Fault tolerance of augmented cubes

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Abstract

The augmented cube AQn, proposed by Choudum and Sunitha [S. A. Choudum, V. Sunitha, Augmented cubes, Networks 40 (2) (2002) 71-84], is a (2n − 1)-regular (2n − 1)-connected graph (n ≥ 4). This paper determines that the 2-extra connectivity of AQn is 6n − 17 for n ≥ 9 and the 2-extra edge-connectivity is 6n − 9 for n ≥ 4. That is, for n ≥ 9 (respectively, n ≥ 4), at least 6n − 17 vertices (respectively, 6n − 9 edges) of AQn have to be removed to get a disconnected graph that contains no isolated vertices and isolated edges. When the augmented cube is used to model the topological structure of a large-scale parallel processing system, these results can provide more accurate measurements for reliability and fault tolerance of the system.

Keywords: connectivity, extra connectivity, fault tolerance, augmented cube.

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1. Introduction

It is well known that the underlying topology of an interconnection network can be modeled by a graph G = (V, E), where V is the set of processors and E is the set of communication links in the network. For all the graph terminologies and notations not defined here, we follow [25]. Then we use graphs and networks interchangeably in this paper.

A set of vertices (respectively, edges) S of G is called a vertex-cut (respectively, an edge-cut) if G − S is disconnected. The connectivity κ(G) (respectively, the edge-connectivity λ(G) of G is the minimum number of vertices (respectively, edges) whose removal renders G disconnected.

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\( \lambda(G) \) of \( G \) is defined as the minimum cardinality of a vertex-cut (respectively, an edge-cut) \( S \). And it is known to all that the connectivity \( \kappa(G) \) and the edge-connectivity \( \lambda(G) \) are two important parameters to measure reliability and fault tolerance of the network. These parameters, however, have an obvious deficiency, that is, they tacitly assume that all elements in any subset of \( G \) can potentially fail at the same time, which is almost impossible in practiced applications of networks. In other words, in the definition of \( \kappa(G) \) and \( \lambda(G) \), absolutely no conditions or restrictions are imposed either on the set \( S \) or on the components of \( G - S \). Consequently, to compensate for these shortcomings, it seems natural to generalize the classical connectivity by adding some conditions or restrictions on the set \( S \) and the components of \( G - S \).

In [11, 12], Esfahanian and Hakimi generalized the notion of connectivity by suggesting the concept of restricted connectivity in point of view of network applications. A set \( S \subseteq V(G) \) (respectively, \( F \subseteq E(G) \)) is called a restricted vertex-set (respectively, edge-set) if it does not contain the neighbor-set of any vertex in \( G \) as its subset. A restricted vertex-set \( S \) (respectively, edge-set \( F \)) is called a restricted vertex-cut (respectively, restricted edge-cut) if \( G - S \) is disconnected. The restricted connectivity \( \kappa'(G) \) (respectively, restricted edge-connectivity \( \lambda'(G) \)) is the minimum cardinality of a restricted vertex-cut (respectively, edge-cut) in \( G \), if any, and does not exist otherwise, denoted by \( +\infty \).

However, the maximum difficult for computing the restricted connectivity of a graph \( G \) is to check that a vertex-cut does not contain the neighbor-set of any vertex in \( G \) as its subset. Thus, only a little knowledge of results has been known on \( \kappa'(G) \) or \( \lambda'(G) \) even for particular classes of graphs. For example, Xu and Xu [30] studied the \( \lambda'(G) \) for a vertex-transitive graph \( G \), Esfahanian [11] determined \( \kappa'(Q_n) = \lambda'(Q_n) = 2n - 2 \) for the hypercube \( Q_n \) and \( n \geq 3 \).

To avoid this difficult, one slightly modified the concept of a restricted vertex-set \( S \) by replacing the term “any vertex in \( G \)” in the condition by the term “any vertex in \( G - S \)”. We call in this sense the connectivity as the super connectivity, denoted by \( \kappa_1(G) \) for the super connectivity and \( \lambda_1(G) \) for the super edge-connectivity (see, for example, [1, 2, 15, 16, 26]). Clearly, \( \kappa_1(G) \leq \kappa'(G) \) and \( \lambda_1(G) = \lambda'(G) \) if they exist. The super connectivity of some graphs have been determined by several authors. For example, for the hypercube \( Q_n \), the twisted cube \( TQ_n \), the cross cube \( CQ_n \), the Möbius cube \( MQ_n \) and the locally twisted cube \( LTQ_n \), Xu et al. [29] showed that their super connectivity and the super edge-connectivity are all \( 2n - 2 \); for the star graph \( S_n \), Hu and Yang [19] proved that \( \kappa_1(S_n) = 2n - 4 \) for \( n \geq 3 \); for the augmented cube \( AQ_n \), Ma, Liu and Xu [21, 22] determined \( \kappa_1(AQ_n) = 4n - 8 \) for \( n \geq 6 \) and \( \lambda_1(AQ_n) = 4n - 4 \) for \( n \geq 2 \); for the \((n, k)\)-star graphs \( S_{n, k} \), Yang et al. [32] proved that \( \kappa_1(S_{n, k}) = n + k - 3 \); for the \( n \)-dimensional alternating group graph \( AG_n \), Cheng et al. [6] determined \( \kappa_1(AG_n) = 4n - 11 \) for \( n \geq 5 \).

Observing that every component of \( G - S \) contains at least two vertices if \( S \) is a restricted vertex-set of \( G \), Fàbrega and Fiol [13] introduced the \( h \)-extra connectivity of \( G \). A vertex-cut (respectively, an edge-cut) \( S \) of \( G \) is called an \( h \)-vertex-cut (respectively, an \( h \)-edge-cut) if every component of \( G - S \) has more than \( h \) vertices. The \( h \)-extra connectivity \( \kappa_{h,k}(G) \)
(respectively, \(h\)-extra edge-connectivity \(\lambda_h(G)\)) defined as \(\min\{|S|: S \text{ is an } h\text{-vertex-cut}\}\). Clearly, \(\kappa_0(G) = \kappa(G)\) and \(\lambda_0(G) = \lambda(G)\) for any graph \(G\) if \(G\) is not a complete graph.

For example, for the hypercube \(Q_n\), Xu et al. [31, 34] determined \(\kappa_2(Q_n) = 3n - 5\) and \(\lambda_2(Q_n) = 3n - 4\) for \(n \geq 4\); for the folded hypercube \(FQ_n\), Zhu et al. [35] determined \(\kappa_2(FQ_n) = 3n - 2\) for \(n \geq 8\) and \(\lambda_2(FQ_n) = 3n - 1\) for \(n \geq 5\); for the star graph \(S_n\), Wan, Zhang [23] determined \(\kappa_2(S_n) = 6(n - 3)\) for \(n \geq 4\); for the \((n, k)\)-star graphs \(S_{n,k}\), Yang et al. [32] proved that \(\kappa_2(S_{n,k}) = n + 2k - 5\) for \(2 \leq k \leq n - 2\); Zhang et al. [33] proved \(\kappa_2(AQ_n) = 6n - 18\) for \(n \geq 5\).

In this paper, we study the augmented cube \(AQ_n\) and determine \(\kappa_2(AQ_n) = 6n - 17\) for \(n \geq 9\) and \(\lambda_2(AQ_n) = 6n - 9\) for \(n \geq 4\).

The rest of this paper is organized as follows. In Section 2, we recall the structure of \(AQ_n\), and some definitions and lemmas. The main results are given in Section 3. Finally, we conclude our paper in Section 4.

## 2. Definitions and lemmas

Let \(n\) be a positive integer. The \(n\)-dimensional augmented cube, denoted by \(AQ_n\), proposed by Choudum and Sunitha [7–9], having \(2^n\) vertices, each labeled by an \(n\)-bit binary string, that is, \(V(AQ_n) = \{x_n x_{n-1} \cdots x_1 : x_i \in \{0,1\}, 1 \leq i \leq n\}\), can be defined recursively as follows.

**Definition 2.1.** \(AQ_1\) is a complete graph \(K_2\) with the vertex set \(\{0,1\}\). For \(n \geq 2\), \(AQ_n\) is obtained by taking two copies of the augmented cube \(AQ_{n-1}\), denoted by \(AQ^0_{n-1}\) and \(AQ^1_{n-1}\), where \(V(AQ^0_{n-1}) = \{0x_{n-1} \cdots x_2 x_1 : x_i \in \{0,1\}, 1 \leq i \leq n - 1\}\) and \(V(AQ^1_{n-1}) = \{1x_{n-1} \cdots x_2 x_1 : x_i \in \{0,1\}, 1 \leq i \leq n - 1\}\), and a vertex \(X = 0x_{n-1} \cdots x_2 x_1\) of \(AQ^0_{n-1}\) being joined to a vertex \(Y = 1y_{n-1} \cdots y_2 y_1\) of \(AQ^1_{n-1}\) if and only if either

(i) \(x_i = y_i\) for \(1 \leq i \leq n - 1\), or

(ii) \(x_i = \overline{y}_i\) for \(1 \leq i \leq n - 1\).

The graphs shown in Figure 1 are the augmented cubes \(AQ_1\), \(AQ_2\) and \(AQ_3\), respectively.
For convenience, we can express the recursive structure of $AQ_n$ as $AQ_n = L \odot R$, where $L = AQ_{n-1}^0$ and $R = AQ_{n-1}^1$. Then we call the edges between $L$ and $R$ crossing edges. Obviously every vertex is incident to exactly two crossing edges. Let $X = x_nx_{n-1} \cdots x_1$ be an $n$-bit binary string. And for $1 \leq i \leq n$, let

$$X_i = x_nx_{n-1} \cdots x_{i+1}\bar{x}_i x_{i-1} \cdots x_1,$$

$$\bar{X}_i = x_nx_{n-1} \cdots x_{i+1}\bar{x}_i \bar{x}_{i-1} \cdots \bar{x}_1.$$

Obviously, $X_1 = \bar{X}_1$, $(X_i)_i = X = (\bar{X}_i)_i$. According to Definition 2.1, we can directly obtain a useful characterization of adjacency.

**Proposition 2.2.** Assume that $X = x_nx_{n-1} \cdots x_1$ and $Y = y_ny_{n-1} \cdots y_1$ are two distinct vertices in $AQ_n$. Then $X$ and $Y$ are adjacent if and only if either

i) there exists an integer $i$ ($1 \leq i \leq n$) such that $Y = X_i$, or

ii) there exists an integer $i$ ($2 \leq i \leq n$) such that $Y = \bar{X}_i$.

By Proposition 2.2, an alternative definition of $AQ_n$ can be stated as follows.

**Definition 2.3.** The augmented cube $AQ_n$ of dimension $n$ has $2^n$ vertices. Each vertex is labeled by a unique $n$-bit binary string as its address. Two vertices $X$ and $Y$ are joined if and only if either

(i) there exists an integer $i$ with $1 \leq i \leq n$ such that $Y = X_i$; in this case, the edge is called a hypercube edge of dimension $i$, denoted by $XX_i$, or

(ii) there exists an integer $i$ with $2 \leq i \leq n$ such that $Y = \bar{X}_i$; in this case, the edge is called a complement edge of dimension $i$, denoted by $X\bar{X}_i$.

**Lemma 2.4.** $[9]$ $AQ_n$ is $(2^n - 1)$-regular $(2^n - 1)$-connected for $n \geq 4$, however, $\kappa(AQ_3) = 4$ for $n = 3$.

**Lemma 2.5.** $[21, 22]$ $\kappa_1(AQ_n) = 4n - 8$ for $n \geq 6$ and $\lambda_1(AQ_n) = 4n - 4$ for $n \geq 2$. 

![Figure 1: Three augmented cubes $AQ_1$, $AQ_2$ and $AQ_3$](image)
Lemma 2.6. Any two adjacent vertices in $AQ_n$ have either two or four common neighbors for $n \geq 3$.

Proof. Let $X$ and $Y$ be two adjacent vertices in $AQ_n$. Then $Y$ is either $X_i$ or $\overline{X_i}$ by Proposition 2.2. If $Y = X_i$ for some $i$ with $1 \leq i \leq n$, that is, $XX_i$ is a hypercube edge of dimension $i$, then we have

$$N_{AQ_n}(X) \cap N_{AQ_n}(X_i) = \begin{cases} \{X_2, \overline{X}_2\} & \text{if } i = 1 \\ \{\overline{X}_i, X_{i-1}\} & \text{if } i > 1, \end{cases}$$

(2.1)

that is, $X$ and $X_i$ have exactly two common neighbors in $AQ_n$. If $Y = \overline{X_i}$ for some $i$ with $2 \leq i \leq n$, that is, $X\overline{X}_i$ is a complement edge of dimension $i$, then we have

$$N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X_i}) = \begin{cases} \{X_i, X_{i+1}, \overline{X}_{i-1}, \overline{X}_{i+1}\} & \text{if } 2 \leq i \leq n-1 \\ \{\overline{X}_{n-1}, X_n\} & \text{if } i = n. \end{cases}$$

(2.2)

In this case, $X$ and $\overline{X}_i$ have four common neighbors for $2 \leq i \leq n-1$ while have two common neighbors for $i = n$. \hfill \square

Lemma 2.7. [21] Any two vertices in $AQ_n$ have at most four common neighbors for $n \geq 3$.

Let $N_{AQ_n}(T) = \bigcup_{U \in V(T)} N_{AQ_n}(U) \setminus V(T)$ and $E_{AQ_n}(T) = \{XY \mid XY \in E(AQ_n) \land X \in V(T), Y \in V(AQ_n) \setminus V(T)\}$ for any subgraph $T$ of $AQ_n$, we have the following consequences.

Lemma 2.8. Let $P = (Y, X, Z)$ be a path of length two in $AQ_n$ between $Y$ and $Z$ for $n \geq 5$. Then $|N_{AQ_n}(X) \cap N_{AQ_n}(Y) \cap N_{AQ_n}(Z)| \leq 1$. Furthermore, if $Z = \overline{X}_n$, we have $|N_{AQ_n}(P)| \geq 6n - 15$.

Proof. According to Proposition 2.2, $Y$ and $Z$ are in $N_{AQ_n}(X) = A \cup B$, where $A = \{X_i \mid 1 \leq i \leq n\}$ and $B = \{\overline{X}_j \mid 2 \leq j \leq n\}$. It is clear that $A \cap B = \emptyset$. Consider the following three cases.

Case 1. $\{Y, Z\} \subset A$.

In this case, $XY$ and $XZ$ are all hypercube edges of some dimensions $i$ and $j$, respectively. Without loss of generality, we may assume that $Y = X_i, Z = X_j$ for $1 \leq i < j \leq n$. By Definition 2.3 and Lemma 2.6, we have

$$N_{AQ_n}(X) \cap N_{AQ_n}(X_i) = \begin{cases} \{X_2, \overline{X}_2\} & \text{if } i = 1 \\ \{\overline{X}_i, X_{i-1}\} & \text{if } i > 1, \end{cases}$$

and

$$N_{AQ_n}(X) \cap N_{AQ_n}(X_j) = \{\overline{X}_j, X_{j-1}\}.$$
Since $AQ_n$ is $(2n-1)$-regular and $(2n-1)$-connected for $n \geq 5$, it is not difficult to check that
\[
|N_{AQ_n}(P)| = \begin{cases} 
6n - 13 & \text{if } i = 1, j = 2, 3 \\
6n - 13 & \text{if } i > 1, j = i + 1 \\
6n - 12 & \text{otherwise.}
\end{cases} \tag{2.3}
\]

**Case 2.** \{Y, Z\} $\subset$ B.

In this case, $XY$ and $XZ$ are all complement edges of some dimensions $i$ and $j$, respectively. Without loss of generality, assume that $Y = \overline{X}_i$ and $Z = \overline{X}_j$ for $2 \leq i < j \leq n$, then we have
\[
N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X}_i) = \{X_i, X_{i+1}, \overline{X}_{i-1}, \overline{X}_{i+1}\},
\]
\[
N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X}_j) = \begin{cases} 
\{X_j, X_{j+1}, \overline{X}_{j-1}, \overline{X}_{j+1}\} & \text{if } j < n \\
\{\overline{X}_{n-1}, X_n\} & \text{if } j = n
\end{cases}
\]
and
\[
N_{AQ_n}(\overline{X}_i) \cap N_{AQ_n}(\overline{X}_j) = \begin{cases} 
\{X, X_{i+1}\} & \text{if } j = i + 1, j < n \\
\{X, X_n\} & \text{if } j = i + 1, j = n \\
\{X, \overline{X}_{i+1}, (\overline{X}_i)_{i+2}, (\overline{X}_i)_{i+2}\} & \text{if } j = i + 2, j < n \\
\{X, \overline{X}_{n-1}, (\overline{X}_{n-2})_n, (\overline{X}_{n-2})_n\} & \text{if } j = i + 2, j = n \\
\{X, (\overline{X}_i)_n\} & \text{if } j \geq i + 3, j < n \\
\{X, (\overline{X}_i)_n\} & \text{if } j \geq i + 3, j = n
\end{cases}
\]
It is easy to compute that
\[
|N_{AQ_n}(P)| = \begin{cases} 
6n - 15 & \text{if } j = i + 1, j < n \\
6n - 13 & \text{if } j = i + 1, j = n \\
6n - 17 & \text{if } j = i + 2, j < n \\
6n - 15 & \text{if } j = i + 2, j = n \\
6n - 16 & \text{if } j \geq i + 3, j < n \\
6n - 14 & \text{if } j \geq i + 3, j = n
\end{cases} \tag{2.4}
\]

**Case 3.** $Y \in A$ and $Z \in B$.

In this case, $XY$ is a hypercube edge of some dimension $i$ and $XZ$ is a complement hypercube edge of some dimension $j$. Assume that $Y = X_i$, $Z = \overline{X}_j$, for $1 \leq i \leq n, 2 \leq j \leq n$, we have
\[
N_{AQ_n}(X) \cap N_{AQ_n}(X_i) = \begin{cases} 
\{X_2, \overline{X}_2\} & \text{if } i = 1 \\
\{\overline{X}_i, X_{i-1}\} & \text{if } i > 1.
\end{cases}
\]
\[
N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X}_j) = \begin{cases} 
\{X_j, X_{j+1}, \overline{X}_{j-1}, \overline{X}_{j+1}\} & \text{if } j < n \\
\{\overline{X}_{n-1}, X_n\} & \text{if } j = n
\end{cases}
\]
and

\[ N_{AQ_n}(X_i) \cap N_{AQ_n}(X_j) = \begin{cases} 
\{X, X_2\} & \text{if } i = 1, j = 2 \\
\{X, X_2, (X_1)_3, (X_1)_3\} & \text{if } i = 1, j = 3 \\
\{X, (X_1)_1\} & \text{if } i = 1, j < n \\
\{X, (X_1)_n\} & \text{if } i = 1, j = n \\
\{X, X_i, (X_i)_n\} & \text{if } i = j = 2 \\
\{X, X_{i-1}, (X_{i-1})_{i-1}, (X_{i-1})_{i-1}\} & \text{if } 3 \leq i = j \leq n - 1 \\
\{X, \overline{X}_{n-1}, (X_n)_{n-1}, (X_n)_{n-1}\} & \text{if } i = j = n \\
\{X, X_i, (X_i)_{i+1}, (X_i)_{i+1}\} & \text{if } j = i - 1, 3 \leq i \leq n - 1 \\
\text{or } j = i + 1, 2 \leq i \leq n - 2 \\
\{X, \overline{X}_n\} & \text{if } j = i - 1, i = n \\
\{X, \overline{X}_{n-1}, (X_{n-1})_{n}, (X_{n-1})_{n}\} & \text{if } j = n, i = n - 1 \\
\{X, X_{i-2}, (X_{i-2})_{i-2}\} & \text{if } j = i - 2, 4 \leq i \leq 5 \\
\text{or } j = i + 2, j < n \\
\{X, (X_1)_n\} & \text{if } j = i + 2, j = n.
\end{cases} \]

We can compute that

\[ |N_{AQ_n}(P)| = \begin{cases} 
6n - 13 & \text{if } i = 1, j = 2 \\
6n - 15 & \text{if } i = 1, j = 3 \\
6n - 14 & \text{if } i = 1, 4 \leq j < n \\
6n - 12 & \text{if } i = 1, j = n \\
6n - 13 & \text{if } i = j = 2 \\
6n - 15 & \text{if } 3 \leq i = j \leq n - 1 \\
6n - 13 & \text{if } i = j = n \\
6n - 15 & \text{if } j = i - 1, 3 \leq i \leq n - 1 \\
6n - 13 & \text{if } j = i, i = n \\
6n - 13 & \text{if } j = n, i = n - 1 \\
6n - 14 & \text{if } j = i - 2, i \geq 4 \\
6n - 13 & \text{if } j \leq i - 3, i \geq 5 \\
\text{or } j = i + 2, j < n \\
6n - 12 & \text{if } j = i + 2, j = n.
\end{cases} \tag{2.5} \]

in view of (2.3),(2.4) and (2.5), we derive that \(|N_{AQ_n}(P)| \geq 6n - 17\).

From the above, we can easily check that \(|N_{AQ_n}(X) \cap N_{AQ_n}(Y) \cap N_{AQ_n}(Z)| \leq 1\) in all these cases. By (2.4) and (2.5), we have \(|N_{AQ_n}(P)| \geq 6n - 15\) if \(Z = \overline{X}_n\).

The lemma follows.

**Lemma 2.9.** Let \(P = (Y, X, Z)\) be a path of length two in \(AQ_n\) for \(n \geq 5\). Assume that \(U \in N_{AQ_n}(P)\), then \(|N_{AQ_n}(U, X, Y, Z)| \geq 8n - 31\). If \(Z = \overline{X}_n\), we have \(|N_{AQ_n}(U, X, Y, Z)| \geq 8n - 29\).
Proof. By Lemma 2.8, we have $|N_{AQ_n}(P)| \geq 6n - 17$, and $|N_{AQ_n}(P)| \geq 6n - 15$ if $Z = \overline{X}_n$. Since $U$ is a vertex in $N_{AQ_n}(P)$, $U$ has at most three neighbors in $P$

If $U$ has exactly one neighbor in $P$, by Lemma 2.7 we have $|N_{AQ_n}(U)\cap N_{AQ_n}(X, Y, Z)| \leq 11$. In this case we can easily compute that $|N_{AQ_n}(U, X, Y, Z)| \geq 6n - 17 - 1 + 2n - 1 - 1 - 11 = 8n - 31$, and $|N_{AQ_n}(U, X, Y, Z)| \geq 6n - 15 - 1 + 2n - 1 - 1 - 11 = 8n - 29$ if $Z = \overline{X}_n$.

If $U$ has exactly two neighbors in $P$, by Lemma 2.7, we have $|N_{AQ_n}(U)\cap N_{AQ_n}(X, Y, Z)| \leq 10$. In this case we arrive at $|N_{AQ_n}(U, X, Y, Z)| \geq 6n - 17 - 1 + 2n - 1 - 2 - 10 = 8n - 31$, and $|N_{AQ_n}(U, X, Y, Z)| \geq 6n - 15 - 1 + 2n - 1 - 2 - 10 = 8n - 29$ if $Z = \overline{X}_n$.

If $U$ has exactly three neighbors in $P$, by Lemma 2.7, we have $|N_{AQ_n}(U)\cap N_{AQ_n}(X, Y, Z)| \leq 8$. Therefore, $|N_{AQ_n}(U, X, Y, Z)| \geq 6n - 17 - 1 + 2n - 1 - 3 - 8 = 8n - 30$, and $|N_{AQ_n}(U, X, Y, Z)| \geq 6n - 15 - 1 + 2n - 1 - 3 - 8 = 8n - 28$ if $Z = \overline{X}_n$.

The lemma follows. 

3. Main Results

In this section, we present our main results, that is, we determine the 2-extra connectivity and the 2-extra edge-connectivity of the augmented cube $AQ_n$.

**Theorem 3.1.** $\kappa_2(AQ_n) = 6n - 17$ for $n \geq 9$.

**Proof.** Take a path $P = (\overline{X}_i, X, \overline{X}_{i+2})$ in $AQ_n$, where $2 \leq i \leq n - 3$. Then, by (2.4) in the proof of Lemma 2.8, $|N_{AQ_n}(P)| = 6n - 17$.

Let $H = AQ_n - (P \cup N_{AQ_n}(P))$. Then, for $n \geq 9$,

$$|V(H)| = |V(AQ_n)| - |V(P)| - |N_{AQ_n}(P)|$$

$$= 2^n - 3 - (6n - 17)$$

$$= 2^n - 6n + 14 > 0,$$

that is, $V(H) \neq \emptyset$. By Lemma 2.7, $|N_{AQ_n}(Y)\cap N_{AQ_n}(P)| \leq 12$ for any $Y \in V(H)$ and $|N_{AQ_n}(e)\cap N_{AQ_n}(P)| \leq 24$ for any $e \in E(H)$. It follows that, for $n \geq 9$,

$$|N_{AQ_n}(Y)\cap N_{AQ_n}(P)| \leq 12 < 2n - 1 = |N_{AQ_n}(Y)|,$$

$$|N_{AQ_n}(e)\cap N_{AQ_n}(P)| \leq 24 < 4n - 8 \leq |N_{AQ_n}(e)|,$$

which mean that there is neither isolated vertex nor isolated edge in $AQ_n - N_{AQ_n}(P)$, and so $\kappa_2(AQ_n) \leq 6n - 17$ for $n \geq 9$.

Now we only need to prove $\kappa_2(AQ_n) \geq 6n - 17$ for $n \geq 9$.

Suppose that there is a subset $S \subset V(AQ_n)$ with $|S| \leq 6n - 18$ such that there is neither isolated vertex nor isolated edge in $AQ_n - S$. We want to deduce a contradiction by proving that $AQ_n - S$ is connected.
Let $AQ_n = L \cup R$, where $L = AQ_{n-1}^0$ and $R = AQ_{n-1}^1$. For convenience, let $S_L = S \cap L$ and $S_R = S \cap R$. Without loss of generality we may suppose that $|S_L| \leq |S_R|$. Then

$$|S_L| \leq (6n - 18)/2 = 3n - 9. \tag{3.6}$$

We prove that $AQ_n - S$ is connected by two steps. In step 1, we prove that $L - S_L$ is connected in $AQ_n - S$. And in step 2, we prove that any vertex in $R - S_R$ can be connected to some vertex in $L - S_L$.

**Step 1.** $L - S_L$ is connected in $AQ_n - S$.

In this case, by (3.6) and Lemma 2.5, for $n \geq 9$,

$$|S_L| \leq 3n - 9 < 4n - 12 = \kappa_1(L). \tag{3.7}$$

If there are no isolated vertices in $L - S_L$, then $L - S_L$ is connected by (3.7).

Suppose now that there exist isolated vertices in $L - S_L$. Note that $L$ is $(2n - 3)$-regular since $L \cong AQ_{n-1}$. By Lemma 2.7, any two vertices in $L$ have at most four common neighbors. To isolate two vertices in $L$, we have to remove at least $4n - 12$ vertices. By (3.7), there is exactly one isolated vertex, say $X$ in $L - S_L$. Thus

$$|S_L| \geq |N_L(X)| = 2n - 3, \tag{3.8}$$

and so, $|S_R| = |S| - |S_L| \leq 6n - 18 - (2n - 3) = 4n - 15$, that is,

$$|S_R| \leq 4n - 15. \tag{3.9}$$

Let $S'_L = S_L \cup \{X\}$. Then $L - S'_L$ contains no isolated vertices. By (3.7), $|S'_L| < \kappa_1(L)$ for $n \geq 9$. Thus, $L - S'_L$ is also connected. We only need to show that $X$ can be connected to some vertices in $L - S'_L$ via some vertices in $R - S_R$.

By Definition 2.1, $X$ has two neighbors in $R$, that is, $X_n$ and $\overline{X}_n$. By our hypothesis there are no isolated vertices in $AQ_n - S$, then there is at least one in $\{X_n, \overline{X}_n\}$ is not in $S_R$. Without loss of generality, assume that $X_n$ is not in $S_R$. Consider two cases according as the vertex $\overline{X}_n$ is in $S_R$ or not.

**Case 1.1.** $\overline{X}_n \notin S_R$.

Since $X_n$ and $\overline{X}_n$ are adjacent in $AQ_n$, and $X \overline{X}_n$ is a complement edge of dimension $n$, by (2.2), $N_{AQ_n}(X_n) \cap N_{AQ_n}(\overline{X}_n) = \{X, \overline{X}_{n-1}, (X_n)_{n-1}, (\overline{X}_n)_{n-1}\}$, that is, $|N_R(X_n) \cap N_R(\overline{X}_n)| = 2$. By (3.9), we have

$$|N_R(X_n, \overline{X}_n)| = 2(2n - 3) - 2 - 2 = 4n - 10 > 4n - 15 \geq |S_R|.$$

Thus $|N_R(X_n, \overline{X}_n) \setminus S_R| \geq 5$. Let

$$Y = \{Y^i : Y^i \in N_R(X_n, \overline{X}_n) \setminus S_R\}.$$
Then $|Y| \geq 5$. If there is some vertex $Y^i \in Y$ such that at least one of $Y^i_n$ and $\overline{Y^i}_n$ is not $S_L$, then we are done. So assume that both $Y^i_n$ and $\overline{Y^i}_n$ are in $S_L$ for any $Y^i \in Y$. Take $Y^1, Y^2 \in Y$ such that $(X_n, \overline{X}_n, Y^1)$ (or $(\overline{X}_n, X_n, Y^1)$) is a path and $Y^2$ is adjacent to one in $\{X_n, \overline{X}_n, Y^1\}$. By Lemma 2.9, we have that

$$|N_R(X_n, \overline{X}_n, Y^1, Y^2)| \geq 8(n-1) - 29 = 8n - 37.$$  

Let

$$C = N_R(X_n, \overline{X}_n, Y^1, Y^2) \cup \{\overline{X}_n, Y^1, Y^2\}.$$  

Then $|C| \geq 8n - 34$. Let $E^h_n = \{UU_n : U \in C\}$. Noting that all edges in $E^h_n$ are hypercube edges of dimension $n$, we have that, for $n \geq 9$,

$$|E^h_n| = |C| \geq 8n - 34 > 6n - 18 \geq |S|,$$

which means that there exists an edge, say $UU_n$, in $E^h_n$ such that neither of its two end-vertices is in $S$. Since $X_n, \overline{X}_n, Y^1$ and $Y^2$ are all not in $S$, $X$ can be connected to $L - S_L'$ via vertices in $R - S_R$ and the edge $UU_n$.

**Case 1.2.** $\overline{X}_n \in S_R$.

Since there are no isolated edges in $AQ_n - S$, we have $N_R(X_n) \setminus S_R \neq \emptyset$. If there is some $U \in N_R(X_n) \setminus S_R$ such that at least one in $\{U_n, \overline{U}_n\}$ is not in $S_L$, then we are done. So assume that $U_n$ and $\overline{U}_n$ are both in $S_L$ for any $U \in N_R(X_n) \setminus S_R$. Noting that $\kappa_1(R) = 4(n-1) - 8$ and (3.9), we have that

$$|N_R(X_n, U)| \geq 4n - 12 > 4n - 15 \geq |S_R|,$$

which implies $N_R(X_n, U) \setminus S_R \neq \emptyset$. Let

$$Z = \{Z^i : Z^i \in N_R(X_n, U) \setminus S_R\}.$$  

Then $|Z| \geq 2$. Take $Z^1, Z^2 \in Z$ such that $(X_n, U, Z^1)$ (or $(U, X_n, Z^1)$) is a path and $Z^2$ is adjacent to one in $\{X_n, U, Y^1\}$. By Lemma 2.9, we have that

$$|N_R(X_n, U, Z^1, Z^2)| \geq 8(n-1) - 31 = 8n - 39.$$  

Let

$$D = N_R(X_n, U, Z^1, Z^2) \cup \{X_n, U, Z^1, Z^2\}.$$  

Then $|D| \geq 8n - 35$. Let $E^c_n = \{AA_n : A \in D\}$. Noting that all edges in $E^c_n$ are complement edges of dimension $n$, we have that, for $n \geq 9$,

$$|E^c_n| = |D| \geq 8n - 35 > 6n - 18.$$  

Hence, there exists an edge, say $AA_n$, in $E^c_n$ such that neither of its two end-vertices is $S$. Since $X_n, U, Z^1$ and $Z^2$ are all not in $S$, $X$ can be connected to $L - S_L'$ via vertices in $R - S_R$ and the edge $AA_n$. 
Step 2. Any vertex in $R - S_R$ can be connected to some vertex in $L - S_L$.

Let $U$ be any vertex in $R - S_R$. Consider $U_n$ and $\overline{U}_n$, which are neighbors of $U$ in $L$. If at least one of $U_n$ and $\overline{U}_n$ is not in $S_L$, we are done. So suppose that both $U_n$ and $\overline{U}_n$ are in $S_L$. Consider the neighbor $\overline{U}_{n-1}$ of $U$ in $R$. There are two cases according as $\overline{U}_{n-1}$ is in $S_R$ or not.

Case 2.1. $\overline{U}_{n-1} \notin S_R$.

Note that $U \overline{U}_{n-1}$ is a complement edge of dimension $(n - 1)$ in $AQ_n$ and

$$(N_{AQ_n}(U) \cap N_{AQ_n}(\overline{U}_{n-1})) \cap L = \{U_n, \overline{U}_n\} \subseteq S_L.$$  

Since $U \overline{U}_{n-1}$ is not an isolated edge in $AQ_n - S$, there exists a vertex $V \in N_R(U, \overline{U}_{n-1}) \setminus S_R$. Then $V_n$ and $\overline{V}_n$ are neighbors of $V$ in $L$. If at least one of $V_n$ and $\overline{V}_n$ is not in $S_L$, we are done. So assume that both $V_n$ and $\overline{V}_n$ are in $S_L$. By (2.4) and (2.5) in the proof of Lemma 2.8, we have that

$$|N_R(U, \overline{U}_{n-1}, V)| \geq 6(n - 1) - 15 = 6n - 21. \quad (3.10)$$

Since

$$|S_L| \geq |\{U_n, \overline{U}_n, V_n, \overline{V}_n\}| = 4,$$

we have that

$$|S_R| = |S| - |S_L| \leq 6n - 18 - 4 = 6n - 22, \quad (3.11)$$

By (3.10) and (3.11), we have $|N_R(U, \overline{U}_{n-1}, V)| > |S_R|$, that is,

$$N_R(U, \overline{U}_{n-1}, V) \setminus S_R \neq \emptyset.$$  

If there is some $W \in N_R(U, \overline{U}_{n-1}, V) \setminus S_R$ such that at least one of $W_n$ and $\overline{W}_n$, which are two neighbors of $W$ in $L$, is not in $S_L$, we are done. So assume that both $W_n$ and $\overline{W}_n$ are in $S_L$ for any $W \in N_R(U, \overline{U}_{n-1}, V) \setminus S_R$. By Lemma 2.9, we have that

$$|N_R(U, \overline{U}_{n-1}, V, W)| \geq 8n - 37.$$  

Let

$$C' = N_R(U, \overline{U}_{n-1}, V, W) \cup \{U, \overline{U}_{n-1}, V, W\}.$$  

Then $|C'| \geq 8n - 33$. Let $E'_n = \{AA_n : A \in C'\}$. Noting that all edges in $E'_n$ are hypercube edges of dimension $n$, we have that, for $n \geq 9$,

$$|E'_n| = |C'| \geq 8n - 33 \geq 6n - 18.$$  

There exists at least one edge, say $AA_n$, of $E'_n$ whose two end-vertices both are not in $S$. Since $U, \overline{U}_{n-1}, V$ and $W$ are all not in $S$, this implies that $U$ can be connected to $L - S_L$.

Case 2.2. $\overline{U}_{n-1} \in S_R$.  

Since $U$ is not an isolated vertex in $AQ_n - S$ and two neighbors $U, B$ of $U$ in $L$ are both in $S$, there exists a vertex $B \in N_R(U) \setminus S_R$. Then $B, D$ and $\overline{B}$ are neighbors of $B$ in $L$. If at least one of $B_n$ and $\overline{B}$ is not in $S_L$, we are done. So assume that both $B_n$ and $\overline{B}$ are in $S_L$. If $\overline{B}_{n-1} \notin S_R$, we can obtain a path joining $B$ to some vertex in $L - S_L$ by Case 2.1 by replacing $U$ by $B$. Therefore assume $\overline{B}_{n-1} \in S_R$ below.

Since $UB$ is not an isolated edge in $AQ_n - S$, there exists a vertex $F \in N_R(U, B) \setminus S_R$. Then $F_n$ and $\overline{F}_n$ are two neighbors of $F$ in $L$. If at least one of $F_n$ and $\overline{F}_n$ is not in $S_L$, we are done. So suppose that both $F_n$ and $\overline{F}_n$ are in $S_L$. By Lemma 2.9, we have that

$$|N_R(U, B, F)| \geq 6(n - 1) - 17 = 6n - 23. \quad \text{(3.12)}$$

Since

$$|S_L| \geq |\{U, B, F, R, B_n, F_n, \overline{F}_n\}| = 6,$$

we have that

$$|S_R| = |S| - |S_L| \leq 6n - 18 - 6 = 6n - 24. \quad \text{(3.13)}$$

Comparing (3.12) with (3.13), we have that $N_R(U, B, F) \setminus S_R \neq \emptyset$. Let $Q \in N_R(U, B, F) \setminus S_R$. By Lemma 2.9, we have that

$$|N_R(U, B, F, Q)| \geq 8n - 39.$$

Let

$$C'' = N_R(U, B, F, Q) \cup \{U, B, F, Q\}.$$

Then $|C''| \geq 8n - 35$. Let $E''_n = \{AA_n : A \in C''\}$. Noting that all edges in $E''_n$ are hypercube edges of dimension $n$, we have, for $n \geq 9$,

$$|E''_n| = |C''| \geq 8n - 35 > 6n - 18.$$

There exists an edge, say $AA_n$, of $E''_n$ whose two end-vertices both are not in $S$. Since $U, B, F$ and $Q$ are all not in $S$, thus $U$ can be connected to $L - S_L$.

The proof of the theorem is complete. \hfill \Box

**Theorem 3.2.** $\lambda_2(AQ_n) = 6n - 9$ for $n \geq 4$.

**Proof.** Let $C_3$ be a cycle of length three in $AQ_n$, $U$ be any vertex not in $C_3$, and let $e$ be any edge $e$ not incident with any vertex in $C_3$. Obviously, any vertex in $C_3$ can have at most 3 neighbors in $C_3$. Thus, for $n \geq 4$,

$$|E_{AQ_n}(U) \cap E_{AQ_n}(C_3)| \leq 3 < 2n - 1 = |E_{AQ_n}(U)|,$$

and

$$|E_{AQ_n}(e) \cap E_{AQ_n}(C_3)| \leq 6 < 4n - 4 = |E_{AQ_n}(e)|.$$

So, there are no isolated vertices or isolated edges in $AQ_n - E_{AQ_n}(C_3)$. That is, $E_{AQ_n}(C_3)$ is a 2-edge-cut of $G$. It follows that, for $n \geq 4$,

$$\lambda_2(AQ_n) \leq E_{AQ_n}(C_3) = 6n - 9.$$
In the following, we only need to prove that \( \lambda_2(AQ_n) \geq 6n - 9 \) for \( n \geq 4 \).

Let \( F \) be an arbitrary 2-edge-cut in \( AQ_n \) with \( |F| \leq 6n - 10 \) such that there are neither isolated vertices nor isolated edges in \( AQ_n - F \). Let \( F_L = F \cap L \) and \( F_R = F \cap R \). Without loss of generality we may suppose that \( |F_L| \leq |F_R| \). Then

\[
|F_L| \leq \frac{1}{2}(6n - 10) = 3n - 5.
\]

We will deduce a contradiction by proving that \( AQ_n - F \) is connected by two steps. In step 1, we show that \( L - F_L \) is connected in \( AQ_n - F \). In step 2, we show that any vertex of \( R \) can be connected to \( L \) in \( AQ_n - F \).

**Step 1.** \( L - F_L \) is connected in \( AQ_n - F \).

By our hypothesis and Lemma 2.5, for \( n \geq 4 \), we have that

\[
|F_L| \leq 3n - 5 < 4(n - 1) - 4 = \lambda_1(L). \tag{3.14}
\]

Thus, if there are no isolated vertices in \( L - F_L \), then \( L - F_L \) is connected, and so we are done. In the following discussion, we assume that there exists an isolated vertex \( X \) in \( L - F_L \).

Since \( L \) is \( (2n - 3) \)-regular and any two vertices are incident with at most one edge, to get two isolated vertices in \( L \), we have to remove at least \( 4n - 7 \) edges from \( L \). However, by (3.14), \( |F_L| \leq 3n - 5 < 4n - 7 \) for \( n \geq 4 \). This shows that there is just one isolated vertex \( X \) in \( L - F_L \). Then by Lemma 2.4, we have

\[
\lambda(L - X) \geq \kappa(L - X) \geq \kappa(L) - 1 = \begin{cases} 
4 - 1 = 3 & \text{if } n = 4 \\
2n - 3 - 1 = 2n - 4 & \text{if } n > 4 
\end{cases}
\]

and

\[
|F_L| - |E_L(X)| \leq 3n - 5 - (2n - 3) = n - 2,
\]

which implies that \( |F_L| - |E_L(X)| < \lambda(L - X) \) for \( n \geq 4 \). In other words, the subgraph \( H = (L - X) - F_L = (L - F_L) - X \) is connected. In the following we only need to prove that \( X \) can be connected to \( H \) in \( AQ_n - F \). Since \( AQ_n - F \) contains no isolated vertices, at least one of two edges \( XX_n \) and \( \overline{X} \overline{X}_n \) is not in \( F \). Without loss of generality, we may assume that \( XX_n \) is not in \( F \). Consider two cases according as the edge \( XX_n \) is in \( F \) or not.

**Case 1.1.** \( XX_n \notin F \).

Note that \( X_n, \overline{X}_{n-1} \) and \( \overline{X}_n, \overline{X}_{n-1} \) are edges in \( AQ_n \), where \( \overline{X}_{n-1} \) is in \( L \). If at least one of the two edges is not in \( F \), we are done. So, we can assume that both the two edges are in \( F \). We will construct \( 4n - 8 \) edge disjoint paths joining \( X \) to some vertex in \( L - X \) in the following.

Let

\[
E_1 = \{X_nX^i : X_nX^i \in E_R(X_n) \setminus \{X_n\overline{X}_n\}\}, \\
E_2 = \{\overline{X}_nY^j : \overline{X}_nY^j \in E_R(\overline{X}_n) \setminus \{X_n\overline{X}_n\}\}, \\
F' = F \setminus (E_L(X) \cup \{X_n\overline{X}_{n-1}, \overline{X}_n\overline{X}_{n-1}\}).
\]
Then \(|E_1'| = |E_2'| = 2n - 4, E_1' \cap E_2' = \emptyset\), and
\[|F'| \leq 6n - 10 - (2n - 3) - 2 = 4n - 9.\] (3.15)

Let
\[P_i = (X, X_n, X^i, X^i_n) \quad \text{and} \quad Q_j = (X, \overline{X}_n, Y^j, \overline{Y}_n)\] (3.16)
be a path joining \(X\) to some vertex in \(L - X\), and let
\[\mathcal{P} = \{P_i : 1 \leq i \leq 2n - 4\} \cup \{Q_j : 1 \leq j \leq 2n - 4\}.\]
Then
\[|\mathcal{P}| = 4n - 8.\] (3.17)

Since these paths defined in (3.16) are edges disjoint, comparing (3.15) and (3.17), we have that there exists a path \(P \in \mathcal{P}\) such that \(E(P) \cap F' = \emptyset\). Then \(X\) can be connected to a vertex in \(L\).

**Case 1.2.** \(X \overline{X}_n \in F\).

If \(X_n \overline{X}_{n-1} \notin F\), we are done. So assume that \(X_n \overline{X}_{n-1} \in F\). Since there are no isolated edges in \(AQ_n - F\), we have \(E_R(X_n) - F_R \neq \emptyset\).

If \(X_n \overline{X}_n \notin F_R\), we can obtain a path joining \(X\) via a path to a vertex in \(L - X\) by the \(4n - 8\) paths \(\mathcal{P}\) constructed in Case 1.1. Hence, we can assume \(X_n \overline{X}_n \in F_R\) below.

We will construct \(4n - 9\) edge disjoint paths joining \(X\) to some vertex in \(L - X\) in the following.

Let \(X_nW \in E_R(X_n) - F_R\). Then \(W \neq \overline{X}_n\).

Let
\[E'_1 = \{WX^i : WX^i \in E_R(W) \setminus \{X_nW\}\},\]
\[E'_2 = \{X_nY^j : X_nY^j \in E_R(X_n) \setminus \{X_nW, X_n\overline{X}_n\}\},\]
\[F^* = F \setminus (E_L(X) \cup \{X \overline{X}_n, X_n \overline{X}_{n-1}, X_{n-1} \overline{X}_n\}).\]
Then \(|E'_1| = 2n - 4, |E'_2| = 2n - 5\) and
\[|F^*| \leq 6n - 10 - (2n - 3) - 3 = 4n - 10.\] (3.18)

Let
\[P^*_i = (X, X_n, W, X^i, X^i_n) \quad \text{and} \quad Q^*_j = (X, X_n, Y^j, \overline{Y}_n)\] (3.19)
be a path joining \(X\) to some vertex in \(L - X\), and let
\[\mathcal{P}^* = \{P^*_i : 1 \leq i \leq 2n - 4\} \cup \{Q^*_j : 1 \leq j \leq 2n - 5\}.\]
Then
\[|\mathcal{P}^*| = 4n - 9.\] (3.20)

Since these paths defined in (3.19) are edges disjoint, comparing (3.18) and (3.20), there exists a path \(P^* \in \mathcal{P}^*\) such that \(E(P^*) \cap F^* = \emptyset\). This implies that vertex \(X\) can be connected to a vertex in \(L - X\).
**Step 2.** Any vertex $X$ of $R$ can be connected to $L$ in $AQ_n - F$.

Suppose that $X$ is an arbitrary vertex in $R$, if $\{XX_n, X\overline{X}_n\} \not\subseteq F$, where $X_n$ and $\overline{X}_n$ are both in $L$, we are done. Thus, assume that $\{XX_n, X\overline{X}_n\} \subseteq F$. Since there is neither isolated vertex nor isolated edge in $AQ_n - F$, the vertex $X$ lies on a path $T$ of length 2 in $R - FR$. Assume $V(T) = \{X, Y, Z\}$. If $\{YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n\} \not\subseteq F$, we are done. Hence, assume that $\{YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n\} \subseteq F$ in the following.

Let $F^* = F \setminus \{XX_n, X\overline{X}_n, YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n\}$.

Then

$$|F^*| \leq 6n - 10 - 6 = 6n - 16. \quad (3.21)$$

Note that

$$|E_R(T)| \geq 3 \ast (2n - 5) = 6n - 15. \quad (3.22)$$

By Lemma 2.8, we have $|N_R(X) \cap N_R(Y) \cap N_R(Z)| \leq 1$. We will construct edge disjoint paths joining $V(T)$ to $V(L)$ according to the following two cases.

**Case 2.1** $|N_R(X) \cap N_R(Y) \cap N_R(Z)| = 0$.

In this case, each vertex in $N_R(T)$ is incident to at most two edges in $E_R(T)$. Since every vertex is incident to exactly two crossing edges in $AQ_n$, we can construct $|E_R(T)|$ edge disjoint paths $\mathcal{P}$ joining $V(T)$ to $V(L)$ as follows.

For any vertex $W^i$ in $N_R(T)$, if $W^i$ is incident to exactly one edge $AW^i$ in $E_R(T)$ where $A \in V(T)$, let $P_i = (A, W^i, W^i_i)$; if $W^i$ is incident to exactly two edges $AW^i$ and $BW^i$ in $E_R(T)$ where $A, B \in V(T)$, let $P_i = (A, W^i, W^i_n)$ and $P'_i = (B, W^i, W^i'_n)$.

Comparing (3.21) and (3.22), there exists a path $P \in \mathcal{P}$ such that $E(P) \cap F^* = \emptyset$. This implies that vertex $X$ can be connected to a vertex in $L$.

**Case 2.2** $|N_R(X) \cap N_R(Y) \cap N_R(Z)| = 1$.

In this case, there is a vertex $U$ in $N_R(T)$ incident to exactly three edges in $E_R(T)$ and other vertices in $N_R(T)$ is incident to at most two edges in $E_R(T)$.

Since $|F_R| \leq |F^*|$, comparing (3.21) and (3.22), there exists an edge $e \in E_R(T) \setminus F_R$. Without loss of generality, we may assume $e = XW$. If $\{WW_n, W\overline{W}_n\} \not\subseteq F$, where $W_n$ and $\overline{W}_n$ are both in $L$, we are done. Thus, assume that $\{WW_n, W\overline{W}_n\} \subseteq F$.

Let $F' = F \setminus \{XX_n, X\overline{X}_n, YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n, WW_n, W\overline{W}_n\}$.

Then

$$|F'| \leq 6n - 10 - 8 = 6n - 18. \quad (3.23)$$

Since every vertex is incident to exactly two crossing edges in $AQ_n$, we can construct edge disjoint paths $\mathcal{P}$ joining $V(T)$ to $V(L)$ as follows.

For any vertex $W^i$ in $N_R(T) \setminus \{W^i\}$, if $W^i$ is incident to exactly one edge $AW^i$ in $E_R(T)$ where $A \in V(T)$, let $P_i = (A, W^i, W^i_n)$; if $W^i$ is incident to exactly two edges $AW^i$ and
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Let $A, B \in V(T)$, let $P_i = (A, W_i, W_i^n)$ and $P'_i = (B, W_i, W_i^n)$; if $W_i = U$, let $P_i = (Y, U, U_n)$ and $P'_i = (Z, U, U_n)$.

**Case 2.2.1** The vertex $W$ is incident to one edge in $E_R(T)$. Hence, we construct at least $|E_R(T)| - 2 \geq 6n - 17$ edge disjoint paths $\mathcal{P}$ joining $V(T)$ to $V(L)$. By (3.23), there exists a path $P \in \mathcal{P}$ such that $E(P) \cap F' = \emptyset$. This implies that vertex $X$ can be connected to a vertex in $L$.

**Case 2.2.2** The vertex $W$ is incident to at least two edges in $E_R(T)$. Hence, we construct at least $|E_R(T)| - 3 \geq 6n - 18$ edge disjoint paths $\mathcal{P}$ joining $V(T)$ to $V(L)$. If there exists a path $P \in \mathcal{P}$ such that $E(P) \cap F' = \emptyset$, we are done. Assume $E(P) \cap F' \neq \emptyset$ for every path in $\mathcal{P}$. By (3.23), the faulty edges $F'$ are all in $\mathcal{P}$. Let $P = (X, W, W_j, (W_j)_n)$, where $W_j$ is not in $T$ and $W W_j \not\subseteq F$. Then, $P$ is fault-free. This implies that vertex $X$ can be connected to a vertex in $L$.

We proved that $AQ_n - F$ is connected, which means $\lambda_2(AQ_n) \geq 6n - 9$ for $n \geq 4$.

The theorem follows.

4. Conclusions

In this paper, we explore two stronger measurement parameters for the reliability and the tolerance of networks called the 2-extra connectivity $\kappa_2(G)$ and the 2-extra edge-connectivity $\lambda_2(G)$ of a connected graph $G$, which not only compensate for some shortcomings but also generalize the classical connectivity $\kappa(G)$ and the classical edge-connectivity $\lambda(G)$, and so can provide more accurate measures for the reliability and the tolerance of a large-scale parallel processing system. The augmented cube $AQ_n$, as an important variant of the hypercube $Q_n$, has many desirable properties (for more results, see, for example, [3–5, 7–10, 14, 17, 18, 20–22, 24, 27, 28]). Here, we have showed that $\kappa_2(AQ_n) = 6n - 17$ for $n \geq 9$; and $\lambda_2(AQ_n) = 6n - 9$ for $n \geq 4$. In other words, for $n \geq 9$ (respectively, $n \geq 4$), at least $6n - 17$ vertices (respectively, $6n - 9$ edges) of $AQ_n$ have to be removed to get a disconnected graph that contains no isolated vertices and isolated edges. Compared with previous results, our results enhance the fault tolerant ability of this kind of network theoretically.

References


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