is smaller than the connectivity of the network. When the number of faults is larger than the connectivity, some extensions of connectivity are necessary, since the graph may become disconnected. Some generalizations of connectivity were introduced and investigated for various classes of graphs in [10], including super connectedness and tightly super connectedness, where only one singleton can appear in the surviving network. As the number of faults in the graph is increased, it is desirable that the most part of the surviving network retain connection, with a few processors separated from the rest, since then the network will continue to be able to function. Many interconnection networks have been investigated in this aspect, when the number of faults is roughly twice the connectivity, see [12, 24]. One can even go further and ask what happens when more faulty vertices appear. This has been investigated for the hypercube in [44–46] and for certain Cayley graphs generated by transpositions in [13], and it has been shown that the surviving network has a large connected component containing almost all non-faulty vertices.

The process of identifying faulty processors in a system by analyzing the outcomes of available inter-processor tests is called *system-level diagnosis*. In 1967, Preparata, Metze, and Chien [39] established a foundation of system diagnosis and an original diagnostic model, called the PMC model. Its target is to identify the exact set of all faulty vertices before their repair or replacement. All tests are performed between two adjacent processors, and it is assumed that a test result is reliable (respectively, unreliable) if the processor that initiates the test is fault-free (respectively, faulty). The comparison-based diagnosis models, first proposed by Malek [37] and Chwa and Hakimi [16], are considered to be a practical approach for fault diagnosis in the multiprocessor systems. In these models, the same job is assigned to a pair of processors in the system and their outputs are compared by a central observer. This central observer performs diagnosis using the outcomes of these comparisons. Maeng and Malek [36] extended Malek's comparison approach to allow the comparisons carried out by the processors themselves. Sengupta and Dahbura [40] developed this comparison approach such that the comparisons have no central unit involved.

Lin *et al.* [35] introduced the conditional diagnosis under the comparison model. By evaluating the size of connected components, they obtained that the conditional diagnosability of the star graph S_n under the comparison model is $3n - 7$, which is about three times larger that the classical diagnosability of star graphs. In the same method, Hsu *et al.* [20] have recently proved that the conditional diagnosability of the hypercube Q_n is $3n - 5$. This idea is attributed to Lai *et al.* [25] who are the first to use a conditional diagnosis strategy. They obtained that the conditional diagnosability of the hypercube Q_n is $4n - 7$ under the PMC model. Furthermore, Hsu *et al.* [20] exposed the difference between these two conditional diagnosis models.

The dual-cube, proposed as a generalization of the hypercubes in an attempt to solve the scalability problem of the hypercubes, while preserving its attractive features, has been extensively studied [21–23, 26–32, 41]. Chen *et al.* [6, 7] first showed

that the diagnosability of the distributed system modeled by dual-cube DC_n is $n+1$ under the PMC model, and presented an adaptive diagnosis provided that at most $n+1$ processes are faulty. Based on the fault tolerance of the dual-cube, this paper considers its conditional diagnosability under the comparison diagnosis model.

The rest of this paper is organized as follows. Section 2 introduces some definitions, notations and the structure of the dual-cube DC_n, and shows that DC_n is a Cayley graph based on semi-direct product in group theory. Section 3 is devoted to the fault resiliency of DC_n, and derives the extra connectivity. Section 4 concentrates on the conditional diagnosability of DC_n. Section 5 concludes the paper.

2. Dual-Cubes

An interconnection network is conveniently represented by an undirected graph. The vertices (edges) of the graph represent the nodes (links) of the network. Throughout this paper, the terms vertex and node, edge and link, and graph and network are used interchangeably. For notation and terminology not defined here we follow [42]. Specifically, we use a graph $G = G(V, E)$ to represent an interconnection network, where a vertex $u \in V$ represents a processor and an edge $(u, v) \in E$ represents a link between vertices u and v. If at least one end-vertex of an edge is faulty, the edge is said to be faulty; otherwise, the edge is said to be fault-free.

For any vertex u of the graph $G = (V, E)$, $N(u)$ denotes the set of all neighbors of u, i.e., $N(u) = \{v \mid (u, v) \in E\}$. We also denote, by $|N(u)|$, the degree $d(u)$ of u. The parameters $\Delta(G) = \max\{d(u) \mid u \in V(G)\}$ and $\delta(G) = \min\{d(u) \mid u \in V(G)\}$ are the maximum and the minimum degree of the graph G. Let S be a subset or a subgraph of $V(G)$, whose order is denoted by $|S|$. The subgraph of G induced by S, denoted by $G[S]$, is the graph with the vertex-set S and the edge-set $\{(u, v) \mid (u, v) \in E(G), u, v \in S\}$. Let S be a subgraph of G or a subset of $V(G)$, and let $N(S) = \bigcup_{u \in S} N(u) \setminus S$. We also denote $N[S] = N(S) \cup S$. For brevity, $N[u] = N(u) \cup \{u\}$, $N(\{u, v\})$ and $N[\{u, v\}]$ are written as $N(u, v)$ and $N[u, v]$, respectively. We use $d(u, v)$ to denote the distance between u and v, the length of a shortest path between u and v in G. The diameter of G is defined as the maximum distance between any two vertices in G.

For any subset $F \subset V$, the notation $G - F$ denotes a graph obtained by removing all vertices in F from G and deleting those edges with at least one end-vertex in F, simultaneously. If $G - F$ is disconnected, F is called a *separating set*. A separating set F is called a *k-separating set* if $|F| = k$. The maximal connected subgraphs of $G - F$ are called *components*. The *connectivity* $\kappa(G)$ of G is defined as the minimum k for which G has a k-separating set; otherwise $\kappa(G)$ is defined as $n - 1$ if $G = K_n$. A graph G is said to be *k-connected* if $\kappa(G) \geq k$. A k-separating set is called to be *minimum* if $k = \kappa(G)$.

The interconnection network has been an important research area for parallel and distributed computer systems. Network reliability is one of the major factors in
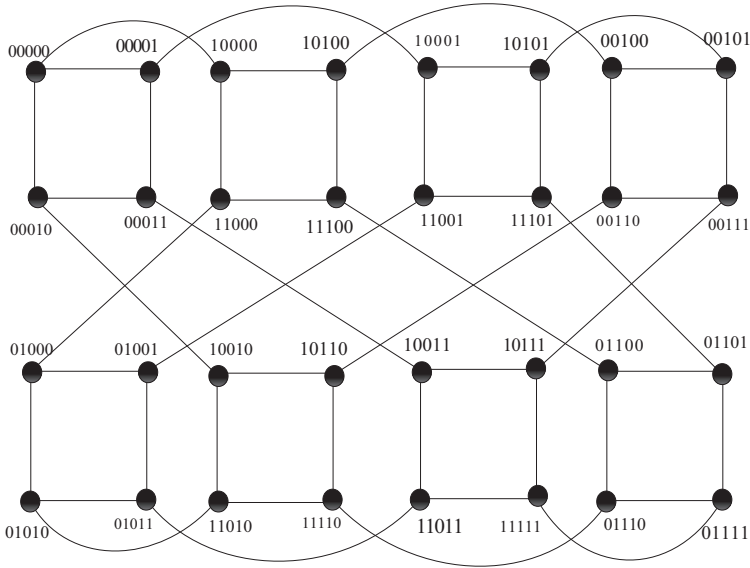
Fig. 1. The dual-cube DC_2.

designing the topology of an interconnection network. Because of its elegant topo-logical properties and the ability to emulate a wide variety of other frequently used networks, the hypercube has been one of the most popular interconnection net-works for parallel computer/communication systems. However, when dealing with the parallel computers of very large scale, the port limitation due to the technology greatly forbids the use of hypercube networks.

The dual-cube DC_n(see Fig. 1 for $n = 2$), first introduced by Li and Peng [27], mitigates the problem of increasing number of links in the large-scale hypercube network while it keeps most of the topological properties of the hypercube network. The number of vertices of an n-dimensional dual-cube DC_n is equal to the number of vertices of a $(2n + 1)$-dimensional hypercube Q_{2n+1}. Each vertex in Q_{2n+1} is adjacent to $2n+1$ neighbors and the total number of edges of Q_{2n+1} is $(2n+1) \times 2^{2n}$, while each vertex in DC_n is adjacent to $n + 1$ neighbors and the total number of edges of DC_n is $(n+1) \times 2^{2n}$. Although any DC_n has much less edges than Q_{2n+1} with the same number of vertices, the diameter of DC_n, $2n+2$, is of the same order of the diameter of Q_{2n+1}, which is $2n + 1$. The dual-cube DC_n has 2^{n+1} copies of Q_n, which are divided into two classes, Class 0 and Class 1. Each class consists of 2^n copies of Q_n and each copy is called a cluster. Every pair of clusters from the opposite classes has an edge.

Definition 1. [27] A dual-cube DC_n consists of 2^{2n+1} vertices, and each vertex is labeled with a unique $(2n + 1)$-bits binary string and has $n + 1$ neighbors. There is a link between two nodes $u = u_{2n}u_{2n-1} \ldots u_0$ and $v = v_{2n}v_{2n-1} \ldots v_0$ if and only

if u and v differ exactly in one bit position i under the the following conditions:
(1) if $0 \leq i \leq n - 1$, then $u_{2n} = v_{2n} = 0$; and
(2) if $n \leq i \leq 2n - 1$, then $u_{2n} = v_{2n} = 1$.

Each node in a DC_n is identified by a unique $(2n + 1)$-bit number, an *id*. Each *id* contains three parts: 1-bit class-*id*, n-bit cluster-*id* and n-bit node-*id*. We use *id*=(class-*id*, cluster-*id*, node-*id*) to denote the node address. The bit-position of cluster-*id* and node-*id* depends on the value of class-*id*. If class-*id*=0 (resp. class-*id*=1), then node-*id* (resp. cluster-*id*) is the rightmost n bits and cluster-*id* (resp. node-*id*) is the next n bits. An edge in a cluster is called a *cube edge*; and an edge connecting two nodes in two clusters of distinct classes is called a *cross edge*. In the other word, $e = (u, v)$ is a cross edge if and only if u and v differ in the leftmost bit position.

The *Hamming weight* of a vertex u, denoted by $w(u)$, is the number of i such that $u_i = 1$. The Hamming distance $h(u, v)$ between two vertices u and v is the number of different bits in the corresponding strings of both vertices. Clearly, $h(u, v) = 1$ if u and v are adjacent. Let $V_b = \{u \mid w(u) \text{ is even}\}$ and $V_w = \{u \mid w(u) \text{ is odd}\}$. Obviously, $V_b \cap V_w = \emptyset$, and $V(DC_n) = V_b \cup V_w$. There is no edge between the clusters of the same class. If two nodes are in one cluster, or in two clusters of distinct classes, the distance between the two nodes is equal to its Hamming distance, the number of bits where the two nodes have distinct values. Otherwise, it is equal to the Hamming distance plus two: one for entering a cluster of another class and one for leaving.

In addition, the following property of DC_n is useful, which can be checked by the definition of DC_n. For any two distinct vertices u and v in DC_n,

$$|N(u) \cap N(v)| \begin{cases} = 0, & \text{if } d(u, v) \geq 3; \\ \leq 2, & \text{if } d(u, v) = 2; \\ = 0, & \text{if } d(u, v) = 1. \end{cases} \quad (1)$$

Recently, Chen and Kao [8] have proposed a more convenient new labelling for vertices of dual-cubes. Now, we modify it as follows.

Definition 2. [8] The dual-cube DC_n consists of two classes, Class 0 and Class 1. For $i \in \{0, 1\}$, Class i has 2^n copies of Q_n, namely, $DC_n^{i,0}, DC_n^{i,1}, \ldots, DC_n^{i,2^n-1}$, and each $DC_n^{i,j}$ is called a cluster. We shall label any vertex in $DC_n^{i,j}$ of DC_n by (i, j, k), where k is the vertex id in Q_n. Two vertices (i, j, k) and (i', j', k') are adjacent in DC_n if and only if one of the following conditions are satisfied:
(1) $i = i'$, $j = j'$ and the vertices k and k' are adjacent in Q_n; and
(2) $|i - i'| = 1$, $j = k'$, and $k = j'$.

Efficient algorithms that find disjoint paths for node-to-node routing, node-to-set routing, and set-to-set routing in dual-cubes are presented by Li and Peng [28], Kaneko and Peng [22], and Kaneko and Peng [23], respectively. Using global and

local information of faulty status, Li *et al.* [31] proposed two efficient algorithms for finding a fault-free routing path between any two fault-free nodes in the dual cube with a large number of faulty nodes, respectively. Li *et al.* [30] showed that the collective communications can be done in dual-cube with almost the same communication times as in hypercube. To avoid collecting global fault information whose broadcasting propagation will incur traffic congestion and even new component failure in the networks, Jiang and Wu [21] proposed a fault tolerant routing based on a limited global information in dual-cube. Li *et al.* [29] showed that DC_n contains a fault-free hamiltonian cycle even if it has up to $n-1$ edge faults for $n \geq 2$. Subsequently, they [32] showed that there exists a fault-free cycle containing at least $2^{2n+1} - 2f$ vertices in DC_n, $n \geq 3$, with $f \leq n$ faulty nodes. Lai and Tsai [26] obtained the vertex bipancyclicity of dual-cube, and showed that dual-cube is bipancyclic even if it has up to $n-1$ faulty edges. Shih *et al.* [41] proved the existence of $n+1$ mutually independent hamiltonian cycles in dual-cube.

Definition 3. [42] Let H be a finite group, and $S \subset H$ be a generating set of H. The right *Cayley graph*, $G = Cay(H, S)$, of H corresponding to S is defined as: $V(G) = H$, $E(G) = \{(h, hs) \mid h \in H, s \in S\}$. $G = Cay(H, S)$ is undirected if S is symmetric, i.e., $S^{-1} = S$; and $G = Cay(H, S)$ has no loop if S does not contains the identity of H.

Let S_2 be the symmetric group of order 2, and Z_2 be the cyclic group of order 2. Obviously, S_2 is isomorphic to Z_2. We denote that $S_2 = \{e, \epsilon\}$ with $\epsilon^2 = e$, the identity of S_2.

Let Γ be the direct product of n cyclic group Z_2's. i.e., $\Gamma = Z_2^n = Z_2 \times Z_2 \times \cdots \times Z_2$. Obviously, Γ is also a group with the identity $e_0 = (0, 0, \ldots, 0)$, and its generating set $S = \{e_1, e_2, \ldots, e_n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, in which the only "1" is in the ith position from the right.

Let S_2 act on the set of the product $\Gamma \times \Gamma$ via

$$(\alpha, \beta)^\gamma = \begin{cases} (\alpha, \beta) & \text{if } \gamma \text{ is the identity of } S_2; \\ (\beta, \alpha) & \text{otherwise.} \end{cases} \tag{2}$$

Define the semi-direct product $(\Gamma \times \Gamma) \rtimes S_2$ such that for any two elements (α, β, γ) and $(\alpha', \beta', \gamma')$, the operation $*$ is defined as follows.

$$\begin{aligned} (\alpha, \beta, \gamma) * (\alpha', \beta', \gamma') &= (\alpha, \beta, \gamma) * ((\alpha', \beta')^\gamma, \gamma') \\ &= \begin{cases} (\alpha\alpha', \beta\beta', \gamma\gamma') & \text{if } \gamma \text{ is the identity of } S_2; \\ (\alpha\beta', \beta\alpha', \gamma\gamma') & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

Let $S = \{(e_0, e_1, e), (e_0, e_2, e), \ldots, (e_0, e_n, e), (e_0, e_0, \epsilon)\}$. Then it is easy to check that the dual-cube DC_n is isomorphic to the Cayley graph of $(\Gamma \times \Gamma) \rtimes S_2$ corresponding to S, that is, $DC_n \cong Cay((\Gamma \times \Gamma) \rtimes S_2, S)$. We state this result as the following theorem.

Theorem 1. DC_n *is a Cayley graph, i.e.,* $DC_n \cong Cay((\Gamma \times \Gamma) \rtimes S_2, S)$, *where* $\Gamma = Z_2^n = Z_2 \times Z_2 \times \cdots \times Z_2$; *and so* DC_n *is vertex transitive.*

3. Fault Tolerance of the Dual-Cubes

The connectivity $\kappa(G)$ of a graph G is an important parameter to measure the fault tolerance of the network, while it has an obvious deficiency in that it tacitly assume that all elements in any subset of G can potentially fail at the same time. To compensate for this shortcoming, it would seem natural to generalize the classical connectivity by introducing some conditions or restrictions on the separating set S and/or the components of $G - S$.

The connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal results in a disconnected or a trivial (one vertex) graph. A k-regular k-connected graph is *super k-connected* if any one of its minimum separating sets is a set of the neighbors of some vertex. If, in addition, the deletion of a minimum separating set results in a graph with two components (one of which has only one vertex), then the graph is *tightly super k-connected*. For example, the complete bipartite graph $K_{n,n}$ is n-super connected but not tightly n-super connected. The notions of super connectedness and tightly super connectedness are first introduced in [1] and [10], respectively.

Esfahanian [17] first introduced the concepts of the restricted separating set and the restricted connectivity of a graph G. A set S of vertices is a *restricted separating set* if $G - S$ is disconnected and $N(x)$ is not completely contained in S for any vertex x in G. The *restricted connectivity* of G, denoted by $\kappa_r(G)$, is the minimum cardinality of a restricted separating set. Considering that it is not easy to examine whether a separating set is restricted, Xu *et al.* [43] formally proposed the super connectivity, a weaker concept than the restricted connectivity. A separating set S of G is *super* if $G - S$ contains no isolated vertices. The *super connectivity* of G, denoted by $\kappa_s(G)$, is the minimum cardinality of a super separating set. Clearly, $\kappa(G) \leq \kappa_s(G) \leq \kappa_r(G)$ if $\kappa_r(G)$ exists.

Fábrega and Fiol [18] generalized the concept of super connectivity to *h-extra connectivity* for an undirected graph. Let G be a connected undirected graph, and h be an integer with $0 \leq h \leq \delta(G)$. A subset $S \subset V(G)$ is said an *h-extra separating set* if $G - S$ is disconnected and every connected component contains at least $h + 1$ vertices. The *h-extra connectivity* $\kappa_o^{(h)}(G)$ is defined as

$$\kappa_o^{(h)}(G) = \min\{|S| \mid S \text{ is an } h - \text{extra separating set of } G\}.$$

It follows from the definition that the h-extra connectivity can provide a more accurate measurement than the traditional connectivity or super connectivity for fault tolerance of a large-scale interconnection network.

Usually, if the surviving graph $G - S$ contains a large connected component C when $G - S$ is not connected, the component C may be used as the functional subsystem, without incurring severe performance degradation. Thus, in evaluating a

distributed system, it is indispensable to estimate the size of the maximal connected components of the underlying graph when the structure begins to lose processors.

Yang *et al.* [44–46] proved that the hypercube Q_n with f faulty processors has a component of size at least $2^n - f - 1$ if $f \leq 2n - 3$, and of size at least $2^n - f - 2$ if $f \leq 3n - 6$. Yang *et al.* [47] also obtained a similar result for the star graph S_n. Cheng *et al.* [11, 14] gave a more detailed result for S_n. The removal of any separating set of at most $2n - 4$ vertices from S_n results in exactly two components, one of them is a single vertex or edge. Cheng and Lipták [13] generalized this result for S_n with linearly many faults. Cheng *et al.* [15] presented a similar result for the 2-tree-generated networks with linearly many faults. In this section, we detail on the fault resilience of the dual-cube DC_n.

Lemma 2. [27, 28, 32] *For $n \geq 3$, DC_n has the following combinatorial properties.*

(1) *DC_n has 2^{2n+1} vertices with regular degree $n + 1$;*

(2) *DC_n has vertex connectivity of $n + 1$, and edge connectivity $n + 1$;*

(3) *Assume that two vertices u and v differ in k bit-positions. Then the distance between u and v is $d(u, v) = k + 2$ if u and v are in different clusters of the same class; otherwise $d(u, v) = k$. DC_n has diameter $2n + 2$;*

(4) *DC_n is bipartite graph.*

Throughout this paper, the notation F denotes a set of faulty vertices in DC_n. A subgraph H of DC_n is called *fault-free* if $V(H) \cap F = \emptyset$. We denote $\langle n \rangle = \{0, 1\} \times \{0, 1, 2, \ldots, 2^n - 1\}$ and let

$$F_{i,j} = DC_n^{i,j} \cap F \quad \text{and} \quad f_{i,j} = |F_{i,j}| \text{ for } (i, j) \in \langle n \rangle; \tag{4}$$

$$I = \{(i, j) \mid f_{i,j} = |F_{i,j}| \geq n \text{ for } (i, j) \in \langle n \rangle\}, \quad J = \langle n \rangle - I. \tag{5}$$

Lemma 3. [20, 44–46] *Let F be a set of faulty vertices in the hypercube Q_n with $|F| \leq 2n - 3$ and $n \geq 3$. If $Q_n - F$ is disconnected, then $Q_n - F$ has two connected components and one of which is an isolated vertex.*

Lemma 4. *Let F be a set of faulty vertices in DC_n with $|F| \leq 3n - 3$ and $n \geq 3$. Then $DC_n^J - F_J$ is connected.*

Proof. For any $(i, j) \in J$, $f_{i,j} \leq n - 1$, $DC_n^{i,j} - F_{i,j}$ is connected.

Note that each class has 2^n clusters. Since $2^n - |F| \geq 2^n - (3n - 3) \geq 2$, there exist some cluster DC_n^{0,j_0} in class 0 and some cluster DC_n^{1,j_1} in class 1, each of which has no vertex in F. Obviously, DC_n^{0,j_0} is connected to DC_n^{1,j_1} for there is a fault-free cross edge between them.

If $|I| \geq 3$, then $|F| \geq 3n$, which contradicts our hypothesis. Thus, $|I| \leq 2$. Now we discuss as follows.

Case 1 There exists exactly one subgraph DC_n^{0,x_0}(respectively, DC_n^{1,y_1}), such that $(0, x_0) \in I$ (respectively, $(1, y_1) \in I$).

Since $2^n - 1 > 2n - 3$, for any subgraph $DC_n^{1,y}$ with $(1, y) \in J$, there exists one fault-free cross edge between $DC_n^{1,y}$ and some $DC_n^{0,x}$, such that $DC_n^{0,x}$ has no

faulty vertex and is connected to DC_n^{1,j_1}. Since $2^n > 2n - 3$, for any $DC_n^{0,x'}$ with $(0, x') \in J$, there exists a fault-free cross edge between $DC_n^{0,x'}$ and some $DC_n^{1,y'}$ such that $DC_n^{1,y'}$ has no faulty vertex and is connected to DC_n^{0,j_0}. Thus, $DC_n^J - F_J$ is connected.

Case 2 There exist exactly two subgraphs DC_n^{i,x_0}, DC_n^{i,x_1} such that $(i, x_0), (i, x_1) \in I$, where i is 0 or 1.

Without loss of generality, we may say that i is 0. Since $2^n - 2 > n - 3$, for any $DC_n^{1,y}$ with $(1, y) \in J$, there exists one fault-free cross edge between $DC_n^{1,y}$ and some $DC_n^{0,x}$, such that $DC_n^{0,x}$ has no faulty vertex and is connected to DC_n^{1,j_1}. Since $2^n > n - 3$, for any $DC_n^{0,x'}$ with $(0, x') \in J$, there exists one fault-free cross edge between $DC_n^{0,x'}$ and some $DC_n^{1,y'}$ such that $DC_n^{1,y'}$ has no faulty vertex and is connected to DC_n^{0,j_0}. Thus, $DC_n^J - F_J$ is connected.

Case 3 There exist exactly two subgraphs DC_n^{0,x_0}, DC_n^{1,y_0} such that $(0, x_0), (1, y_0) \in I$.

Since $2^n - 1 > n - 3$, for any $DC_n^{1,y}$ with $(1, y) \in J$, there exists one fault-free cross edge between $DC_n^{1,y}$ and some $DC_n^{0,x}$, such that $DC_n^{0,x}$ has no faulty vertex and is connected to DC_n^{1,j_1}. Since $2^n - 1 > n - 3$, for any $DC_n^{0,x'}$ with $(0, x') \in J$, there exists one fault-free cross edge between $DC_n^{0,x'}$ and some $DC_n^{1,y'}$ such that $DC_n^{1,y'}$ has no faulty vertex and is connected to DC_n^{0,j_0}. Thus, $DC_n^J - F_J$ is connected. □

Theorem 5. *For $n \geq 3$, DC_n is tightly super $n + 1$-connected.*

Proof. Let F be a minimum separating set in DC_n. Then, using the notations defined in (4), we have that

$$|F| = \sum_{(i,j) \in \langle n \rangle} f_{i,j} = \kappa(DC_n) = n + 1.$$

By the definition of tightly super connectivity, we need to show that $DC_n - F$ has exactly two components, one of them is a single vertex. We consider three cases.

Case 1 There exists some $(i_0, j_0) \in \langle n \rangle$ such that $f_{i_0,j_0} = n + 1$.

In this case, by Lemma 2, $f_{i,j} = 0$ for any $(i, j) \in \langle n \rangle$ and $(i, j) \neq (i_0, j_0)$, $DC_n^{i,j}$ is connected. $DC_n - DC_n^{i_0,j_0}$ is still connected by Lemma 4. Every vertex of $DC_n^{i_0,j_0} - F_{i_0,j_0}$ has exactly one fault-free neighbor vertex in $DC_n - DC_n^{i_0,j_0}$, so $DC_n - F$ is still connected, a contradiction.

Case 2 $f_{i_0,j_0} = n$ for some $(i_0, j_0) \in \langle n \rangle$.

By the hypothesis, there exists some $(i_1, j_1) \in \langle n \rangle$ with $(i_1, j_1) \neq (i_0, j_0)$ such that $f_{i_1,j_1} = 1$. Since $DC_n^{i,j}$ is isomorphic to the n-dimensional hypercube Q_n which is n-connected, $DC_n^{i,j}$ is still connected for any $(i, j) \in \langle n \rangle$ with $(i, j) \neq (i_0, j_0)$. As $DC_n^{i_0,j_0}$, which is isomorphic to the hypercube Q_n, is tightly super n-connected by Lemma 3, $DC_n^{i_0,j_0} - F_{i_0,j_0}$ has at most one vertex isolated from $DC_n - (V(DC_n^{i_0,j_0}) \cup (F - F_{i_0,j_0}))$. Since $f_{i_1,j_1} = 1$, $DC_n - F$ has exactly two connected components, one of which is an isolated vertex.

Case 3 $f_{i,j} \leq n - 1$ for any $(i, j) \in \langle n \rangle$.

Obviously, $DC_n^{i,j}$ is still connected. Since $2^n > n + 1$, there exist some DC_n^{0,j_0} and DC_n^{1,j_1}, each of which has no vertex in F. Obviously, DC_n^{0,j_0} is connected to DC_n^{1,j_1} for there is a fault-free cross edge between them.

Since $2^n > n + 1$, for any $DC_n^{1,y}$ with $(1, y) \in J$, there exists one fault-free cross edge between $DC_n^{1,y}$ and some $DC_n^{0,x}$, which has no faulty vertex and is connected to DC_n^{1,j_1}. Since $2^n > n + 1$, for any $DC_n^{0,x'}$ with $(0, x') \in J$, there exists one fault-free cross edge between $DC_n^{0,x'}$ and some $DC_n^{1,y'}$ which has no faulty vertex and is connected to DC_n^{0,j_0}. Thus, $DC_n - F$ is connected, a contradiction. $\qquad\square$

Lemma 6. *Let F be a separating set of DC_n with $|F| \leq 3n - 3$ and $n \geq 3$. If there is some $(i_0, j_0) \in \langle n \rangle$ such that $|F| - f_{i_0, j_0} \leq 1$, then $DC_n - F$ has exactly two components, one of which is a single vertex.*

Proof. We use the notations defined in (4) and (5) in the following. By the hypothesis, for any $(i, j) \in \langle n \rangle - \{(i_0, j_0)\}$,

$$f_{i,j} \leq |F| - f_{i_0, j_0} \leq 1.$$

Since $DC_n - F$ is disconnected, and $DC_n - (DC_n^{i_0, j_0} \cup F)$ is connected by Lemma 4, there is a component of $DC_n - F$ that contains no vertices in $DC_n^J - F_J$. Let H be a union of such components of $DC_n - F$. Thus $N_{DC_n - DC_n^{i_0, j_0}}(H) \subseteq F \setminus F_{i_0, j_0}$, and we have that

$$|V(H)| \leq |F| - f_{i_0, j_0} \leq 1,$$

which yields $|V(H)| \leq 1$, that is, H is a single vertex, say u. By the choice of H, other components of $DC_n - F$ must be contained in $DC_n^J - F_J$. Since $DC_n^J - F_J$ is connected by Lemma 4, $DC_n - (F \cup \{u\})$ is connected. It follows that $DC_n - F$ has exactly two components, one of which is a single vertex. The lemma follows. $\square$

Lemma 7. *Let F be a separating set of DC_n with $|F| \leq 3n - 3$ and $n \geq 3$, and let H be the union of connected components of $DC_n - F$, whose vertices are totally distributed in $DC_n^{i,j} - F_{i,j}$ for some $(i, j) \in \langle n \rangle$. If $N_{DC_n^{i,j}}(H) \subseteq F_{i,j}$, then $|V(H)| \leq 2$.*

Proof. Let $h = |V(H)|$. We want to prove $h \leq 2$. Suppose to the contrary that $h \geq 3$. Take a subset $T \subseteq V(H)$ with $|T| = 3$. Let $T' = V(H - T)$. By the hypothesis, $N_{DC_n^{i,j}}(T) \setminus T' \subseteq F_{i,j}$. Note that $DC_n^{i,j}$ is isomorphic to hypercube Q_n. We denote $T = DC_n^{i,j}[x, y, z]$, and discuss as follows.

If $H[T]$ has no edges, then $N_{DC_n^{i,j}}(u) - T = N_{DC_n^{i,j}}(u)$ for any vertex $u \in \{x, y, z\}$ with $|N_{DC_n^{i,j}}(u)| = n$. Note that $DC_n^{i,j}$ is isomorphic to the hypercube Q_n, $|N_{DC_n^{i,j}}(u) \cap N_{DC_n^{i,j}}(v)| \leq 2$ for any two distinct vertices $u, v \in \{x, y, z\}$. Furthermore, if

$$|N_{DC_n^{i,j}}(x) \cap N_{DC_n^{i,j}}(y)| = |N_{DC_n^{i,j}}(x) \cap N_{DC_n^{i,j}}(z)| = |N_{DC_n^{i,j}}(y) \cap N_{DC_n^{i,j}}(z)| = 2,$$

then

$$|N_{DC_n^{i,j}}(x) \cap N_{DC_n^{i,j}}(y) \cap N_{DC_n^{i,j}}(z)| = 1.$$

Thus, by the principle of inclusion and exclusion, we have

$$|N_{DC_n^{i,j}}(T)| = \sum_{u \in T} |N_{DC_n^{i,j}}(u) - T| - (\sum_{u \neq v \in T} |(N_{DC_n^{i,j}}(u) - T) \cap (N_{DC_n^{i,j}}(v)| - T|$$
$$+ |(N_{DC_n^{i,j}}(x) - T) \cap (N_{DC_n^{i,j}}(y) - T) \cap (N_{DC_n^{i,j}}(z) - T)|$$
$$\geq 3n - 5.$$

If $H[T]$ has only one edge, say $e = (x, y)$, then x and y have no common neighbors, z and x (resp. y) have at most two common neighbors by (1), but two cases can not occur meanwhile as there are no cycles of odd length. It follows that

$$|N_{DC_n^{i,j}}(T)| \geq 3n - 4.$$

If $H[T]$ has two edges, we deduce, by (1), that

$$|N_{DC_n^{i,j}}(T)| \geq 3n - 5.$$

Summing up all cases above, we have that

$$f_{i,j} \geq |N_{DC_n^{i,j}}(T) \setminus T'|$$
$$\geq |N_{DC_n^{i,j}}(T)| - (h - 3)$$
$$\geq 3n - 5 - (h - 3)$$
$$= 3n - 2 - h,$$

that is,

$$f_{i,j} \geq 3n - 2 - h. \tag{6}$$

By the definition of H, we have $N_{DC_n - DC_n^{i,j}}(H) \subseteq F - F_{i,j}$ and $|F| - f_{i,j} \geq h$. Thus, we deduce that

$$f_{i,j} \leq |F| - h \leq 3n - 3 - h,$$

that is,

$$f_{i,j} \leq 3n - 3 - h. \tag{7}$$

Combining (6) with (7), we deduce a contradiction. Thus, we have $h \leq 2$. $\square$

Theorem 8. *The 1-extra connectivity of DC_n ($n \geq 3$) is $\kappa_o^{(1)}(DC_n) = 2n$.*

Proof. We choose an edge (u, v) in some subgraph $DC_n^{i,j}$. Obviously, $|N(u, v)| = 2n$, $DC_n - N[u, v]$ is still connected by Lemma 4. Each connected component of $DC_n - N(u, v)$ has order at least two. Thus, we have $\kappa_o^{(1)}(DC_n) \leq 2n$.

Now we show that $\kappa_o^{(1)}(DC_n) > 2n - 1$. Let F be an arbitrary set of faulty vertices in DC_n with $|F| \leq 2n - 1$ such that $DC_n - F$ is disconnected.

If $|I| \geq 2$, then $|F| \geq 2n$, a contradiction. Now, we set $I = \{(i, j)\}$.

Let H be the union of components of $DC_n - F$ that contain no vertex in $DC_n^J - F_J$. Thus, H is in $DC_n^{i,j}$. By the choice of H, other components of $DC_n - F$ must be contained in $DC_n^J - F_J$. Since $DC_n^J - F_J$ is connected, $DC_n - (F \cup V(H))$ is connected. Thus, to complete the proof of the theorem, we only need to show that $|H| = 1$. By Lemma 7, we only need to show that $|H| = 2$ is not possible. Suppose to the contrary that $H = DC_n^{i,j}[u, v]$. Obviously, $N(u, v) \subseteq F$.

If u is not adjacent to v, then $d(u, v) \geq 2$, and $|N(u) \cap N(v)| \leq 2$ by (1). Thus, we have

$$
\begin{aligned}
|F| &\geq |N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \\
&= 2(n+1) - |N(u) \cap N(v)| \\
&\geq 2(n+1) - 2 > |F|,
\end{aligned}
$$

a contradiction.

If (u, v) is an edge of DC_n, then $|N(u) \cap N(v)| = 0$ by (1). Thus, we have

$$
\begin{aligned}
|F| &\geq |N(u, v)| = |N(u)| + |N(v)| - |\{u, v\}| \\
&= 2(n+1) - 2 \\
&> |F|,
\end{aligned}
$$

a contradiction. $\qquad\square$

We now discuss the fault tolerance of DC_n with more faulty vertices, up to $3n - 3$, when $n \geq 3$.

Lemma 9. *Let F be an arbitrary set of faulty vertices in DC_n ($n \geq 3$) with $|F| \leq 3n - 3$. If $DC_n - F$ is disconnected, then it either has two components, one of which is an isolated vertex or an isolated edge, or has three components, two of which are isolated vertices.*

Proof. Since $DC_n - F$ is disconnected, F is a separating set of DC_n.

If there exists some $(i, j) \in \langle n \rangle$ such that $f_{i,j} \geq 3n - 4$, and so

$$
|F| - f_{i,j} \leq 1,
$$

by Lemma 6, $DC_n - F$ has exactly two components, one of which is a single vertex, and so the theorem holds. Now, we consider that $f_{i,j} \leq 3n - 5$ for any $(i, j) \in \langle n \rangle$.

Let H be the union of components of $DC_n - F$ that contain no vertex in $DC_n^J - F_J$, and let $h = |V(H)|$. Since $DC_n^J - F_J$ is connected, H is in $DC_n^I - F_I$. By the choice of H, other components of $DC_n - F$ must be contained in $DC_n^J - F_J$. Since $DC_n^J - F_J$ is connected, $DC_n - (F \cup V(H))$ is connected. Thus, to complete the proof of the theorem, we only need to show that $h \leq 2$.

If $|I| \geq 3$, then $|F| \geq 3n > 3n - 3 \geq |F|$, a contradiction. Now, we set $1 \leq |I| \leq 2$ in the following.

If $|I| = 1$, then $h \leq 2$ by Lemma 7. Now we suppose that $I = \{(i_1, j_1), (i_2, j_2)\}$, and let h_1 and h_2 be the numbers of vertices of H that lie in $DC_n^{i_1,j_1}$ and $DC_n^{i_2,j_2}$, respectively.

Obviously, $f_{i,j} \leq 2n - 3$ for any $(i, j) \in I$; otherwise, $|F| \geq 3n - 2$, which is a contradiction. We have $h_1 \leq 1$ and $h_2 \leq 1$ by Lemma 3. Thus, $h = h_1 + h_2 \leq 2$. $\square$

Theorem 10. *The* 2-*extra connectivity of* DC_n $(n \geq 3)$ *is* $\kappa_o^{(2)}(DC_n) = 3n - 2$.

Proof. By Lemma 9, we have that $\kappa_o^{(2)}(DC_n) > 3n - 3$. It suffices to show that $\kappa_o^{(2)}(DC_n) \leq 3n - 2$. We choose a cycle $C = (x, y, u, v, x)$, of length four, in some cluster $DC_n^{i,j}$.

Since the cluster $DC_n^{i,j}$ is isomorphic the hypercube Q_n, $|N_{DC_n^{i,j}}(x, y, u)| = 3n - 5$. By the definition of dual-cubes, every vertex of $\{x, y, u\}$ has exactly one neighbor outside $DC_n^{i,j}$, and these three neighbors (say, $\overline{x}$, $\overline{y}$ and $\overline{u}$) are in different clusters whose class are different from i. Thus, $|N[x, y, u]| = 3n - 5 + 3 = 3n - 2$.

Furthermore, $DC_n - DC_n^{i,j} - \{\overline{x}, \overline{y}, \overline{u}\}$ is still connected. In fact, there exists some cluster $DC_n^{i,l}$, which is different from $DC_n^{i,j}$, where $l \neq i$. Except for at most one vertex of $\{\overline{x}, \overline{y}, \overline{u}\}$, each of the clusters $DC_n^{\overline{i},0}, DC_n^{\overline{i},1}, \ldots, DC_n^{\overline{i},2^n-1}$ is connected, and has exactly one neighbor in $DC_n^{i,l}$. Similarly, Each cluster $DC_n^{i,s}$ (where $s \neq j, l$) has exactly one vertex adjacent to $DC_n^{\overline{i},0}$. Thus, $DC_n - DC_n^{i,j} - \{\overline{x}, \overline{y}, \overline{u}\}$ is connected.

Since every vertex of $DC_n^{i,j} - N_{DC_n^{i,j}}[x, y, u]$ has exactly one neighbor in $DC_n - DC_n^{i,j} - \{\overline{x}, \overline{y}, \overline{u}\}$. Thus, $DC_n - N(x, y, u)$ has two connected components, one is the path $P = P(x, y, u)$, the other is $DC_n - N[x, y, u]$. Obviously, each of these two components has order at least three.

From the discussion above, we have $\kappa_o^{(2)}(DC_n) = 3n - 2$ for $n \geq 3$. $\square$

4. Diagnosability of Dual-Cubes

The comparison diagnosis strategy of a graph $G = (V, E)$ can be modeled as a multi-graph $M = (V, C)$, where C is a set of labelled edges. If the processors u and v can be compared by the processor w, there exists a labelled edge (u, v) in C, denoted by $(u, v)_w$. We call w the comparator of u and v. Since different comparators can compare the same pair of processors, M is a multi-graph. Denote the comparison result as $\sigma((u, v)_w)$ such that $\sigma((u, v)_w) = 0$ if the outputs of u and v agree, and $\sigma((u, v)_w) = 1$ if the outputs disagree. If the comparator w is fault-free and $\sigma((u, v)_w) = 0$, the processors u and v are fault-free; while if $\sigma((u, v)_w) = 1$, at least one of the three processors u, v and w is faulty. The collection of the comparison results defined as a function $\sigma : C \to \{0, 1\}$, is called the *syndrome* of the diagnosis. If the comparator w is faulty, the comparison result is unreliable. A faulty comparator can lead to unreliable results, so a set of faulty vertices may produce different syndromes. A subset $F \subsetneq V$ is said to be *compatible* with a syndrome σ if σ can arise from the circumstance that all vertices in F are faulty and all vertices in $V - F$ are fault-free. A multiprocessor system G is said to be

diagnosable if, for every syndrome σ, there is a unique $F \subset V$ that is compatible with σ. A system is said to be t-diagnosable if the system is diagnosable as long as the number of faulty vertices does not exceed t. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, which is denoted by $t(G)$. Let $\sigma_F = \{\sigma \mid \sigma \text{ is compatible with } F\}$. Two distinct subsets F_1 and F_2 of $V(G)$ are said to be *indistinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_2} \neq \phi$, and *distinguishable* otherwise [20, 35, 40]. There are several different ways to verify whether a system is t-diagnosable under the comparison approach. The following lemma obtained by Sengupta and Dahbura [40] gives necessary and sufficient conditions to ensure distinguishability.

Lemma 11. [40] *Let G be a graph, F_1 and F_2 be two distinct subsets of vertices in G. The pair (F_1, F_2) is distinguishable if and only if at least one of the following conditions is satisfied.*

(1) *There are two distinct vertices u and $w \in V(G - F_1 \cup F_2)$ and a vertex $v \in F_1 \Delta F_2$ such that $(u, v)_w \in C$, where $F_1 \Delta F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$;*
(2) *There are two distinct vertices u and $v \in F_1 \setminus F_2$ (or $F_2 \setminus F_1$) and a vertex $w \in V(G - F_1 \cup F_2)$ such that $(u, v)_w \in C$.*

Lin *et al.* [35] introduced the so-called conditional diagnosability of a multi-processor system under the situation that no set of faulty vertices can contain all neighbors of any vertex in the system. A fault-set $F \subset V(G)$ is called a *conditional fault-set* if $N(v)$ is not subset of the faulty set F for every vertex v in $V(G)$. A system $G(V, E)$ is said to be conditionally t-diagnosable if F_1 and F_2 are distinguishable for each pair (F_1, F_2) of distinct conditional fault-sets in G with $|F_1| \leq t$ and $|F_2| \leq t$. The *conditional diagnosability* of G, denoted by $t_c(G)$ is defined as the maximum value of t for which G is conditionally t-diagnosable. Clearly, $t_c(G) \geq t(G)$. This section will focus on the conditional diagnosability of dual-cubes.

Lemma 12. *Let F_1 and F_2 be any two distinct conditional fault-sets of $|F_1| \leq 3n - 2$, $|F_2| \leq 3n - 2$ for $n \geq 3$. Denote by H the maximum component of $DC_n - F_1 \cap F_2$. Then, for every vertex $u \in F_1 \Delta F_2$, $u \in H$.*

Proof. Without loss of generality, we assume that $u \in F_1 - F_2$. Since F_2 is a conditional fault-set, there is a vertex $v \in (DC_n - F_2) - \{u\}$ such that $(u, v) \in E(DC_n)$. Suppose that u is not a vertex of H. Then v is not in H, so u and v are in a small component of $DC_n - F_1 \cap F_2$. Since F_1 and F_2 are distinct, we have

$$|F_1 \cap F_2| \leq 3n - 3.$$

Hence $\{u, v\}$ forms a component K_2 in $DC_n - F_1 \cap F_2$ by Lemma 9, that is to say, the vertex u is the unique neighbor of v in $DC_n - F_1 \cap F_2$. This is a contradiction since F_1 is a conditional fault-set, but all the neighbors of v are faulty in F_1. $\square$

Lemma 13. [35] *Let G be a graph with $\delta(G) \geq 2$, and let F_1 and F_2 be any two distinct conditional fault-sets of G with $F_1 \subset F_2$. Then, (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.*

Theorem 14. $t_c(DC_n) = 3n - 2$ for $n \geq 3$.

Proof. We first prove that $t_c(DC_n) \leq 3n - 2$ for $n \geq 3$. In fact, when $n \geq 3$, we select four vertices $x, y, z, u \in V(DC_n)$, such that (x, y, z, u) be a cycle of length four. Set $A = N[x, y, z]$, $F_1 = A - \{y, z\}$, and $F_2 = A - \{x, y\}$. We get

$$|F_1| = |F_2| = 3n - 1, \text{ and } |F_1 - F_2| = |F_2 - F_1| = 1.$$

It is easy to check that both F_1 and F_2 are two conditional fault-sets, and F_1 and F_2 are indistinguishable. Thus, we have

$$t_c(DC_n) \leq 3n - 2.$$

Now, we prove that $t_c(DC_n) \geq 3n - 2$ for $n \geq 3$.

Let F_1 and F_2 be any two distinct conditional fault-sets of DC_n with $|F_1| \leq 3n - 2$, $|F_2| \leq 3n - 2$ for $n \geq 3$. We need only to prove that (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.

By Lemma 13, (F_1, F_2) is a distinguishable conditional pair if $F_1 \subset F_2$ or $F_2 \subset F_1$. Now, we assume that $|F_1 - F_2| \geq 1$, and $|F_2 - F_1| \geq 1$. Let $S = F_1 \cap F_2$. Then we have $|S| \leq 3n - 3$ for $n \geq 3$. Let H be the largest connected component of $DC_n - F_1 \cap F_2$. By Lemma 12, every vertex in $F_1 \Delta F_2$ is in H.

We claim that H has a vertex u outside $F_1 \cup F_2$ that has no neighbor in S. We need only to estimate the lower bound on the number, say γ, of candidate nodes for u.

Since every vertex has degree $n + 1$, the vertices in S can have at most $(n+1)|S|$ neighbors in H. There are at most $|F_1| + |F_2| - |S|$ vertices in $F_1 \cup F_2$ and at most two vertices of $DC_n - S$ may not belong to H by Lemma 9. So we have

$$\begin{aligned}
\gamma &\geq |H| - |F_1 \Delta F_2| - (n+1)|S| \\
&\geq 2^{2n+1} - (n+1)|S| - (|F_1| + |F_2| - |S|) - 2 \\
&\geq 2^{2n+1} - (n+2) \times (3n-3) - 2 \\
&\geq 1 \text{ for } n \geq 3.
\end{aligned}$$

Thus, there exists some vertex of H outside $F_1 \cup F_2$, which has no neighbors in S. Let u be such a vertex.

If u has no neighbor in $F_1 \cup F_2$, then we can find a path of length at least two within H to a vertex v in $F_1 \cup F_2$. We may assume that v is the first vertex of $F_1 \Delta F_2$ on this path, and let q and w be the two vertices on this path immediately before v (we may have $u = q$), so q and w are not in $F_1 \cup F_2$. The existence of the edges (q, w) and (w, v) ensures that (F_1, F_2) is a distinguishable conditional pair of DC_n by Lemma 11. Now we assume that u has a neighbor in $F_1 \Delta F_2$. Since the degree of u is at least 3, and u has no neighbor in S, there are three possibilities:

(1) u has two neighbors in $F_1 \setminus F_2$; or

(2) u has two neighbors in $F_2 \setminus F_1$; or

(3) u has at least one neighbor outside $F_1 \cup F_2$.

In each sub-case above, Lemma 11 implies that (F_1, F_2) is a distinguishable conditional pair of DC_n under the comparison diagnosis model, and so the proof is complete. □

5. Conclusion

The paper derives the fault resiliency of dual-cubes, and then uses the fault resiliency to evaluate the conditional fault diagnosability of dual-cubes under the comparison model. The ordinary diagnosability of DC_n under the comparison model is only $n + 1$, while the conditional diagnosability of DC_n is $3n - 2$, which is about three times of the traditional diagnosability under the comparison model. The fault resiliency of dual-cubes may also reveal its conditional connectivity of high order.

The dual-cube is a special case of metacube [4, 34] and its generalization-recursive dual-net [33], two of which are versatile families of interconnection networks that can connect an extremely large number of nodes with a small number of links per node and keep the diameter rather low. The perfect hierarchical hypercubes [3, 5], the hierarchical hypercubes [38], and the hierarchical cubic networks [2, 19] are very similar to dual-cubes: they also connect n-dimensional hypercubes each and all have a regular degree of $n + 1$. The main idea of this paper can be also applied to all of these complex network structure.

Acknowledgments

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