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On Sullivan's Conjecture on Cycles in 4-free and 5-free Digraphs

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Abstract For a simple digraph G, let $\beta(G)$ be the size of the smallest subset $X \subseteq E(G)$ such that $G-X$ has no directed cycles, and let $\gamma(G)$ be the number of unordered pairs of nonadjacent vertices in G. A digraph G is called k-free if G has no directed cycles of length at most k. This paper proves that $\beta(G) \leq 0.3819\gamma(G)$ if G is a 4-free digraph, and $\beta(G) \leq 0.2679\gamma(G)$ if G is a 5-free digraph. These improve the results of Sullivan in 2008.

Keywords Digraph, directed cycle

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1 Introduction

Let $G = (V, E)$ be a digraph without loops and parallel edges, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.

It is well known that the cycle rank of an undirected graph G is the minimum number of edges that must be removed in order to eliminate all of cycles in the graph. That is, if G has υ vertices, ε edges, and ω connected components, then the minimum number of edges whose deletion from G leaves an acyclic graph equals the cycle rank (or Betti number) $\rho(G) = \varepsilon - \upsilon + \omega$ (see Xu [1]). However, the same problem for a digraph is quite difficult.

A digraph G is called to be *k-free* if there is no directed cycle of G with length at most k. A digraph is *acyclic* if it has no directed cycles. For a digraph G, let $\beta(G)$ be the size of the smallest subset $X \subseteq E(G)$ such that $G-X$ is acyclic, and let $\gamma(G)$ be the number of unordered pairs of nonadjacent vertices in G, called the *number of missing edges* of G.

Chudnovsky et al. [2] proved that $\beta(G) \leq \gamma(G)$ if G is a 3-free digraph and gave the following conjecture.

Conjecture 1.1 *If G is a 3-free digraph, then $\beta(G) \leq \frac{1}{2}\gamma(G)$.*

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Concerning this conjecture, Dunkum et al. [3] proved that $\beta(G) \le 0.88\gamma(G)$. Very recently, Chen et al. [4] improved the result to $\beta(G) \le 0.8616\gamma(G)$. Conjecture 1.1 is closely related to the following special case of the conjecture proposed by Caccetta and Häggkvist [5].

Conjecture 1.2 *Any digraph on n vertices with minimum out-degree at least $n/3$ contains a directed triangle.*

Short of proving the conjecture, one may seek as small value of c as possible such that every digraph on n vertices with minimum out-degree at least cn contains a triangle. This was the strategy of Caccetta and Häggkvist [5], who obtained the value $c \le 0.3819$. Bondy [6] showed that $c \le 0.3797$, and Shen [7] improved it to $c \le 0.3542$. Hamburger, et al. [8] improved it to 0.35312. Very recently, Hladky et al. [9] further improved this bound to 0.3465. Namely, any digraph on n vertices with minimum out-degree at least $0.3465\,n$ contains a directed triangle.

Generalizing Conjecture 1.1, Sullivan [10] proposed the following conjecture, and gave an example showing that this would be best possible if this conjecture is true, noting that Conjecture 1.1 is a special case of this when $m = 3$.

Conjecture 1.3 *If G is an m-free digraph with $m \ge 3$, then*

$$\beta(G) \le \frac{2}{(m+1)(m-2)}\gamma(G).$$

Sullivan proved partial results of Conjecture 1.3, and showed that $\beta(G) \le \frac{1}{m-2}\gamma(G)$ for $m = 4, 5$. In this article, we improve these two results, which are summarized in the following theorems.

Theorem 1.4 *If G is a 4-free digraph, then $\beta(G) \le \frac{3-\sqrt{5}}{2}\gamma(G) \approx 0.3819\gamma(G)$.*

Theorem 1.5 *If G is a 5-free digraph, then $\beta(G) \le (2-\sqrt{3})\gamma(G) \approx 0.2679\gamma(G)$.*

The proofs of the two results are in Section 3. We proceed by induction on $|V(G)|$ by refining Sullivan et al.'s methods used in [2, 10] and using some computation techniques. In Section 2, we give some notations and known results used in our proofs.

2 Preliminaries

Let G be a simple digraph. For two disjoint subsets $A, B \subseteq V(G)$, let $E(A, B)$ denote the set of directed edges (a, b) with $a \in A$ and $b \in B$. Similarly, let $\bar{E}(A, B)$ be the missing edges between A and B. It follows that

$$|\bar{E}(A, B)| = |\bar{E}(B, A)| = |A| \cdot |B| - |E(A, B)| - |E(B, A)|.$$

We say $P = (x, y, z)$ an *induced directed 2-path* (*2-path* for short), if $(x, y), (y, z) \in E(G)$ and x, z are nonadjacent, where x, y, z are called the original, internal and terminal vertices of P, respectively. For each $v \in V(G)$, let $f(v)$, $g(v)$ and $h(v)$ be the number of 2-paths with the original vertex v, the internal vertex v, and the terminal vertex v, respectively. Let N be the number of 2-paths of G. Then

$$N = \sum_{v \in V(G)} f(v) = \sum_{v \in V(G)} g(v) = \sum_{v \in V(G)} h(v). \tag{2.1}$$

Let $N_i^+(v)$ be the set of vertices u such that the shortest directed path starting with v and ending with u has length i. Similarly, let $N_i^-(v)$ be the set of vertices whose shortest directed path to v has length i. It follows that

$$\begin{cases} f(v) = |E(N_1^+(v), N_2^+(v))|, \\ g(v) = |\bar{E}(N_1^-(v), N_1^+(v))|, \\ h(v) = |E(N_2^-(v), N_1^-(v))|. \end{cases} \tag{2.2}$$

Let $P(v)$ be the number of triples of distinct vertices (x, y, z) such that for some $u \in V(G)$, (x, u, y, z) is an induced directed path with the original vertex $x = v$. Similarly, let $Q(v)$ be the number of triples of distinct vertices (x, y, z) such that for some $u \in V(G)$, (x, u, y, z) is an induced directed path with the internal vertex $y = v$, and $R(v)$ be the number of such triples with $z = v$. Also, let $P'(v)$ be the number of triples of distinct vertices (x, y, z) which makes (x, y, u, z) be an induced directed path with $x = v$ for some $u \in V(G)$. Let $Q'(v)$ and $R'(v)$ be the number of such triples with $y = v$ and $z = v$, respectively.

From the above definitions, we can verify

$$\sum_{v\in V(G)} P(v) = \sum_{v\in V(G)} Q(v) = \sum_{v\in V(G)} R(v) \tag{2.3}$$

and

$$\sum_{v\in V(G)} P'(v) = \sum_{v\in V(G)} Q'(v) = \sum_{v\in V(G)} R'(v). \tag{2.4}$$

Finally, set $C(v)$ be the vertices whose shortest directed path to or from v has length at least three, that is $C(v) = V(G)\backslash(\{v\} \cup N_1^+(v) \cup N_2^+(v) \cup N_1^-(v) \cup N_2^-(v))$. We have the following bounds on $P(v), Q(v), R(v), P'(v), Q'(v), R'(v)$ in terms of $C(v)$ and $N_i^+(v)$, $N_i^-(v)$ for $i = 1, 2$.

Proposition 2.1 ([10]) *If G is a 4-free digraph, then for any $v \in V(G)$,*

$$\begin{cases} P(v) = |E(N_2^+(v), C(v) \cup N_2^-(v))|, \\ Q(v) \le |\bar{E}(N_2^-(v), N_1^+(v))|, \\ Q'(v) \le |\bar{E}(N_2^+(v), N_1^-(v))|, \\ R'(v) = |E(C(v) \cup N_2^+(v), N_2^-(v))|. \end{cases}$$

Proposition 2.2 ([10]) *If G is a 5-free digraph, then for any $v \in V(G)$,*

$$\begin{cases} P(v) = |E(N_2^+(v), C(v)|, \\ Q(v) \le |\bar{E}(N_2^-(v), N_1^+(v))|, \\ R(v) \le |\bar{E}(C(v), N_1^-(v))|, \\ P'(v) \le |\bar{E}(C(v), N_1^+(v))|, \\ Q'(v) \le |\bar{E}(N_2^+(v), N_1^-(v))|, \\ R'(v) = |E(C(v), N_2^-(v))|. \end{cases}$$

3 Proofs of Main Results

In this section, we will give proofs of Theorems 1.4–1.5, respectively. We first prove Theorem 1.4 starting with some notations and lemmas.

Lemma 3.1 *$a_i \geq 0, b_i \geq 0$ and $\lambda_i > 0, i = 1, 2, \ldots, n$, and $\sum_{i=1}^n b_i > 0$. Then*

$$\min_{1\leq i\leq n}\left\{\frac{a_i}{b_i}\right\} \leq \frac{\sum_{i=1}^n \lambda_i a_i}{\sum_{i=1}^n \lambda_i b_i}.$$

Proof Suppose that $\min_{1\leq i\leq n}\{\frac{a_i}{b_i}\} = \frac{a_1}{b_1}$, without loss of generality. Let $\frac{a_i}{b_i} = +\infty$ if $b_i = 0$. Then $\frac{a_1}{b_1} \leq \frac{a_i}{b_i}$, and so $\frac{a_1}{b_1} \cdot b_i \leq a_i$ (the inequality holds even if $b_i = 0$) for each $i = 1, 2, \ldots, n$. Thus, we have

$$\sum_{i=1}^n \lambda_i a_i \geq \sum_{i=1}^n \lambda_i b_i \cdot \frac{a_1}{b_1} = \frac{a_1}{b_1}\sum_{i=1}^n \lambda_i b_i.$$

Since $\sum_{i=1}^n b_i > 0$ and $\lambda_i > 0$ for each $i = 1, 2, \ldots, n$, we have $\sum_{i=1}^n \lambda_i b_i > 0$. It follows that

$$\min_{1\leq i\leq n}\left\{\frac{a_i}{b_i}\right\} = \frac{a_1}{b_1} \leq \frac{\sum_{i=1}^n \lambda_i a_i}{\sum_{i=1}^n \lambda_i b_i}$$

as desired, and so the lemma follows. □

Let G be a 4-free digraph. For each $v \in V(G)$, set

$$\begin{cases} e_1(v) = |E(N_1^+(v), N_2^+(v))|, \\ e_2(v) = |E(N_2^-(v), N_1^-(v))|, \\ e_3(v) = |E(N_2^+(v), C(v) \cup N_2^-(v))|, \\ e_4(v) = |E(C(v) \cup N_2^+(v), N_2^-(v))|, \\ \bar{e}_1(v) = |\bar{E}(N_1^-(v), N_1^+(v))|, \\ \bar{e}_2(v) = |\bar{E}(N_1^+(v), N_2^-(v))|, \\ \bar{e}_3(v) = |\bar{E}(N_1^-(v), N_2^+(v))|, \end{cases} \tag{3.1}$$

and

$$\begin{cases} k_1(v) = \dfrac{e_1(v)}{\bar{e}_1(v) + \bar{e}_2(v)}, \\ k_2(v) = \dfrac{e_2(v)}{\bar{e}_1(v) + \bar{e}_3(v)}, \\ k_3(v) = \dfrac{e_3(v)}{\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v)}, \\ k_4(v) = \dfrac{e_4(v)}{\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v)}. \end{cases} \tag{3.2}$$

Lemma 3.2 *If G is a 4-free digraph, then there exists some $v \in V(G)$ such that*

$$\min_{1\leq i\leq 4}\{k_i(v)\} \leq \frac{3-\sqrt{5}}{2}.$$

Proof From (2.1)–(2.4), (3.1) and Proposition 2.1, we have

$$\begin{cases} \sum\limits_{v\in V(G)} e_1(v) = \sum\limits_{v\in V(G)} e_2(v) = \sum\limits_{v\in V(G)} \bar{e}_1(v), \\ \sum\limits_{v\in V(G)} e_3(v) = \sum\limits_{v\in V(G)} P(v) = \sum\limits_{v\in V(G)} Q(v) \leq \sum\limits_{v\in V(G)} \bar{e}_2(v), \\ \sum\limits_{v\in V(G)} e_4(v) = \sum\limits_{v\in V(G)} R'(v) = \sum\limits_{v\in V(G)} Q'(v) \leq \sum\limits_{v\in V(G)} \bar{e}_3(v). \end{cases} \tag{3.3}$$

From (3.3), for all $\lambda > 0$, we have

$$\sum_{v\in V(G)} [e_1(v) + e_2(v) + \lambda(e_3(v) + e_4(v))] \leq \sum_{v\in V(G)} [2\bar{e}_1(v) + \lambda(\bar{e}_2(v) + \bar{e}_3(v))]. \tag{3.4}$$

The inequality (3.4) implies that there must exist some $v \in V(G)$ such that

$$e_1(v) + e_2(v) + \lambda(e_3(v) + e_4(v)) \leq 2\bar{e}_1(v) + \lambda(\bar{e}_2(v) + \bar{e}_3(v)). \tag{3.5}$$

It follows from (3.2), (3.5) and Lemma 3.1 that

$$\begin{aligned}\min_{1\leq i\leq 4}\{k_i(v)\} &\leq (e_1(v) + e_2(v) + \lambda e_3(v) + \lambda e_4(v))/((\bar{e}_1(v) + \bar{e}_2(v)) \\ &\quad + (\bar{e}_1(v) + \bar{e}_3(v)) + \lambda(\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v)) + \lambda(\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v))) \\ &= \frac{e_1(v) + e_2(v) + \lambda(e_3(v) + e_4(v))}{(2 + 2\lambda)\bar{e}_1(v) + (2\lambda + 1)(\bar{e}_2(v) + \bar{e}_3(v))} \\ &\leq \frac{2\bar{e}_1(v) + \lambda(\bar{e}_2(v) + \bar{e}_3(v))}{(2 + 2\lambda)\bar{e}_1(v) + (2\lambda + 1)(\bar{e}_2(v) + \bar{e}_3(v))}.\end{aligned} \tag{3.6}$$

Let $\lambda = \frac{1+\sqrt{5}}{2}$. Then

$$\frac{2}{2 + 2\lambda} = \frac{\lambda}{2\lambda + 1} = \frac{3 - \sqrt{5}}{2}. \tag{3.7}$$

Substituting (3.7) into (3.6) yields the desired inequality

$$\min_{1\leq i\leq 4}\{k_i(v)\} \leq \frac{3 - \sqrt{5}}{2},$$

and so the lemma follows. □

Proof of Theorem 1.4 We proceed by induction on $|V(G)|$. Clearly, Theorem 1.4 holds for $|V(G)| \leq 5$. Assume that Theorem 1.4 holds for all digraphs with $|V(G)| < n$. Let G be a 4-free digraph with $|V(G)| = n$.

If there exists some $v \in V(G)$ such that $N_1^+(v) = \emptyset$ or $N_1^-(v) = \emptyset$, then v is not in a directed cycle. By the induction hypothesis, we can choose $X \subseteq E(G - v)$ with $|X| \leq \frac{3-\sqrt{5}}{2}\gamma(G - v)$ such that $(G - v) - X$ is acyclic, then $G - X$ has no directed cycles. It follows that $\beta(G) \leq |X| \leq \frac{3-\sqrt{5}}{2}\gamma(G - v) \leq \frac{3-\sqrt{5}}{2}\gamma(G)$, and so the theorem follows.

Thus, in the following discussion, we assume that $N_1^+(v) \neq \emptyset$ and $N_1^-(v) \neq \emptyset$ for any $v \in V(G)$.

Let v be the vertex satisfying Lemma 3.2. Now we prove that for each $i = 1, 2, 3, 4$, if $k_i(v) \leq \frac{3-\sqrt{5}}{2}$, we can find $X \subseteq E(G)$ satisfying $|X| \leq \frac{3-\sqrt{5}}{2}\gamma(G)$ and $G - X$ has no directed cycles. We consider four cases, respectively, according to Lemma 3.2, which $k_i(v)$ defined in (3.2) is at most $\frac{3-\sqrt{5}}{2}$ for $i \in \{1, 2, 3, 4\}$.

Case 1 $k_1(v) = \frac{e_1(v)}{\bar{e}_1(v)+\bar{e}_2(v)} \leq \frac{3-\sqrt{5}}{2}$.

We consider the partition of $V(G)$ as follows

$$V(G_1) = N_1^+(v),\ V(G_2) = \{v\} \cup N_2^+(v) \cup C(v) \cup N_2^-(v) \cup N_1^-(v).$$

The number of missing edges between $V(G_1)$ and $V(G_2)$ satisfies

$$|\bar{E}(V(G_1), V(G_2))| \geq |\bar{E}(N_1^+(v), N_1^-(v))| + |\bar{E}(N_1^+(v), N_2^-(v))| = \bar{e}_1(v) + \bar{e}_2(v).$$

It follows that

$$\gamma(G) = \gamma(G_1) + \gamma(G_2) + |\bar{E}(V(G_1), V(G_2))| \geq \gamma(G_1) + \gamma(G_2) + \bar{e}_1(v) + \bar{e}_2(v).$$

For $i = 1, 2$, since $0 < |V(G_i)| < n$, by the induction hypothesis, $\beta(G_1) \leq \frac{3-\sqrt{5}}{2}\gamma(G_1)$ and $\beta(G_2) \leq \frac{3-\sqrt{5}}{2}\gamma(G_2)$, we can choose $X_i \subseteq E(G_i)$ with $|X_i| \leq \frac{3-\sqrt{5}}{2}\gamma(G_i)$ such that $G_i - X_i$ is acyclic. Let X_3 be the set of all edges from $N_1^+(v)$ to $N_2^+(v)$. Then $|X_3| = e_1(v)$. Since there is no edge from $N_1^+(v)$ to $\{v\} \cup C(v) \cup N_2^-(v) \cup N_1^-(v)$ (because G is 4-free), every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X = X_1 \cup X_2 \cup X_3$. Then $G - X$ has no directed cycles. Thus,

$$\begin{aligned}
\beta(G) &\leq |X_1| + |X_2| + |X_3| \\
&= |X_1| + |X_2| + e_1(v) \\
&\leq \frac{3-\sqrt{5}}{2}\gamma(G_1) + \frac{3-\sqrt{5}}{2}\gamma(G_2) + \frac{3-\sqrt{5}}{2}(\bar{e}_1(v) + \bar{e}_2(v)) \\
&\leq \frac{3-\sqrt{5}}{2}\gamma(G)
\end{aligned}$$

as desired.

Case 2 $k_2(v) = \frac{e_2(v)}{\bar{e}_1(v)+\bar{e}_3(v)} \leq \frac{3-\sqrt{5}}{2}$.

Using the following partition of $V(G)$,

$$V(G_1) = \{v\} \cup N_1^+(v) \cup N_2^+(v) \cup C(v) \cup N_2^-(v),\ V(G_2) = N_1^-(v),$$

we get

$$|\bar{E}(V(G_1), V(G_2))| \geq |\bar{E}(N_1^+(v), N_1^-(v))| + |\bar{E}(N_2^+(v), N_1^-(v))| = \bar{e}_1(v) + \bar{e}_3(v),$$

which derives that

$$\gamma(G) = \gamma(G_1) + \gamma(G_2) + |\bar{E}(V(G_1), V(G_2))| \geq \gamma(G_1) + \gamma(G_2) + \bar{e}_1(v) + \bar{e}_3(v).$$

By the induction hypothesis, $\beta(G_1) \leq \frac{3-\sqrt{5}}{2}\gamma(G_1)$ and $\beta(G_2) \leq \frac{3-\sqrt{5}}{2}\gamma(G_2)$, we can choose $X_i \subseteq E(G_i)$ with $|X_i| \leq \frac{3-\sqrt{5}}{2}\gamma(G_i)$ such that $G_i - X_i$ is acyclic. Let X_3 be the set of all edges from $N_2^-(v)$ to $N_1^-(v)$. Then $|X_3| = e_2(v)$. Since there is no edge from $\{v\} \cup N_1^+(v) \cup N_2^+(v) \cup C(v)$ to $N_1^-(v)$ (because G is 4-free), every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X = X_1 \cup X_2 \cup X_3$. Then $G - X$ has no directed cycles. Thus,

$$\begin{aligned}
\beta(G) &\leq |X_1| + |X_2| + |X_3| \\
&= |X_1| + |X_2| + e_2(v) \\
&\leq \frac{3-\sqrt{5}}{2}\gamma(G_1) + \frac{3-\sqrt{5}}{2}\gamma(G_2) + \frac{3-\sqrt{5}}{2}(\bar{e}_1(v) + \bar{e}_3(v)) \\
&\leq \frac{3-\sqrt{5}}{2}\gamma(G)
\end{aligned}$$

as desired.

Case 3 $k_3(v) = \frac{e_3(v)}{\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v)} \leq \frac{3-\sqrt{5}}{2}$.

Consider the following partition of $V(G)$,

$$V(G_1) = N_1^+(v) \cup N_2^+(v), \quad V(G_2) = \{v\} \cup C(v) \cup N_2^-(v) \cup N_1^-(v).$$

The number of missing edges between $V(G_1)$ and $V(G_2)$ satisfies

$$\begin{aligned}&|\bar{E}(V(G_1),V(G_2))|\\&\quad\geq|\bar{E}(N_1^+(v),N_1^-(v))|+|\bar{E}(N_1^+(v),N_2^-(v))|+|\bar{E}(N_2^+(v),N_1^-(v))|\\&\quad=\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v).\end{aligned}$$

It follows that

$$\gamma(G)=\gamma(G_1)+\gamma(G_2)+|\bar{E}(V(G_1),V(G_2))|\geq\gamma(G_1)+\gamma(G_2)+\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v).$$

By the induction hypothesis, $\beta(G_1)\leq\frac{3-\sqrt{5}}{2}\gamma(G_1)$ and $\beta(G_2)\leq\frac{3-\sqrt{5}}{2}\gamma(G_2)$, we can choose $X_i\subseteq E(G_i)$ with $|X_i|\leq\frac{3-\sqrt{5}}{2}\gamma(G_i)$ such that G_i-X_i is acyclic. Let X_3 be the set of all edges from $N_2^+(v)$ to $C(v)\cup N_2^-(v)$. Then $|X_3|=e_3(v)$. Since there is no edge from $N_1^+(v)$ to $\{v\}\cup C(v)\cup N_2^-(v)\cup N_1^-(v)$ and from $N_2^+(v)$ to $\{v\}\cup N_1^-(v)$ (because G is 4-free), every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X=X_1\cup X_2\cup X_3$. Then $G-X$ has no directed cycles. Thus, we get

$$\begin{aligned}\beta(G)&\leq|X_1|+|X_2|+|X_3|\\&=|X_1|+|X_2|+e_3(v)\\&\leq\frac{3-\sqrt{5}}{2}\gamma(G_1)+\frac{3-\sqrt{5}}{2}\gamma(G_2)+\frac{3-\sqrt{5}}{2}(\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v))\\&\leq\frac{3-\sqrt{5}}{2}\gamma(G)\end{aligned}$$

as desired.

Case 4 $k_4(v)=\frac{e_4(v)}{\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v)}\leq\frac{3-\sqrt{5}}{2}$.

Using the following partition of $V(G)$,

$$V(G_1)=\{v\}\cup N_1^+(v)\cup N_2^+(v)\cup C(v),\quad V(G_2)=N_2^-(v)\cup N_1^-(v),$$

we obtain

$$\begin{aligned}&|\bar{E}(V(G_1),V(G_2))|\\&\quad\geq|\bar{E}(N_1^+(v),N_1^-(v))|+|\bar{E}(N_1^+(v),N_2^-(v))|+|\bar{E}(N_2^+(v),N_1^-(v))|\\&\quad=\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v).\end{aligned}$$

It follows that

$$\gamma(G)=\gamma(G_1)+\gamma(G_2)+|\bar{E}(V(G_1),V(G_2))|\geq\gamma(G_1)+\gamma(G_2)+\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v).$$

By the induction hypothesis, $\beta(G_1)\leq\frac{3-\sqrt{5}}{2}\gamma(G_1)$ and $\beta(G_2)\leq\frac{3-\sqrt{5}}{2}\gamma(G_2)$, we can choose $X_i\subseteq E(G_i)$ with $|X_i|\leq\frac{3-\sqrt{5}}{2}\gamma(G_i)$ such that G_i-X_i is acyclic. Let X_3 be the set of all edges from $N_2^+(v)\cup C(v)$ to $N_2^-(v)$. Then $|X_3|=e_4(v)$. Since there is no edge from $\{v\}\cup N_1^+(v)\cup N_2^+(v)\cup C(v)$ to $N_1^-(v)$ and from $N_2^+(v)$ to $\{v\}\cup N_1^-(v)$ (because G is 4-free), every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X=X_1\cup X_2\cup X_3$. Then $G-X$ has no directed cycles and

$$\begin{aligned}\beta(G)&\leq|X_1|+|X_2|+|X_3|\\&=|X_1|+|X_2|+e_4(v)\end{aligned}$$

$$\leq \frac{3-\sqrt{5}}{2}\gamma(G_1) + \frac{3-\sqrt{5}}{2}\gamma(G_2) + \frac{3-\sqrt{5}}{2}(\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v))$$
$$\leq \frac{3-\sqrt{5}}{2}\gamma(G)$$

as desired.

For each case there exists $X \subseteq E(G)$ satisfying $|X| \leq \frac{3-\sqrt{5}}{2}\gamma(G)$ and $G - X$ has no directed cycles. This implies that $\beta(G) \leq |X| \leq \frac{3-\sqrt{5}}{2}\gamma(G)$. Theorem 1.4 follows. □

We now prove Theorem 1.5 starting with some notations and a lemma used in our proofs.

Let G be a 5-free digraph, for each $v \in V(G)$, set

$$\begin{cases} e_1(v) = |E(N_1^+(v), N_2^+(v))|, \\ e_2(v) = |E(N_2^-(v), N_1^-(v))|, \\ e_3(v) = |E(N_2^+(v), C(v))|, \\ e_4(v) = |E(C(v), N_2^-(v))|, \\ \bar{e}_1(v) = |\bar{E}(N_1^-(v), N_1^+(v))|, \\ \bar{e}_2(v) = |\bar{E}(N_1^+(v), N_2^-(v))|, \\ \bar{e}_3(v) = |\bar{E}(N_1^-(v), N_2^+(v))|, \\ \bar{e}_4(v) = |\bar{E}(N_1^+(v), C(v))|, \\ \bar{e}_5(v) = |\bar{E}(N_1^-(v), C(v))|, \end{cases} \tag{3.8}$$

and

$$\begin{cases} k_1(v) = \dfrac{e_1(v)}{\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_4(v)}, \\ k_2(v) = \dfrac{e_2(v)}{\bar{e}_1(v) + \bar{e}_3(v) + \bar{e}_5(v)}, \\ k_3(v) = \dfrac{e_3(v)}{\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v) + \bar{e}_4(v)}, \\ k_4(v) = \dfrac{e_4(v)}{\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v) + \bar{e}_5(v)}. \end{cases} \tag{3.9}$$

Lemma 3.3 *If G is a 4-free digraph, there exists some $v \in V(G)$ such that*

$$\min_{1 \leq i \leq 4}\{k_i(v)\} \leq 2 - \sqrt{3}.$$

Proof From (2.1)–(2.4), (3.9) and Proposition 2.2, we have

$$\begin{cases} \sum\limits_{v \in V(G)} e_1(v) = \sum\limits_{v \in V(G)} e_2(v) = \sum\limits_{v \in V(G)} \bar{e}_1(v), \\ \sum\limits_{v \in V(G)} e_3(v) = \sum\limits_{v \in V(G)} P(v) = \sum\limits_{v \in V(G)} Q(v) \leq \sum\limits_{v \in V(G)} \bar{e}_2(v), \\ \sum\limits_{v \in V(G)} e_3(v) = \sum\limits_{v \in V(G)} P(v) = \sum\limits_{v \in V(G)} R(v) \leq \sum\limits_{v \in V(G)} \bar{e}_5(v), \\ \sum\limits_{v \in V(G)} e_4(v) = \sum\limits_{v \in V(G)} R'(v) = \sum\limits_{v \in V(G)} Q'(v) \leq \sum\limits_{v \in V(G)} \bar{e}_3(v), \\ \sum\limits_{v \in V(G)} e_4(v) = \sum\limits_{v \in V(G)} R'(v) = \sum\limits_{v \in V(G)} P'(v) \leq \sum\limits_{v \in V(G)} \bar{e}_4(v). \end{cases} \tag{3.10}$$

It follows from (3.10) that, for all $\alpha, \beta > 0$,

$$\sum_{v\in V(G)} [e_1(v) + e_2(v) + (\alpha+\beta)(e_3(v) + e_4(v))] \\ \leq \sum_{v\in V(G)} [2\bar{e}_1(v) + \alpha(\bar{e}_2(v) + \bar{e}_3(v)) + \beta(\bar{e}_4(v) + \bar{e}_5(v))]. \quad (3.11)$$

Thus, the inequality (3.11) implies that there exists some $v \in V(G)$ such that

$$e_1(v) + e_2(v) + (\alpha+\beta)(e_3(v) + e_4(v)) \\ \leq 2\bar{e}_1(v) + \alpha(\bar{e}_2(v) + \bar{e}_3(v)) + \beta(\bar{e}_4(v) + \bar{e}_5(v)). \quad (3.12)$$

By (3.9), (3.12) and Lemma 3.1, we have

$$\min_{1\leq i\leq 4}\{k_i(v)\} \\ \leq (e_1(v) + e_2(v) + (\alpha+\beta)(e_3(v) + e_4(v)))/((\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_4(v)) + (\bar{e}_1(v) + \bar{e}_3(v) \\ + \bar{e}_5(v)) + (\alpha+\beta)[(\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v) + \bar{e}_4(v)) + (\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v) + \bar{e}_5(v))]) \\ = (e_1(v) + e_2(v) + \alpha(e_3(v) + e_4(v)) + \beta(e_3(v) + e_4(v)))/ \\ ([2 + 2(\alpha+\beta)]\bar{e}_1(v) + [2(\alpha+\beta) + 1](\bar{e}_2(v) + \bar{e}_3(v)) + [(\alpha+\beta) + 1](\bar{e}_4(v) + \bar{e}_5(v))) \\ \leq (2\bar{e}_1(v) + \alpha(\bar{e}_2(v) + \bar{e}_3(v)) + \beta(\bar{e}_4(v) + \bar{e}_5(v)))/([2 + 2(\alpha+\beta)]\bar{e}_1(v) \\ + [2(\alpha+\beta) + 1](\bar{e}_2(v) + \bar{e}_3(v)) + [(\alpha+\beta) + 1](\bar{e}_4(v) + \bar{e}_5(v))). \quad (3.13)$$

Let $\alpha = \sqrt{3}$, $\beta = 1$. Then

$$\frac{2}{2 + 2(\alpha+\beta)} = \frac{\alpha}{2(\alpha+\beta) + 1} = \frac{\beta}{(\alpha+\beta) + 1} = 2 - \sqrt{3}. \quad (3.14)$$

Substituting (3.14) into (3.13) yields the desired inequality

$$\min_{1\leq i\leq 4}\{k_i(v)\} \leq 2 - \sqrt{3},$$

and so the lemma follows. □

Proof of Theorem 1.5 We proceed by induction on $|V(G)|$. Clearly, Theorem 1.5 holds for $|V(G)| \leq 6$. Assume that Theorem 1.5 holds for all digraphs with $|V(G)| < n$. Let G be a 5-free digraph with $|V(G)| = n$. We may assume that $N_1^+(v) \neq \emptyset$ and $N_1^-(v) \neq \emptyset$ for any $v \in V(G)$.

Let v be the vertex satisfying Lemma 3.3. Now we prove that for each $i = 1, 2, 3, 4$, if $k_i(v) \leq 2 - \sqrt{3}$, we can find $X \subseteq E(G)$ satisfying $|X| \leq (2-\sqrt{3})\gamma(G)$ and $G - X$ has no directed cycles. It will derive that $\beta(G) \leq |X| \leq (2-\sqrt{3})\gamma(G)$. We consider four cases, respectively, according to Lemma 3.3, which $k_i(v)$ defined in (3.9) is at most $2 - \sqrt{3}$ for $i \in \{1, 2, 3, 4\}$.

Case 1 $k_1(v) = \frac{e_1(v)}{\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_4(v)} \leq 2 - \sqrt{3}$.

Consider the following partition of $V(G)$,

$$V(G_1) = N_1^+(v), \qquad V(G_2) = \{v\} \cup N_2^+(v) \cup C(v) \cup N_2^-(v) \cup N_1^-(v).$$

The number of missing edges between $V(G_1)$ and $V(G_2)$ satisfies

$$|\bar{E}(V(G_1), V(G_2))| \\ \geq |\bar{E}(N_1^+(v), N_1^-(v))| + |\bar{E}(N_1^+(v), N_2^-(v))| + |\bar{E}(N_1^+(v), C(v))|$$

$$= \bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_4(v).$$

Thus,

$$\begin{aligned}\gamma(G) &= \gamma(G_1) + \gamma(G_2) + |\bar{E}(V(G_1), V(G_2))| \\ &\geq \gamma(G_1) + \gamma(G_2) + \bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_4(v).\end{aligned}$$

For $i = 1, 2$, since $0 < |V(G_i)| < n$, by the inductive hypothesis, $\beta(G_1) \leq (2-\sqrt{3})\gamma(G_1)$ and $\beta(G_2) \leq (2-\sqrt{3})\gamma(G_2)$, we can choose $X_i \subseteq E(G_i)$ with $|X_i| \leq (2-\sqrt{3})\gamma(G_i)$ such that $G_i - X_i$ is acyclic. Let X_3 be the set of all edges from $N_1^+(v)$ to $N_2^+(v)$. Then $|X_3| = e_1(v)$. Since there is no edge from $N_1^+(v)$ to $\{v\} \cup C(v) \cup N_2^-(v) \cup N_1^-(v)$ (because G is 5-free), every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X = X_1 \cup X_2 \cup X_3$. Then $G - X$ has no directed cycles. Thus, we can deduce the desired inequality

$$\begin{aligned}\beta(G) &\leq |X_1| + |X_2| + |X_3| \\ &= |X_1| + |X_2| + e_1(v) \\ &\leq (2-\sqrt{3})\gamma(G_1) + (2-\sqrt{3})\gamma(G_2) + (2-\sqrt{3})(\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_4(v)) \\ &\leq (2-\sqrt{3})\gamma(G).\end{aligned}$$

Case 2 $k_2(v) = \frac{e_2(v)}{\bar{e}_1(v)+\bar{e}_3(v)+\bar{e}_5(v)} \leq 2-\sqrt{3}$.

Using the following partition of $V(G)$,

$$V(G_1) = \{v\} \cup N_1^+(v) \cup N_2^+(v) \cup C(v) \cup N_2^-(v), \quad V(G_2) = N_1^-(v),$$

we have

$$\begin{aligned}&|\bar{E}(V(G_1), V(G_2))| \\ &\quad \geq |\bar{E}(N_1^+(v), N_1^-(v))| + |\bar{E}(N_2^+(v), N_1^-(v))| + |\bar{E}(C(v), N_1^-(v))| \\ &\quad = \bar{e}_1(v) + \bar{e}_3(v) + \bar{e}_5(v).\end{aligned}$$

It follows that

$$\begin{aligned}\gamma(G) &= \gamma(G_1) + \gamma(G_2) + |\bar{E}(V(G_1), V(G_2))| \\ &\geq \gamma(G_1) + \gamma(G_2) + \bar{e}_1(v) + \bar{e}_3(v) + \bar{e}_5(v).\end{aligned}$$

By the induction hypothesis, $\beta(G_1) \leq (2-\sqrt{3})\gamma(G_1)$ and $\beta(G_2) \leq (2-\sqrt{3})\gamma(G_2)$, we can choose $X_i \subseteq E(G_i)$ with $|X_i| \leq (2-\sqrt{3})\gamma(G_i)$ such that $G_i - X_i$ is acyclic. Let X_3 be the set of all edges from $N_2^-(v)$ to $N_1^-(v)$. Then $|X_3| = e_2(v)$. Since there is no edge from $\{v\} \cup N_1^+(v) \cup N_2^+(v) \cup C(v)$ to $N_1^-(v)$, every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X = X_1 \cup X_2 \cup X_3$. Then $G - X$ has no directed cycles. We can deduce the desired inequality

$$\begin{aligned}\beta(G) &\leq |X_1| + |X_2| + |X_3| \\ &= |X_1| + |X_2| + e_2(v) \\ &\leq (2-\sqrt{3})\gamma(G_1) + (2-\sqrt{3})\gamma(G_2) + (2-\sqrt{3})(\bar{e}_1(v) + \bar{e}_3(v) + \bar{e}_5(v)) \\ &\leq (2-\sqrt{3})\gamma(G).\end{aligned}$$

Case 3 $k_3(v) = \frac{e_3(v)}{\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v)+\bar{e}_4(v)} \leq 2-\sqrt{3}$.

Consider the partition of $V(G)$,

$$V(G_1) = N_1^+(v) \cup N_2^+(v), \quad V(G_2) = \{v\} \cup C(v) \cup N_2^-(v) \cup N_1^-(v).$$

The number of missing edges between $V(G_1)$ and $V(G_2)$ satisfies

$$\begin{aligned}&|\bar{E}(V(G_1), V(G_2))| \\ &\quad \geq |\bar{E}(N_1^+(v), N_1^-(v))| + |\bar{E}(N_1^+(v), N_2^-(v))| \\ &\qquad + |\bar{E}(N_1^+(v), C(v))| + |\bar{E}(N_2^+(v), N_1^-(v))| \\ &\quad = \bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_4(v) + \bar{e}_3(v).\end{aligned}$$

Thus,

$$\begin{aligned}\gamma(G) &= \gamma(G_1) + \gamma(G_2) + |\bar{E}(V(G_1), V(G_2))| \\ &\geq \gamma(G_1) + \gamma(G_2) + \bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_4(v) + \bar{e}_3(v).\end{aligned}$$

By the induction hypothesis, $\beta(G_1) \leq (2-\sqrt{3})\gamma(G_1)$ and $\beta(G_2) \leq (2-\sqrt{3})\gamma(G_2)$, we can choose $X_i \subseteq E(G_i)$ with $|X_i| \leq (2-\sqrt{3})\gamma(G_i)$ such that $G_i - X_i$ is acyclic. Let X_3 be the set of all edges from $N_2^+(v)$ to $C(v)$. Then $|X_3| = e_3(v)$. Since there is no edge from $N_1^+(v)$ to $\{v\} \cup C(v) \cup N_2^-(v) \cup N_1^-(v)$ and from $N_2^+(v)$ to $\{v\} \cup N_1^-(v) \cup N_2^-(v)$ (because G is 5-free), every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X = X_1 \cup X_2 \cup X_3$. Then $G - X$ has no directed cycles. We can deduce the desired inequality

$$\begin{aligned}\beta(G) &\leq |X_1| + |X_2| + |X_3| \\ &= |X_1| + |X_2| + e_3(v) \\ &\leq (2-\sqrt{3})\gamma(G_1) + (2-\sqrt{3})\gamma(G_2) + (2-\sqrt{3})(\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v) + \bar{e}_4(v)) \\ &\leq (2-\sqrt{3})\gamma(G).\end{aligned}$$

Case 4 $\quad k_4(v) = \frac{e_4(v)}{\bar{e}_1(v)+\bar{e}_2(v)+\bar{e}_3(v)} \leq 2-\sqrt{3}$.

Using the partitions of $V(G)$,

$$V(G_1) = \{v\} \cup N_1^+(v) \cup N_2^+(v) \cup C(v), \quad V(G_2) = N_2^-(v) \cup N_1^-(v),$$

we have

$$\begin{aligned}&|\bar{E}(V(G_1), V(G_2))| \\ &\quad \geq |\bar{E}(N_1^+(v), N_1^-(v))| + |\bar{E}(N_1^+(v), N_2^-(v))| \\ &\qquad + |\bar{E}(N_1^-(v), C(v))| + |\bar{E}(N_2^+(v), N_1^-(v))| \\ &\quad = \bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_5(v) + \bar{e}_3(v),\end{aligned}$$

which derives

$$\begin{aligned}\gamma(G) &= \gamma(G_1) + \gamma(G_2) + |\bar{E}(V(G_1), V(G_2))| \\ &\geq \gamma(G_1) + \gamma(G_2) + \bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_5(v) + \bar{e}_3(v).\end{aligned}$$

By the induction hypothesis, $\beta(G_1) \leq (2-\sqrt{3})\gamma(G_1)$ and $\beta(G_2) \leq (2-\sqrt{3})\gamma(G_2)$, we can choose $X_i \subseteq E(G_i)$ with $|X_i| \leq (2-\sqrt{3})\gamma(G_i)$ such that $G_i - X_i$ is acyclic. Let X_3 be the set of all edges from $C(v)$ to $N_2^-(v)$. Then $|X_3| = e_4(v)$. Since there are no edges from

$\{v\} \cup N_1^+(v) \cup N_2^+(v) \cup C(v)$ to $N_1^-(v)$ and from $\{v\} \cup N_1^+(v) \cup N_2^+(v)$ to $N_2^-(v)$ (because G is 5-free), every edge from $V(G_1)$ to $V(G_2)$ belongs to X_3. Let $X = X_1 \cup X_2 \cup X_3$, we can get that $G \backslash X$ has no directed cycles and

$$\begin{aligned}\beta(G) &\leq |X_1| + |X_2| + |X_3| \\ &= |X_1| + |X_2| + e_4(v) \\ &\leq (2-\sqrt{3})\gamma(G_1) + (2-\sqrt{3})\gamma(G_2) + (2-\sqrt{3})(\bar{e}_1(v) + \bar{e}_2(v) + \bar{e}_3(v) + \bar{e}_5(v)) \\ &\leq (2-\sqrt{3})\gamma(G).\end{aligned}$$

The proof of Theorem 1.5 is complete. □

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