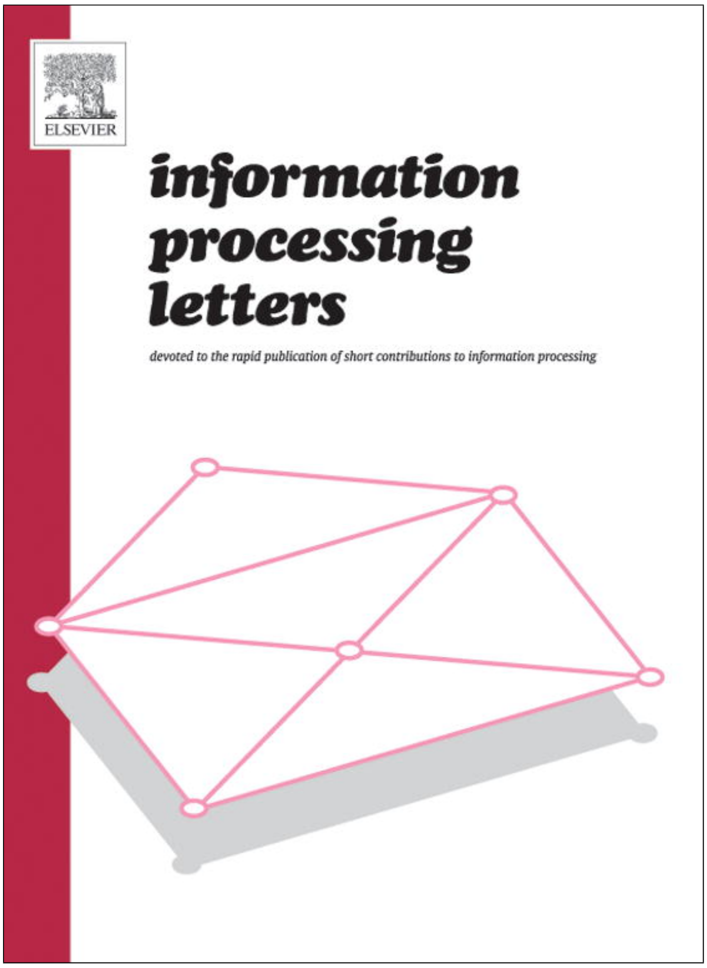
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Minimum feedback arc set of m-free digraphs ☆

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ABSTRACT

For a simple digraph G, let $\beta(G)$ be the size of the smallest subset $X \subseteq E(G)$ such that $G - X$ has no directed cycles, and let $\gamma(G)$ be the number of unordered pairs of nonadjacent vertices in G. A digraph G is called m-free if G has no directed cycles of length at most m. This paper proves that $\beta(G) \leqslant \frac{1}{m-2}\gamma(G)$ for any m-free digraph G, which generalizes some known results.

1. Introduction

Let $G = (V, E)$ be a digraph without loops and parallel edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. A graph G is said to be *acyclic* if it has no directed cycles. A subset $X \subseteq E(G)$ is said a *feedback arc set* of G if $G - X$ is acyclic. Let $\beta(G)$ be the cardinality of the *minimum feedback arc set* of G.

The minimum feedback arc set problem for a digraph was proven by Karp to be NP-complete (see the 8th of 21 problems in [8]). But the analogue for undirected graph is much easier. It is well known that the cycle rank of an undirected graph G is the minimum number of edges that must be removed in order to eliminate all of the cycles in the graph. That is, if G has υ vertices, ε edges, and ω connected components, then the minimum number of edges whose deletion from G leaves an acyclic graph equals the cycle rank (or Betti number) $\rho(G) = \varepsilon - \upsilon + \omega$ (see Xu [12]).

A digraph G is called to be *m-free* if there is no directed cycle of G with length at most m. Let $\gamma(G)$ be the number of unordered pairs of nonadjacent vertices in G, called the *number of missing edges* of G.

Chudnovsky, Seymour, and Sullivan [4] proved that $\beta(G) \leqslant \gamma(G)$ if G is a 3-free digraph and gave the following conjecture.

Conjecture 1.1. *If G is a 3-free digraph, then $\beta(G) \leqslant \frac{1}{2}\gamma(G)$.*

Concerning this conjecture, Dunkum, Hamburger, and Pór [5] proved that $\beta(G) \leqslant 0.88\gamma(G)$. Very recently, Chen et al. [3] improved the result to $\beta(G) \leqslant 0.8616\gamma(G)$. Conjecture 1.1 is closely related to the following special case of the conjecture proposed by Caccetta and Häggkvist [2].

Conjecture 1.2. *Any digraph on n vertices with minimum out-degree at least $n/3$ contains a directed triangle.*

Short of proving the conjecture, one may seek as small a value of c as possible such that every digraph on n vertices with minimum out-degree at least cn contains a triangle. This was the strategy of Caccetta and Häggkvist [2], who obtained the value $c \leqslant 0.3819$. Bondy [1] showed that

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$c \leqslant 0.3797$, and Shen [10] improved it to $c \leqslant 0.3542$. Hamburger, Haxell, and Kostochka [6] improved it to 0.35312. In 2009, Hladký et al. [7] further improved this bound to 0.3465. Namely, any digraph on n vertices with minimum out-degree at least $0.3465n$ contains a directed triangle.

More generally, Sullivan [11] proposed the following conjecture, and gave an example showing that this would be best possible if this conjecture is true. Conjecture 1.1 is the special case when $m = 3$.

Conjecture 1.3. *If G is an m-free digraph with $m \geqslant 3$, then*

$$\beta(G) \leqslant \frac{2}{(m+1)(m-2)}\gamma(G).$$

Sullivan proved partial results of Conjecture 1.3, and showed that $\beta(G) \leqslant \frac{1}{m-2}\gamma(G)$ for $m = 4, 5$. Very recently, we have improved these results to $\beta(G) \leqslant \frac{3-\sqrt{5}}{2}\gamma(G)$ for $m = 4$ and $\beta(G) \leqslant (2-\sqrt{3})\gamma(G)$ for $m = 5$ [9]. In this article, we prove the following theorem, which extends Sullivan's result to more general m-free digraphs for $m \geqslant 4$.

Theorem 1.4. *If G is an m-free digraph with $m \geqslant 3$, then $\beta(G) \leqslant \frac{1}{m-2}\gamma(G)$.*

2. Some lemmas

Let G be a simple digraph. For two disjoint subsets $A, B \subseteq V(G)$, let $E(A, B)$ denote the set of directed edges from A to B, that is, $E(A, B) = \{(a, b) \mid a \in A,\ b \in B\}$. Let $\bar{E}(A, B)$ be the missing edges between A and B. It follows that

$$\left|\bar{E}(A, B)\right| = \left|\bar{E}(B, A)\right| = |A| \cdot |B| - \left|E(A, B)\right| - \left|E(B, A)\right|.$$

A directed (v_0, v_k)-path P in G is a sequence of distinct vertices $(v_0, v_1, \ldots, v_{k-1}, v_k)$, where (v_i, v_{i+1}) is a directed edge for each $i = 0, \ldots, k-1$, its length is k. Clearly, the subsequence $(v_1, \ldots, v_{k-1})$ is a (v_1, v_{k-1})-path, denoted by P'. We can denote $P = (v_0, P', v_k)$. A directed path P is said to be *induced* if every edge in the subgraph induced by vertices of P is contained in P.

For $v \in V(G)$, let $N_i^+(v)$ be the set of vertices u such that the shortest directed (v, u)-path has length i. Similarly, let $N_i^-(v)$ be the set of vertices whose shortest directed path to v has length i. An induced directed (v_0, v_k)-path is called to be *shortest* if $v_k \in N_k^+(v_0)$. From definition, we immediately have the following result.

Lemma 2.1. *If $(v_0, v_1, \ldots, v_{k-1}, v_k)$ is a shortest induced directed (v_0, v_k)-path, then for any i and j with $0 \leqslant i < j \leqslant k$,*

$$v_j \in N_{j-i}^+(v_i) \quad \text{and} \quad v_i \in N_{j-i}^-(v_j).$$

Let $\mathscr{P}(G)$ be the set of shortest induced directed paths of G, and m be an integer with $m \geqslant 4$. For any $v \in V(G)$ and integer k with $1 \leqslant k \leqslant m-3$, let $P_k(v)$ be the set of the triples (x, y, z) of vertices of G with $x = v$ and so that there exists a path $P \in \mathscr{P}(G)$ of length $k-1$, such that $(x, P, y, z) \in \mathscr{P}(G)$. Similarly, let $Q_k(v)$ be the set of the triples (x, y, z) of vertices of G with $y = v$ and so that there exists a path $P \in \mathscr{P}(G)$ of length $k-1$, such that $(x, P, y, z) \in \mathscr{P}(G)$, and $R_k(v)$ be the set of such triples with $z = v$. Also, let $P'_k(v)$ be the set of the triples (x, y, z) of vertices of G with $x = v$ and so that there exists a path $P \in \mathscr{P}(G)$ of length $k-1$, such that $(x, y, P, z) \in \mathscr{P}(G)$. Let $Q'_k(v)$ and $R'_k(v)$ be the set of such triples with $y = v$ and $z = v$, respectively. Set

$$p_k(v) = \left|P_k(v)\right|, \qquad q_k(v) = \left|Q_k(v)\right|,$$
$$r_k(v) = \left|R_k(v)\right|,$$

and

$$p'_k(v) = \left|P'_k(v)\right|, \qquad q'_k(v) = \left|Q'_k(v)\right|,$$
$$r'_k(v) = \left|R'_k(v)\right|.$$

Lemma 2.2. *For any integer k with $1 \leqslant k \leqslant m-3$, we have*

$$\sum_{v \in V(G)} p_k(v) = \sum_{v \in V(G)} q_k(v) = \sum_{v \in V(G)} r_k(v), \tag{2.1}$$

and

$$\sum_{v \in V(G)} p'_k(v) = \sum_{v \in V(G)} q'_k(v) = \sum_{v \in V(G)} r'_k(v). \tag{2.2}$$

Proof. For each integer k with $1 \leqslant k \leqslant m-3$,

$$\sum_{v \in V(G)} p_k(v), \qquad \sum_{v \in V(G)} q_k(v), \qquad \sum_{v \in V(G)} r_k(v)$$

are all equal to the number of triples (x, y, z) of distinct vertices so that there exists a path $P \in \mathscr{P}(G)$ of length $k-1$, such that $(x, P, y, z) \in \mathscr{P}(G)$. Thus (2.1) holds. The proof of (2.2) is similar. □

Lemma 2.3. *If G is an m-free digraph, then for any $v \in V(G)$ and any integer k with $1 \leqslant k \leqslant m-3$,*

$$\begin{cases} p_k(v) = \left|E\left(N_{k+1}^+(v), N_{k+2}^+(v)\right)\right|, \\ q_k(v) \leqslant \left|\bar{E}\left(N_{k+1}^-(v), N_1^+(v)\right)\right|, \\ r_k(v) \leqslant \left|\bar{E}\left(N_1^-(v), N_{k+2}^-(v)\right)\right|, \\ p'_k(v) \leqslant \left|\bar{E}\left(N_1^+(v), N_{k+2}^+(v)\right)\right|, \\ q'_k(v) \leqslant \left|\bar{E}\left(N_{k+1}^+(v), N_1^-(v)\right)\right|, \\ r'_k(v) = \left|E\left(N_{k+2}^-(v), N_{k+1}^-(v)\right)\right|. \end{cases}$$

Proof. We prove the three first inequalities. By definition, for each edge $(u, w) \in E(N_{k+1}^+(v), N_{k+2}^+(v))$, there exists $v_i \in N_i^+(v)$, for each $i = 1, 2, \ldots, k$, such that $(v, v_1, \ldots, v_{k-1}, v_k, u, w)$ is a directed (v, w)-path of length $k+2$. Since G is m-free and $1 \leqslant k \leqslant m-3$, it is easy to see that $(v, v_1, \ldots, v_{k-1}, v_k, u, w)$ is a shortest induced directed path. It follows that $(v, u, w) \in P_k(v)$ and

$$p_k(v) \geqslant \left|E\left(N_{k+1}^+(v), N_{k+2}^+(v)\right)\right|. \tag{2.3}$$

On the other hand, for each $(v, u, w) \in P_k(v)$, from the definition of $P_k(v)$ and Lemma 2.1, $u \in N_{k+1}^+(v)$ and

$w \in N^+_{k+2}(v)$. Thus $(u, w) \in E(N^+_{k+1}(v), N^+_{k+2}(v))$. It follows that

$$p_k(v) \leqslant \left|E\left(N^+_{k+1}(v), N^+_{k+2}(v)\right)\right|. \tag{2.4}$$

Combining (2.3) and (2.4), we have that

$$p_k(v) = \left|E\left(N^+_{k+1}(v), N^+_{k+2}(v)\right)\right|.$$

For each $(u, v, w) \in Q_k(v)$, from the definition of $Q_k(v)$ and Lemma 2.1, we have $u \in N^-_{k+1}(v)$, $w \in N^+_1(v)$ and $uw \notin E(G)$. Since G is m-free, we have $(w, u) \notin E(G)$. If not, there exists a directed cycle $(v, w, u, \ldots, v)$ with length $l = k + 3 \leqslant m$, a contradiction. So $(u, w) \in \bar{E}(N^-_{k+1}(v), N^+_1(v))$. Thus, $q_k(v) \leqslant |\bar{E}(N^-_{k+1}(v), N^+_1(v))|$.

For each $(u, w, v) \in R_k(v)$, from the definition of $R_k(v)$ and Lemma 2.1, we have $u \in N^-_{k+2}(v)$, $w \in N^-_1(v)$ and $(u, w) \notin E(G)$. Since G is m-free, $(w, u) \notin E(G)$. Otherwise, there exists a directed cycle $(w, u, \ldots, w)$ with length $l = k + 2 \leqslant m - 1$, a contradiction. Thus we have $(u, w) \in \bar{E}(N^-_1(v), N^-_{k+2}(v))$. It derives that $r_k(v) \leqslant |\bar{E}(N^-_1(v), N^-_{k+2}(v))|$.

The proof of the three last inequalities is exactly the same as the three first ones applied to G' which obtained from G by reversing all the arcs. □

For any $v \in V(G)$ and any integer k with $1 \leqslant k \leqslant m - 3$, set

$$\alpha_k(v) = \frac{p_k(v)}{s_k(v)} \quad \text{and} \quad \alpha'_k(v) = \frac{r'_k(v)}{t_k(v)}.$$

Here

$$s_k(v) = \sum_{i=k}^{m-3} p'_i(v) + \sum_{i=1}^{k} q'_i(v) \quad \text{and}$$
$$t_k(v) = \sum_{i=k}^{m-3} r_i(v) + \sum_{i=1}^{k} q_i(v). \tag{2.5}$$

The following lemma is obvious.

Lemma 2.4. *If $a_i \geqslant 0$, $b_i \geqslant 0$ for each $i = 1, 2, \ldots, n$, and $\sum_{i=1}^n b_i > 0$, then*

$$\min_{1 \leqslant i \leqslant n} \left\{\frac{a_i}{b_i}\right\} \leqslant \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}.$$

Let

$$\alpha = \min_{\substack{v \in V(G) \\ 1 \leqslant k \leqslant m-3}} \left\{\alpha_k(v)\right\} \quad \text{and}$$
$$\alpha' = \min_{\substack{v \in V(G) \\ 1 \leqslant k \leqslant m-3}} \left\{\alpha'_k(v)\right\}. \tag{2.6}$$

Applying Lemma 2.4, we obtain the following bound about α and α'.

Lemma 2.5. *If G is an m-free digraph, then*

$$\min\{\alpha, \alpha'\} \leqslant \frac{1}{m-2}.$$

Proof. Applying Lemma 2.4 to the collections of quotients $\{\alpha_k(v) \mid v \in V(G),\ 1 \leqslant k \leqslant m - 3\} \cup \{\alpha'_k(v) \mid v \in V(G),\ 1 \leqslant k \leqslant m - 3\}$, we immediately have that

$$\min\{\alpha, \alpha'\} \leqslant \frac{\sum_{k=1}^{m-3}(\sum_{v \in V(G)} p_k(v) + \sum_{v \in V(G)} r'_k(v))}{\sum_{k=1}^{m-3}(\sum_{v \in V(G)} s_k(v) + \sum_{v \in V(G)} t_k(v))}. \tag{2.7}$$

Summing $s_k(v)$ and $t_k(v)$ over all $v \in V(G)$ and noting (2.5), we have

$$\begin{aligned}
&\sum_{k=1}^{m-3} \sum_{v \in V(G)} s_k(v) \\
&= \sum_{k=1}^{m-3}\left(\sum_{i=k}^{m-3} \sum_{v \in V(G)} p'_i(v)\right) + \sum_{k=1}^{m-3}\left(\sum_{i=1}^{k} \sum_{v \in V(G)} q'_i(v)\right) \\
&= \sum_{k=1}^{m-3}\left(\sum_{i=k}^{m-3} \sum_{v \in V(G)} r'_i(v)\right) + \sum_{k=1}^{m-3}\left(\sum_{i=1}^{k} \sum_{v \in V(G)} r'_i(v)\right) \\
&= \sum_{k=1}^{m-3}\left(\sum_{i=1}^{m-3} \sum_{v \in V(G)} r'_i(v) + \sum_{v \in V(G)} r'_k(v)\right) \\
&= (m-2) \sum_{k=1}^{m-3} \sum_{v \in V(G)} r'_k(v)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{m-3} \sum_{v \in V(G)} t_k(v) \\
&= \sum_{k=1}^{m-3}\left(\sum_{i=k}^{m-3} \sum_{v \in V(G)} r_i(v)\right) + \sum_{k=1}^{m-3}\left(\sum_{i=1}^{k} \sum_{v \in V(G)} q_i(v)\right) \\
&= \sum_{k=1}^{m-3}\left(\sum_{i=k}^{m-3} \sum_{v \in V(G)} p_i(v)\right) + \sum_{k=1}^{m-3}\left(\sum_{i=1}^{k} \sum_{v \in V(G)} p_i(v)\right) \\
&= \sum_{k=1}^{m-3}\left(\sum_{i=1}^{m-3} \sum_{v \in V(G)} p_i(v) + \sum_{v \in V(G)} p_k(v)\right) \\
&= (m-2) \sum_{k=1}^{m-3} \sum_{v \in V(G)} p_k(v).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{k=1}^{m-3}\left(\sum_{v \in V(G)} s_k(v) + \sum_{v \in V(G)} t_k(v)\right) \\
&= (m-2) \sum_{k=1}^{m-3}\left(\sum_{v \in V(G)} p_k(v) + \sum_{v \in V(G)} r'_k(v)\right).
\end{aligned}$$

Substituting this equality into (2.7) yields

$$\min\{\alpha, \alpha'\} \leqslant \frac{1}{m-2}.$$

The lemma follows. □

3. Proof of Theorem 1.4

Clearly Theorem 1.4 holds for $|V(G)| \leqslant m$. We proceed the proof by induction on $|V(G)|$ under the assumption that Theorem 1.4 holds for all digraphs with $|V(G)| < n$, here $n > m$. Now let G be an m-free digraph with $|V(G)| = n$, we may assume that for any $v \in V(G)$, $N_1^+(v) \neq \emptyset$ and $N_1^-(v) \neq \emptyset$. Otherwise, if there exists $v \in V(G)$ such that $N_1^+(v) = \emptyset$ or $N_1^-(v) = \emptyset$, then v is not in a directed cycle. From the inductive hypothesis, we can choose $X \subseteq E(G-v)$ with $|X| \leqslant \frac{1}{m-2}\gamma(G-v)$ such that $(G-v)-X$ is acyclic, then $G-X$ has no directed cycles. It follows that

$$\beta(G) \leqslant |X| \leqslant \frac{1}{m-2}\gamma(G-v) \leqslant \frac{1}{m-2}\gamma(G).$$

From Lemma 2.5, we have that $\alpha \leqslant \frac{1}{m-2}$ or $\alpha' \leqslant \frac{1}{m-2}$. For each case, we prove that there exists $X \subseteq E(G)$ satisfying $|X| \leqslant \frac{1}{m-2}\gamma(G)$ and $G-X$ has no directed cycles.

Case 1. $\alpha \leqslant \frac{1}{m-2}$.

By (2.6), there exists a vertex $v \in V(G)$ and an integer k with $1 \leqslant k \leqslant m-3$ such that

$$\alpha = \alpha_k(v) = \frac{p_k(v)}{s_k(v)} \leqslant \frac{1}{m-2}.$$

We consider the partition $\{V_1, V_2\}$ of $V(G)$, where

$$V_1 = \bigcup_{i=1}^{k+1} N_i^+(v), \qquad V_2 = V(G) \setminus V_1.$$

Clearly, $\bigcup_{i=k+2}^{m-1} N_i^+(v) \subset V_2$. Since G is an m-free digraph, it follows that $N_1^-(v) \subset V_2$. We claim

$$N_1^-(v) \cap \bigcup_{i=1}^{m-1} N_i^+(v) = \emptyset.$$

Otherwise, let $u \in N_1^-(v) \cap \bigcup_{i=1}^{m-1} N_i^+(v)$. Then $(u,v) \in E(G)$ and there exists a directed (v,u)-path P with length $l_1 \leqslant m-1$. Then $P + (u,v)$ is a directed cycle with length $l_1 + 1 \leqslant m$, a contradiction.

By Lemma 2.3, the number of missing edges between V_1 and V_2 satisfies

$$\begin{aligned}
&\left|\bar{E}(V_1, V_2)\right| \\
&\geqslant \left|\bar{E}\left(\bigcup_{i=1}^{k+1} N_i^+(v), N_1^-(v) \cup \left(\bigcup_{i=k+2}^{m-1} N_i^+(v)\right)\right)\right| \\
&\geqslant \sum_{i=k+2}^{m-1} \left|\bar{E}\left(N_1^+(v), N_i^+(v)\right)\right| + \sum_{i=2}^{k+1} \left|\bar{E}\left(N_i^+(v), N_1^-(v)\right)\right| \\
&\geqslant \sum_{i=k}^{m-3} p_i'(v) + \sum_{i=1}^{k} q_i'(v) \\
&= s_k(v).
\end{aligned}$$

Let G_i be the subgraph induced by V_i for each $i = 1, 2$. It follows that

$$\begin{aligned}
\gamma(G) &= \gamma(G_1) + \gamma(G_2) + \left|\bar{E}(V_1, V_2)\right| \\
&\geqslant \gamma(G_1) + \gamma(G_2) + s_k(v).
\end{aligned} \tag{3.1}$$

Since $|V_1| < n$ and $|V_2| < n$, from the inductive hypothesis, we have $\beta(G_1) \leqslant \frac{1}{m-2}\gamma(G_1)$ and $\beta(G_2) \leqslant \frac{1}{m-2}\gamma(G_2)$. We can choose $X_i \subseteq E(G_i)$ with

$$|X_i| \leqslant \frac{1}{m-2}\gamma(G_i) \quad \text{for each } i = 1, 2 \tag{3.2}$$

such that $G_i - X_i$ is acyclic.

Let $X_3 = E(V_1, V_2)$. Then

$$X_3 = E\left(N_{k+1}^+(v), V_2\right) = E\left(N_{k+1}^+(v), N_{k+2}^+(v)\right),$$

and

$$|X_3| = \left|E\left(N_{k+1}^+(v), N_{k+2}^+(v)\right)\right| = p_k(v). \tag{3.3}$$

Let $X = X_1 \cup X_2 \cup X_3$. Then $G-X$ has no directed cycles and, by (3.1)–(3.3),

$$\begin{aligned}
|X| &= |X_1| + |X_2| + |X_3| \\
&= |X_1| + |X_2| + p_k(v) \\
&\leqslant \frac{1}{m-2}\gamma(G_1) + \frac{1}{m-2}\gamma(G_2) + \frac{1}{m-2}s_k(v) \\
&= \frac{1}{m-2}\left(\gamma(G_1) + \gamma(G_2) + s_k(v)\right) \\
&\leqslant \frac{1}{m-2}\gamma(G).
\end{aligned}$$

Case 2. $\alpha' \leqslant \frac{1}{m-2}$.

This case can be immediately reduced to Case 1 applied to G' the graph obtained by G by reversing the orientation of every arc.

For each case, there exists $X \subseteq E(G)$ satisfying $|X| \leqslant \frac{1}{m-2}\gamma(G)$ and $G-X$ has no directed cycles. This implies that $\beta(G) \leqslant |X| \leqslant \frac{1}{m-2}\gamma(G)$, and Theorem 1.4 follows.

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