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Fault tolerance of augmented cubes *

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Abstract

The augmented cube AQ_n, proposed by Choudum and Sunitha [S. A. Choudum, V. Sunitha, Augmented cubes, Networks 40 (2) (2002) 71-84], is a $(2n-1)$-regular $(2n-1)$-connected graph ($n \geq 4$). This paper determines that the 2-extra connectivity of AQ_n is $6n-17$ for $n \geq 9$ and the 2-extra edge-connectivity is $6n-9$ for $n \geq 4$. That is, for $n \geq 9$ (respectively, $n \geq 4$), at least $6n-17$ vertices (respectively, $6n-9$ edges) of AQ_n have to be removed to get a disconnected graph that contains no isolated vertices and isolated edges. When the augmented cube is used to model the topological structure of a large-scale parallel processing system, these results can provide more accurate measurements for reliability and fault tolerance of the system.

Keywords: connectivity, extra connectivity, fault tolerance, augmented cube.

2010 Mathematics Subject Classification: 05C40, 68M10, 68R10

1. Introduction

It is well known that the underlying topology of an interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network. For all the graph terminologies and notations not defined here, we follow [25]. Then we use graphs and networks interchangeably in this paper.

A set of vertices (respectively, edges) S of G is called a *vertex-cut* (respectively, an *edge-cut*) if $G - S$ is disconnected. The *connectivity* $\kappa(G)$ (respectively, the *edge-connectivity*

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$\lambda(G)$) of G is defined as the minimum cardinality of a vertex-cut (respectively, an edge-cut) S. And it is known to all that the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are two important parameters to measure reliability and fault tolerance of the network. These parameters, however, have an obvious deficiency, that is, they tacitly assume that all elements in any subset of G can potentially fail at the same time, which is almost impossible in practiced applications of networks. In other words, in the definition of $\kappa(G)$ and $\lambda(G)$, absolutely no conditions or restrictions are imposed either on the set S or on the components of $G - S$. Consequently, to compensate for these shortcomings, it seems natural to generalize the classical connectivity by adding some conditions or restrictions on the set S and the components of $G - S$.

In [11, 12], Esfahanian and Hakimi generalized the notion of connectivity by suggesting the concept of restricted connectivity in point of view of network applications. A set $S \subset V(G)$ (respectively, $F \subset E(G)$) is called a *restricted vertex-set* (respectively, *edge-set*) if it does not contain the neighbor-set of any vertex in G as its subset. A restricted vertex-set S (respectively, edge-set F) is called a *restricted vertex-cut* (respectively, *restricted edge-cut*) if $G-S$ is disconnected. The *restricted connectivity* $\kappa'(G)$ (respectively, *restricted edge-connectivity* $\lambda'(G)$) is the minimum cardinality of a restricted vertex-cut (respectively, edge-cut) in G, if any, and does not exist otherwise, denoted by $+\infty$.

However, the maximum difficult for computing the restricted connectivity of a graph G is to check that a vertex-cut does not contain the neighbor-set of any vertex in G as its subset. Thus, only a little knowledge of results has been known on $\kappa'(G)$ or $\lambda'(G)$ even for particular classes of graphs. For example, Xu and Xu [30] studied the $\lambda'(G)$ for a vertex-transitive graph G, Esfahanian [11] determined $\kappa'(Q_n) = \lambda'(Q_n) = 2n - 2$ for the hypercube Q_n and $n \geq 3$.

To avoid this difficult, one slightly modified the concept of a restricted vertex-set S by replacing the term "any vertex in G" in the condition by the term "any vertex in $G - S$". We call in this sense the connectivity as the *super connectivity*, denoted by $\kappa_1(G)$ for the super connectivity and $\lambda_1(G)$ for the *super edge-connectivity* (see, for example, [1, 2, 15, 16, 26]). Clearly, $\kappa_1(G) \leq \kappa'(G)$ and $\lambda_1(G) = \lambda'(G)$ if they exist. The super connectivity of some graphs have been determined by several authors. For example, for the hypercube Q_n, the twisted cube TQ_n, the cross cube CQ_n, the Möbius cube MQ_n and the locally twisted cube LTQ_n, Xu *et al.* [29] showed that their super connectivity and the super edge-connectivity are all $2n-2$; for the star graph S_n, Hu and Yang [19] proved that $\kappa_1(S_n) = 2n - 4$ for $n \geq 3$; for the augmented cube AQ_n, Ma, Liu and Xu [21, 22] determined $\kappa_1(AQ_n) = 4n - 8$ for $n \geq 6$ and $\lambda_1(AQ_n) = 4n - 4$ for $n \geq 2$; for the (n, k)-star graphs $S_{n,k}$, Yang et al. [32] proved that $\kappa_1(S_{n,k}) = n + k - 3$; for the n-dimensional alternating group graph AG_n, Cheng et al. [6] determined $\kappa_1(AG_n) = 4n - 11$ for $n \geq 5$.

Observing that every component of $G-S$ contains at least two vertices if S is a restricted vertex-set of G, Fàbrega and Fiol [13] introduced the h-extra connectivity of G. A vertex-cut (respectively, an edge-cut) S of G is called an *h-vertex-cut* (respectively, an *h-edge-cut*) if every component of $G - S$ has more than h vertices. The *h-extra connectivity* $\kappa_h(G)$

(respectively, *h-extra edge-connectivity* $\lambda_h(G)$) defined as $\min\{|S|$: S is an h-vertex-cut (respectively, h-edge-cut) of $G\}$. Clearly, $\kappa_0(G) = \kappa(G)$ and $\lambda_0(G) = \lambda(G)$ for any graph G if G is not a complete graph. Thus, as a new measurement for reliability and fault tolerance of the large-scale parallel processing system, the h-extra connectivity is more accurate than the classical connectivity and has recently received much attention. For example, for the hypercube Q_n, Xu *et al.* [31, 34] determined $\kappa_2(Q_n) = 3n - 5$ and $\lambda_2(Q_n) = 3n - 4$ for $n \geq 4$; for the folded hypercube FQ_n, Zhu *et al.* [35] determined $\kappa_2(FQ_n) = 3n-2$ for $n \geq 8$ and $\lambda_2(FQ_n) = 3n-1$ for $n \geq 5$; for the star graph S_n, Wan, Zhang [23] determined $\kappa_2(S_n) = 6(n-3)$ for $n \geq 4$; for the (n,k)-star graphs $S_{n,k}$, Yang *et al.* [32] proved that $\kappa_2(S_{n,k}) = n + 2k - 5$ for $2 \leq k \leq n-2$; Zhang *et al.* [33] proved $\kappa_2(AG_n) = 6n - 18$ for $n \geq 5$.

In this paper, we study the augmented cube AQ_n and determine $\kappa_2(AQ_n) = 6n - 17$ for $n \geq 9$ and $\lambda_2(AQ_n) = 6n - 9$ for $n \geq 4$.

The rest of this paper is organized as follows. In Section 2, we recall the structure of AQ_n, and some definitions and lemmas. The main results are given in Section 3. Finally, we conclude our paper in Section 4.

2. Definitions and lemmas

Let n be a positive integer. The *n-dimensional augmented cube*, denoted by AQ_n, proposed by Choudum and Sunitha [7–9], having 2^n vertices, each labeled by an n-bit binary string, that is, $V(AQ_n) = \{x_n x_{n-1} \cdots x_1 :\ x_i \in \{0,1\},\ 1 \leq i \leq n\}$, can be defined recursively as follows.

Definition 2.1. *AQ_1 is a complete graph K_2 with the vertex set $\{0,1\}$. For $n \geq 2$, AQ_n is obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, where $V(AQ^0_{n-1}) = \{0x_{n-1} \ldots x_2x_1 : x_i \in \{0,1\},\ 1 \leq i \leq n-1\}$ and $V(AQ^1_{n-1}) = \{1x_{n-1} \ldots x_2x_1 : x_i \in \{0,1\},\ 1 \leq i \leq n-1\}$, and a vertex $X = 0x_{n-1} \ldots x_2x_1$ of AQ^0_{n-1} being joined to a vertex $Y = 1y_{n-1} \ldots y_2y_1$ of AQ^1_{n-1} if and only if either*

(i) $x_i = y_i$ for $1 \leq i \leq n-1$, or

(ii) $x_i = \bar{y}_i$ for $1 \leq i \leq n-1$.

The graphs shown in Figure 1 are the augmented cubes AQ_1, AQ_2 and AQ_3, respectively.

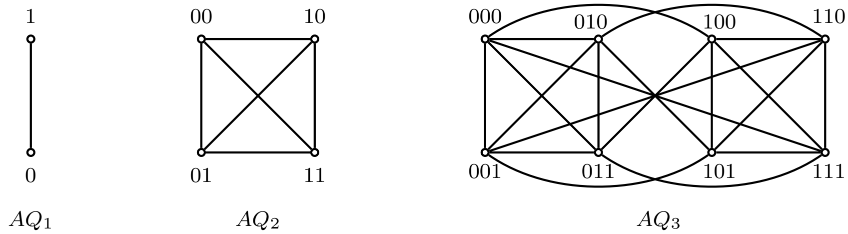

Figure 1: Three augmented cubes AQ_1, AQ_2 and AQ_3

For convenience, we can express the recursive structure of AQ_n as $AQ_n = L \odot R$, where $L = AQ^0_{n-1}$ and $R = AQ^1_{n-1}$. Then we call the edges between L and R *crossing edges*. Obviously every vertex is incident to exactly two crossing edges. Let $X = x_n x_{n-1} \cdots x_1$ be an n-bit binary string. And for $1 \le i \le n$, let

$$X_i = x_n x_{n-1} \cdots x_{i+1} \bar{x}_i x_{i-1} \cdots x_1,$$
$$\overline{X}_i = x_n x_{n-1} \cdots x_{i+1} \bar{x}_i \bar{x}_{i-1} \cdots \bar{x}_1.$$

Obviously, $X_1 = \overline{X}_1$, $(X_i)_i = X = \overline{(\overline{X}_i)_i}$. According to Definition 2.1, we can directly obtain a useful characterization of adjacency.

Proposition 2.2. *Assume that $X = x_n x_{n-1} \cdots x_1$ and $Y = y_n y_{n-1} \cdots y_1$ are two distinct vertices in AQ_n. Then X and Y are adjacent if and only if either*

i) there exists an integer i $(1 \le i \le n)$ such that $Y = X_i$, or

ii) there exists an integer i $(2 \le i \le n)$ such that $Y = \overline{X}_i$.

By Proposition 2.2, an alternative definition of AQ_n can be stated as follows.

Definition 2.3. *The augmented cube AQ_n of dimension n has 2^n vertices. Each vertex is labeled by a unique n-bit binary string as its address. Two vertices X and Y are joined if and only if either*

(i) there exists an integer i with $1 \le i \le n$ such that $Y = X_i$; in this case, the edge is called a hypercube edge of dimension i, denoted by XX_i, or

(ii) there exists an integer i with $2 \le i \le n$ such that $Y = \overline{X}_i$; in this case, the edge is called a complement edge of dimension i, denoted by $X\overline{X}_i$.

Lemma 2.4. [9] *AQ_n is $(2n-1)$-regular $(2n-1)$-connected for $n \ge 4$, however, $\kappa(AQ_3) = 4$ for $n = 3$.*

Lemma 2.5. [21, 22] *$\kappa_1(AQ_n) = 4n - 8$ for $n \ge 6$ and $\lambda_1(AQ_n) = 4n - 4$ for $n \ge 2$.*

Lemma 2.6. *Any two adjacent vertices in AQ_n have either two or four common neighbors for $n \geq 3$.*

Proof. Let X and Y be two adjacent vertices in AQ_n. Then Y is either X_i or $\overline{X}_i$ by Proposition 2.2. If $Y = X_i$ for some i with $1 \leq i \leq n$, that is, XX_i is a hypercube edge of dimension i, then we have

$$N_{AQ_n}(X) \cap N_{AQ_n}(X_i) = \begin{cases} \{X_2, \overline{X}_2\} & \text{if } i = 1 \\ \{\overline{X}_i, \overline{X}_{i-1}\} & \text{if } i > 1, \end{cases} \tag{2.1}$$

that is, X and X_i have exactly two common neighbors in AQ_n. If $Y = \overline{X}_i$ for some i with $2 \leq i \leq n$, that is, $X\overline{X}_i$ is a complement edge of dimension i, then we have

$$N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X}_i) = \begin{cases} \{X_i, X_{i+1}, \overline{X}_{i-1}, \overline{X}_{i+1}\} & \text{if } 2 \leq i \leq n-1 \\ \{\overline{X}_{n-1}, X_n\} & \text{if } i = n. \end{cases} \tag{2.2}$$

In this case, X and $\overline{X}_i$ have four common neighbors for $2 \leq i \leq n-1$ while have two common neighbors for $i = n$. $\square$

Lemma 2.7. [21] *Any two vertices in AQ_n have at most four common neighbors for $n \geq 3$.*

Let $N_{AQ_n}(T) = \cup_{U \in V(T)} N_{AQ_n}(U) \setminus V(T)$ and $E_{AQ_n}(T) = \{XY \mid XY \in E(AQ_n)$ and $X \in V(T), Y \in V(AQ_n) \setminus V(T)\}$ for any subgraph T of AQ_n, we have the following consequences.

Lemma 2.8. *Let $P = (Y, X, Z)$ be a path of length two in AQ_n between Y and Z for $n \geq 5$. Then $|N_{AQ_n}(P)| \geq 6n - 17$ and $|N_{AQ_n}(X) \cap N_{AQ_n}(Y) \cap N_{AQ_n}(Z)| \leq 1$. Furthermore, if $Z = \overline{X}_n$, we have $|N_{AQ_n}(P)| \geq 6n - 15$.*

Proof. According to Proposition 2.2, Y and Z are in $N_{AQ_n}(X) = A \cup B$, where $A = \{X_i \mid 1 \leq i \leq n\}$ and $B = \{\overline{X}_j \mid 2 \leq j \leq n\}$. It is clear that $A \cap B = \emptyset$. Consider the following three cases.

Case 1. $\{Y, Z\} \subset A$.

In this case, XY and XZ are all hypercube edges of some dimensions i and j, respectively. Without loss of generality, we may assume that $Y = X_i$, $Z = X_j$ for $1 \leq i < j \leq n$. By Definition 2.3 and Lemma 2.6, we have

$$\begin{aligned} &N_{AQ_n}(X) \cap N_{AQ_n}(X_i) = \begin{cases} \{X_2, \overline{X}_2\} & \text{if } i = 1 \\ \{\overline{X}_i, \overline{X}_{i-1}\} & \text{if } i > 1, \end{cases} \\ &N_{AQ_n}(X) \cap N_{AQ_n}(X_j) = \{\overline{X}_j, \overline{X}_{j-1}\} \end{aligned}$$

and

$$N_{AQ_n}(X_i) \cap N_{AQ_n}(X_j) = \begin{cases} \{X, \overline{X}_2, (X_1)_3, \overline{(X_1)_3}\} & \text{if } i = 1, j = 2, 3 \\ \{\overline{X}_i, X, (X_i)_{i+1}, \overline{(X_i)}_{i+1}\} & \text{if } i > 1, j = i + 1 \\ \{X, (X_i)_j\} & \text{otherwise.} \end{cases}$$

Since AQ_n is $(2n-1)$-regular and $(2n-1)$-connected for $n \geq 5$, it is not difficult to check that

$$|N_{AQ_n}(P)| = \begin{cases} 6n-13 & \text{if } i=1, j=2,3 \\ 6n-13 & \text{if } i>1, j=i+1 \\ 6n-12 & \text{otherwise.} \end{cases} \tag{2.3}$$

Case 2. $\{Y, Z\} \subset B$.

In this case, XY and XZ are all complement edges of some dimensions i and j, respectively. Without loss of generality, assume that $Y = \overline{X}_i$ and $Z = \overline{X}_j$ for $2 \leq i < j \leq n$, then we have

$$N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X}_i) = \{X_i, X_{i+1}, \overline{X}_{i-1}, \overline{X}_{i+1}\},$$
$$N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X}_j) = \begin{cases} \{X_j, X_{j+1}, \overline{X}_{j-1}, \overline{X}_{j+1}\} & \text{if } j<n \\ \{\overline{X}_{n-1}, X_n\} & \text{if } j=n \end{cases}$$

and

$$N_{AQ_n}(\overline{X}_i) \cap N_{AQ_n}(\overline{X}_j) = \begin{cases} \{X, X_{i+1}\} & \text{if } j=i+1, j<n \\ \{X, X_n\} & \text{if } j=i+1, j=n \\ \{X, \overline{X}_{i+1}, (\overline{X}_i)_{i+2}, \overline{(\overline{X}_i)_{i+2}}\} & \text{if } j=i+2, j<n \\ \{X, \overline{X}_{n-1}, (\overline{X}_{n-2})_n, \overline{(\overline{X}_{n-2})_n}\} & \text{if } j=i+2, j=n \\ \{X, \overline{(\overline{X}_i)_j}\} & \text{if } j \geq i+3, j<n \\ \{X, \overline{(\overline{X}_i)_n}\} & \text{if } j \geq i+3, j=n \end{cases}$$

It is easy to compute that

$$|N_{AQ_n}(P)| = \begin{cases} 6n-15 & \text{if} j=i+1, j<n \\ 6n-13 & \text{if} j=i+1, j=n \\ 6n-17 & \text{if} j=i+2, j<n \\ 6n-15 & \text{if} j=i+2, j=n \\ 6n-16 & \text{if} j \geq i+3, j<n \\ 6n-14 & \text{if} j \geq i+3, j=n. \end{cases} \tag{2.4}$$

Case 3. $Y \in A$ and $Z \in B$.

In this case, XY is a hypercube edge of some dimension i and XZ is a complement hypercube edge of some dimension j. Assume that $Y = X_i$, $Z = \overline{X}_j$, for $1 \leq i \leq n, 2 \leq j \leq n$, we have

$$N_{AQ_n}(X) \cap N_{AQ_n}(X_i) = \begin{cases} \{X_2, \overline{X}_2\} & \text{if } i=1 \\ \{\overline{X}_i, \overline{X}_{i-1}\} & \text{if } i>1, \end{cases}$$
$$N_{AQ_n}(X) \cap N_{AQ_n}(\overline{X}_j) = \begin{cases} \{X_j, X_{j+1}, \overline{X}_{j-1}, \overline{X}_{j+1}\} & \text{if } j<n \\ \{\overline{X}_{n-1}, X_n\} & \text{if } j=n \end{cases}$$

and

$$N_{AQ_n}(X_i)\cap N_{AQ_n}(\overline{X}_j) = \begin{cases} \{X, X_2\} & \text{if } i=1, j=2 \\ \{X, \overline{X}_2, (X_1)_3, \overline{(X_1)_3}\} & \text{if } i=1, j=3 \\ \{X, \overline{(X_1)_j}\} & \text{if } i=1, 4\le j<n \\ \{X, \overline{(X_1)_n}\} & \text{if } i=1, j=n \\ \{X, X_1\} & \text{if } i=j=2 \\ \{X, \overline{X}_{i-1}, (X_i)_{i-1}, (\overline{X}_i)_{i-1}\} & \text{if } 3\le i=j\le n-1 \\ \{X, \overline{X}_{n-1}, (X_n)_{n-1}, (\overline{X}_n)_{n-1}\} & \text{if } i=j=n \\ \{X, \overline{X}_i, (X_i)_{i+1}, \overline{(X_i)_{i+1}}\} & \text{if } j=i-1, 3\le i\le n-1 \\ & \quad \text{or } j=i+1, 2\le i\le n-2 \\ \{X, \overline{X}_n\} & \text{if } j=i-1, i=n \\ \{X, \overline{X}_{n-1}, (X_{n-1})_n, \overline{(X_{n-1})_n}\} & \text{if } j=n, i=n-1 \\ \{X, \overline{X}_{i-1}, (X_i)_{i-1}, \overline{(X_i)_{i-2}}\} & \text{if } j=i-2, i\ge 4 \\ \{X, \overline{(X_i)_j}\} & \text{if } j\le i-3, i\ge 5 \\ & \quad \text{or } j\ge i+2, j<n \\ \{X, \overline{(X_i)_n}\} & \text{if } j\ge i+2, j=n. \end{cases}$$

We can compute that

$$|N_{AQ_n}(P)| = \begin{cases} 6n-13 & \text{if } i=1, j=2 \\ 6n-15 & \text{if } i=1, j=3 \\ 6n-14 & \text{if } i=1, 4\le j<n \\ 6n-12 & \text{if } i=1, j=n \\ 6n-13 & \text{if } i=j=2 \\ 6n-15 & \text{if } 3\le i=j\le n-1 \\ 6n-13 & \text{if } i=j=n \\ 6n-15 & \text{if } j=i-1, 3\le i\le n-1 \\ & \quad \text{or } j=i+1, 2\le i\le n-2 \\ 6n-13 & \text{if } j=i-1, i=n \\ 6n-13 & \text{if } j=n, i=n-1 \\ 6n-15 & \text{if } j=i-2, i\ge 4 \\ 6n-14 & \text{if } j\le i-3, i\ge 5 \\ & \quad \text{or } j\ge i+2, j<n \\ 6n-12 & \text{if } j\ge i+2, j=n. \end{cases} \tag{2.5}$$

in view of (2.3),(2.4) and (2.5), we derive that $|N_{AQ_n}(P)| \ge 6n-17$.

From the above, we can easily check that $|N_{AQ_n}(X)\cap N_{AQ_n}(Y)\cap N_{AQ_n}(Z)| \le 1$ in all these cases. By (2.4) and (2.5), we have $|N_{AQ_n}(P)| \ge 6n-15$ if $Z = \overline{X}_n$.

The lemma follows. $\square$

Lemma 2.9. *Let $P = (Y, X, Z)$ be a path of length two in AQ_n for $n \ge 5$. Assume that $U \in N_{AQ_n}(P)$, then $|N_{AQ_n}(U, X, Y, Z)| \ge 8n-31$. If $Z = \overline{X}_n$, we have $|N_{AQ_n}(U, X, Y, Z)| \ge 8n-29$.*

Proof. By Lemma 2.8, we have $|N_{AQ_n}(P)| \geq 6n-17$, and $|N_{AQ_n}(P)| \geq 6n-15$ if $Z=\overline{X}_n$. Since U is a vertex in $N_{AQ_n}(P)$, U has at most three neighbors in P

If U has exactly one neighbor in P, by Lemma 2.7 we have $|N_{AQ_n}(U) \cap N_{AQ_n}(X,Y,Z)| \leq 11$. In this case we can easily compute that $|N_{AQ_n}(U,X,Y,Z)| \geq 6n-17-1+2n-1-1-11 = 8n-31$, and $|N_{AQ_n}(U,X,Y,Z)| \geq 6n-15-1+2n-1-1-11 = 8n-29$ if $Z=\overline{X}_n$.

If U has exactly two neighbors in P, by Lemma 2.7, we have $|N_{AQ_n}(U) \cap N_{AQ_n}(X,Y,Z)| \leq 10$. In this case we arrive at $|N_{AQ_n}(U,X,Y,Z)| \geq 6n-17-1+2n-1-2-10 = 8n-31$, and $|N_{AQ_n}(U,X,Y,Z)| \geq 6n-15-1+2n-1-2-10 = 8n-29$ if $Z=\overline{X}_n$.

If U has exactly three neighbors in P, by Lemma 2.7, we have $|N_{AQ_n}(U) \cap N_{AQ_n}(X,Y,Z)| \leq 8$. Therefore, $|N_{AQ_n}(U,X,Y,Z)| \geq 6n-17-1+2n-1-3-8 = 8n-30$, and $|N_{AQ_n}(U,X,Y,Z)| \geq 6n-15-1+2n-1-3-8 = 8n-28$ if $Z=\overline{X}_n$.

The lemma follows. □

3. Main Results

In this section, we present our main results, that is, we determine the 2-extra connectivity and the 2-extra edge-connectivity of the augmented cube AQ_n.

Theorem 3.1. $\kappa_2(AQ_n) = 6n-17$ *for* $n \geq 9$.

Proof. Take a path $P=(\overline{X}_i, X, \overline{X}_{i+2})$ in AQ_n, where $2 \leq i \leq n-3$. Then, by (2.4) in the proof of Lemma 2.8, $|N_{AQ_n}(P)| = 6n-17$.

Let $H = AQ_n - (P \cup N_{AQ_n}(P))$. Then, for $n \geq 9$,

$$\begin{aligned} |V(H)| &= |V(AQ_n)| - |V(P)| - |N_{AQ_n}(P)| \\ &= 2^n - 3 - (6n-17) \\ &= 2^n - 6n + 14 > 0, \end{aligned}$$

that is, $V(H) \neq \emptyset$. By Lemma 2.7, $|N_{AQ_n}(Y) \cap N_{AQ_n}(P)| \leq 12$ for any $Y \in V(H)$ and $|N_{AQ_n}(e) \cap N_{AQ_n}(P)| \leq 24$ for any $e \in E(H)$. It follows that, for $n \geq 9$,

$$\begin{aligned} &|N_{AQ_n}(Y) \cap N_{AQ_n}(P)| \leq 12 < 2n-1 = |N_{AQ_n}(Y)|, \text{ and} \\ &|N_{AQ_n}(e) \cap N_{AQ_n}(P)| \leq 24 < 4n-8 \leq |N_{AQ_n}(e)|, \end{aligned}$$

which mean that there is neither isolated vertex nor isolated edge in $AQ_n - N_{AQ_n}(P)$, and so $\kappa_2(AQ_n) \leq 6n-17$ for $n \geq 9$.

Now we only need to prove $\kappa_2(AQ_n) \geq 6n-17$ for $n \geq 9$.

Suppose that there is a subset $S \subset V(AQ_n)$ with $|S| \leq 6n-18$ such that there is neither isolated vertex nor isolated edge in $AQ_n - S$. We want to deduce a contradiction by proving that $AQ_n - S$ is connected.

Let $AQ_n = L \odot R$, where $L = AQ_{n-1}^0$ and $R = AQ_{n-1}^1$. For convenience, let $S_L = S \cap L$ and $S_R = S \cap R$. Without loss of generality we may suppose that $|S_L| \leq |S_R|$. Then

$$|S_L| \leq (6n-18)/2 = 3n-9. \tag{3.6}$$

We prove that $AQ_n - S$ is connected by two steps. In step 1, we prove that $L - S_L$ is connected in $AQ_n - S$. And in step 2, we prove that any vertex in $R - S_R$ can be connected to some vertex in $L - S_L$.

Step 1. $L - S_L$ is connected in $AQ_n - S$.

In this case, by (3.6) and Lemma 2.5, for $n \geq 9$,

$$|S_L| \leq 3n-9 < 4n-12 = \kappa_1(L). \tag{3.7}$$

If there are no isolated vertices in $L - S_L$, then $L - S_L$ is connected by (3.7).

Suppose now that there exist isolated vertices in $L - S_L$. Note that L is $(2n-3)$-regular since $L \cong AQ_{n-1}$. By Lemma 2.7, any two vertices in L have at most four common neighbors. To isolate two vertices in L, we have to remove at least $4n-12$ vertices. By (3.7), there is exactly one isolated vertex, say X in $L - S_L$. Thus

$$|S_L| \geq |N_L(X)| = 2n-3, \tag{3.8}$$

and so, $|S_R| = |S| - |S_L| \leq 6n-18-(2n-3) = 4n-15$, that is,

$$|S_R| \leq 4n-15. \tag{3.9}$$

Let $S'_L = S_L \cup \{X\}$. Then $L - S'_L$ contains no isolated vertices. By (3.7), $|S'_L| < \kappa_1(L)$ for $n \geq 9$. Thus, $L - S'_L$ is also connected. We only need to show that X can be connected to some vertices in $L - S'_L$ via some vertices in $R - S_R$.

By Definition 2.1, X has two neighbors in R, that is, X_n and $\overline{X}_n$. By our hypothesis there are no isolated vertices in $AQ_n - S$, then there is at least one in $\{X_n, \overline{X}_n\}$ is not in S_R. Without loss of generality, assume that X_n is not in S_R. Consider two cases according as the vertex $\overline{X}_n$ is in S_R or not.

Case 1.1. $\overline{X}_n \notin S_R$.

Since X_n and $\overline{X}_n$ are adjacent in AQ_n, and $X\overline{X}_n$ is a complement edge of dimension n, by (2.2), $N_{AQ_n}(X_n) \cap N_{AQ_n}(\overline{X}_n) = \{X, \overline{X}_{n-1}, (X_n)_{n-1}, (\overline{X}_n)_{n-1}\}$, that is, $|N_R(X_n) \cap N_R(\overline{X}_n)| = 2$. By (3.9), we have

$$|N_R(X_n, \overline{X}_n)| = 2(2n-3)-2-2 = 4n-10 > 4n-15 \geq |S_R|.$$

Thus $|N_R(X_n, \overline{X}_n) \setminus S_R| \geq 5$. Let

$$Y = \{Y^i : Y^i \in N_R(X_n, \overline{X}_n) \setminus S_R\}.$$

Then $|Y| \geq 5$. If there is some vertex $Y^i \in Y$ such that at least one of Y^i_n and $\overline{Y^i}_n$ is not S_L, then we are done. So assume that both Y^i_n and $\overline{Y^i}_n$ are in S_L for any $Y^i \in Y$. Take $Y^1, Y^2 \in Y$ such that $(X_n, \overline{X}_n, Y^1)$ (or $(\overline{X}_n, X_n, Y^1)$) is a path and Y^2 is adjacent to one in $\{X_n, \overline{X}_n, Y^1\}$. By Lemma 2.9, we have that

$$|N_R(X_n, \overline{X}_n, Y^1, Y^2)| \geq 8(n-1) - 29 = 8n - 37.$$

Let

$$C = N_R(X_n, \overline{X}_n, Y^1, Y^2) \cup \{\overline{X}_n, Y^1, Y^2\}.$$

Then $|C| \geq 8n - 34$. Let $E^h_n = \{UU_n : U \in C\}$. Noting that all edges in E^h_n are hypercube edges of dimension n, we have that, for $n \geq 9$,

$$|E^h_n| = |C| \geq 8n - 34 > 6n - 18 \geq |S|,$$

which means that there exists an edge, say UU_n, in E^h_n such that neither of its two end-vertices is in S. Since X_n, $\overline{X}_n$, Y^1 and Y^2 are all not in S, X can be connected to $L - S'_L$ via vertices in $R - S_R$ and the edge UU_n.

Case 1.2. $\overline{X}_n \in S_R$.

Since there are no isolated edges in $AQ_n - S$, we have $N_R(X_n) \setminus S_R \neq \emptyset$. If there is some $U \in N_R(X_n) \setminus S_R$ such that at least one in $\{U_n, \overline{U}_n\}$ is not in S_L, then we are done. So assume that U_n and $\overline{U}_n$ are both in S_L for any $U \in N_R(X_n) \setminus S_R$. Noting that $\kappa_1(R) = 4(n-1) - 8$ and (3.9), we have that

$$|N_R(X_n, U)| \geq 4n - 12 > 4n - 15 \geq |S_R|,$$

which implies $N_R(X_n, U) \setminus S_R \neq \emptyset$. Let

$$Z = \{Z^i : Z^i \in N_R(X_n, U) \setminus S_R\}.$$

Then $|Z| \geq 2$. Take $Z^1, Z^2 \in Z$ such that (X_n, U, Z^1) (or (U, X_n, Z^1)) is a path and Z^2 is adjacent to one in $\{X_n, U, Y^1\}$. By Lemma 2.9, we have that

$$|N_R(X_n, U, Z^1, Z^2)| \geq 8(n-1) - 31 = 8n - 39.$$

Let

$$D = N_R(X_n, U, Z^1, Z^2) \cup \{X_n, U, Z^1, Z^2\}.$$

Then $|D| \geq 8n - 35$. Let $E^c_n = \{A\bar{A}_n : A \in D\}$. Noting that all edges in E^c_n are complement edges of dimension n, we have that, for $n \geq 9$,

$$|E^c_n| = |D| \geq 8n - 35 > 6n - 18.$$

Hence, there exists an edge, say $A\bar{A}_n$, in E^c_n such that neither of its two end-vertices is S. Since X_n, U, Z^1 and Z^2 are all not in S, X can be connected to $L - S'_L$ via vertices in $R - S_R$ and the edge $A\bar{A}_n$.

Step 2. Any vertex in $R - S_R$ can be connected to some vertex in $L - S_L$.

Let U be any vertex in $R - S_R$. Consider U_n and $\overline{U}_n$, which are neighbors of U in L. If at least one of U_n and $\overline{U}_n$ is not in S_L, we are done. So suppose that both U_n and $\overline{U}_n$ are in S_L. Consider the neighbor $\overline{U}_{n-1}$ of U in R. There are two cases according as $\overline{U}_{n-1}$ is in S_R or not.

Case 2.1. $\overline{U}_{n-1} \notin S_R$.

Note that $U\overline{U}_{n-1}$ is a complement edge of dimension $(n-1)$ in AQ_n and

$$(N_{AQ_n}(U) \cap N_{AQ_n}(\overline{U}_{n-1})) \cap L = \{U_n, \overline{U}_n\} \subseteq S_L.$$

Since $U\overline{U}_{n-1}$ is not an isolated edge in $AQ_n - S$, there exists a vertex $V \in N_R(U, \overline{U}_{n-1}) \backslash S_R$. Then V_n and $\overline{V}_n$ are neighbors of V in L. If at least one of V_n and $\overline{V}_n$ is not in S_L, we are done. So assume that both V_n and $\overline{V}_n$ are in S_L. By (2.4) and (2.5) in the proof of Lemma 2.8, we have that

$$|N_R(U, \overline{U}_{n-1}, V)| \geq 6(n-1) - 15 = 6n - 21. \tag{3.10}$$

Since

$$|S_L| \geq |\{U_n, \overline{U}_n, V_n, \overline{V}_n\}| = 4,$$

we have that

$$|S_R| = |S| - |S_L| \leq 6n - 18 - 4 = 6n - 22, \tag{3.11}$$

By (3.10) and (3.11), we have $|N_R(U, \overline{U}_{n-1}, V)| > |S_R|$, that is,

$$N_R(U, \overline{U}_{n-1}, V) \setminus S_R \neq \emptyset.$$

If there is some $W \in N_R(U, \overline{U}_{n-1}, V) \setminus S_R$ such that at least one of W_n and $\overline{W}_n$, which are two neighbors of W in L, is not in S_L, we are done. So assume that both W_n and $\overline{W}_n$ are in S_L for any $W \in N_R(U, \overline{U}_{n-1}, V) \setminus S_R$. By Lemma 2.9, we have that

$$|N_R(U, \overline{U}_{n-1}, V, W)| \geq 8n - 37.$$

Let

$$C' = N_R(U, \overline{U}_{n-1}, V, W) \cup \{U, \overline{U}_{n-1}, V, W\}.$$

Then $|C'| \geq 8n - 33$. Let $E'_n = \{AA_n : A \in C'\}$. Noting that all edges in E'_n are hypercube edges of dimension n, we have that, for $n \geq 9$,

$$|E'_n| = |C'| \geq 8n - 33 > 6n - 18.$$

There exists at least one edge, say AA_n, of E'_n whose two end-vertices both are not in S. Since U, $\overline{U}_{n-1}$, V and W are all not in S, this implies that U can be connected to $L - S_L$.

Case 2.2. $\overline{U}_{n-1} \in S_R$.

Since U is not an isolated vertex in $AQ_n - S$ and two neighbors U_n and $\overline{U}_n$ of U in L are both in S, there exists a vertex $B \in N_R(U) \setminus S_R$. Then B_n and $\overline{B}_n$ are neighbors of B in L. If at least one of B_n and $\overline{B}_n$ is not in S_L, we are done. So assume that both B_n and $\overline{B}_n$ are in S_L. If $\overline{B}_{n-1} \notin S_R$, we can obtain a path joining B to some vertex in $L - S_L$ by Case 2.1 by replacing U by B. Therefore assume $\overline{B}_{n-1} \in S_R$ below.

Since UB is not an isolated edge in $AQ_n - S$, there exists a vertex $F \in N_R(U, B) \setminus S_R$. Then F_n and $\overline{F}_n$ are two neighbors of F in L. If at least one of F_n and $\overline{F}_n$ is not in S_L, we are done. So suppose that both F_n and $\overline{F}_n$ are in S_L. By Lemma 2.8, we have that

$$|N_R(U, B, F)| \geq 6(n-1) - 17 = 6n - 23. \tag{3.12}$$

Since

$$|S_L| \geq |\{U_n, \overline{U}_n, B_n, \overline{B}_n, F_n, \overline{F}_n\}| = 6,$$

we have that

$$|S_R| = |S| - |S_L| \leq 6n - 18 - 6 = 6n - 24. \tag{3.13}$$

Comparing (3.12) with (3.13), we have that $N_R(U, B, F) \setminus S_R \neq \emptyset$. Let $Q \in N_R(U, B, F) \setminus S_R$. By Lemma 2.9, we have that

$$|N_R(U, B, F, Q)| \geq 8n - 39.$$

Let

$$C'' = N_R(U, B, F, Q) \cup \{U, B, F, Q\}.$$

Then $|C''| \geq 8n - 35$. Let $E_n'' = \{AA_n : A \in C''\}$. Noting that all edges in E_n'' are hypercube edges of dimension n, we have, for $n \geq 9$,

$$|E_n''| = |C''| \geq 8n - 35 > 6n - 18.$$

There exists an edge, say AA_n, of E_n'' whose two end-vertices both are not in S. Since U, B, F and Q are all not in S, thus U can be connected to $L - S_L$.

The proof of the theorem is complete. □

Theorem 3.2. $\lambda_2(AQ_n) = 6n - 9$ *for* $n \geq 4$.

Proof. Let C_3 be a cycle of length three in AQ_n, U be any vertex not in C_3, and let e be any edge e not incident with any vertex in C_3. Obviously, any vertex not in C_3 can have at most 3 neighbors in C_3. Thus, for $n \geq 4$,

$$|E_{AQ_n}(U) \cap E_{AQ_n}(C_3)| \leq 3 < 2n - 1 = |E_{AQ_n}(U)|,$$

and

$$|E_{AQ_n}(e) \cap E_{AQ_n}(C_3)| \leq 6 < 4n - 4 = |E_{AQ_n}(e)|.$$

So, there are no isolated vertices or isolated edges in $AQ_n - E_{AQ_n}(C_3)$. That is, $E_{AQ_n}(C_3)$ is a 2-edge-cut of G. It follows that, for $n \geq 4$,

$$\lambda_2(AQ_n) \leq E_{AQ_n}(C_3) = 6n - 9.$$

In the following, we only need to prove that $\lambda_2(AQ_n) \geq 6n-9$ for $n \geq 4$.

Let F be an arbitrary 2-edge-cut in AQ_n with $|F| \leq 6n-10$ such that there are neither isolated vertices nor isolated edges in $AQ_n - F$. Let $F_L = F \cap L$ and $F_R = F \cap R$. Without loss of generality we may suppose that $|F_L| \leq |F_R|$. Then

$$|F_L| \leq \frac{1}{2}(6n-10) = 3n-5.$$

We will deduce a contradiction by proving that $AQ_n - F$ is connected by two steps. In step 1, we show that $L - F_L$ is connected in $AQ_n - F$. In step 2, we show that any vertex of R can be connected to L in $AQ_n - F$.

Step 1. $L - F_L$ is connected in $AQ_n - F$.

By our hypothesis and Lemma 2.5, for $n \geq 4$, we have that

$$|F_L| \leq 3n-5 < 4(n-1)-4 = \lambda_1(L). \tag{3.14}$$

Thus, if there are no isolated vertices in $L - F_L$, then $L - F_L$ is connected, and so we are done. In the following discussion, we assume that there exists an isolated vertex X in $L - F_L$.

Since L is $(2n-3)$-regular and any two vertices are incident with at most one edge, to get two isolated vertices in L, we have to remove at least $4n-7$ edges from L. However, by (3.14), $|F_L| \leq 3n-5 < 4n-7$ for $n \geq 4$. This shows that there is just one isolated vertex X in $L - F_L$. Then by Lemma 2.4, we have

$$\lambda(L-X) \geq \kappa(L-X) \geq \kappa(L)-1 = \begin{cases} 4-1=3 & \text{if } n=4 \\ 2n-3-1=2n-4 & \text{if } n>4 \end{cases}$$

and

$$|F_L| - |E_L(X)| \leq 3n-5-(2n-3) = n-2,$$

which implies that $|F_L| - |E_L(X)| < \lambda(L-X)$ for $n \geq 4$. In other words, the subgraph $H = (L-X) - F_L = (L-F_L) - X$ is connected. In the following we only need to prove that X can be connected to H in $AQ_n - F$. Since $AQ_n - F$ contains no isolated vertices, at least one of two edges XX_n and $X\overline{X}_n$ is not in F. Without loss of generality, we may assume that XX_n is not F. Consider two cases according as the edge $X\overline{X}_n$ is in F or not.

Case 1.1. $X\overline{X}_n \notin F$.

Note that $X_n\overline{X}_{n-1}$ and $\overline{X}_n\overline{X}_{n-1}$ are edges in AQ_n, where $\overline{X}_{n-1}$ is in L. If at least one of the two edges is not in F, we are done. So, we can assume that both the two edges are in F. We will construct $4n-8$ edge disjoint paths joining X to some vertex in $L - X$ in the following.

Let

$$\begin{aligned} E_1 &= \{X_nX^i:\ X_nX^i \in E_R(X_n)\backslash\{X_n\overline{X}_n\}\}, \\ E_2 &= \{\overline{X}_nY^j:\ \overline{X}_nY^j \in E_R(\overline{X}_n)\backslash\{X_n\overline{X}_n\}\}, \\ F' &= F \setminus (E_L(X) \cup \{X_n\overline{X}_{n-1}, \overline{X}_n\overline{X}_{n-1}\}). \end{aligned}$$

Then $|E_1| = |E_2| = 2n-4$, $E_1 \cap E_2 = \emptyset$, and

$$|F'| \le 6n - 10 - (2n-3) - 2 = 4n - 9. \tag{3.15}$$

Let

$$P_i = (X, X_n, X^i, X_n^i) \text{ and } Q_j = (X, \overline{X}_n, Y^j, \overline{Y^j}_n) \tag{3.16}$$

be a path joining X to some vertex in $L - X$, and let

$$\mathscr{P} = \{P_i : 1 \le i \le 2n-4\} \cup \{Q_j : 1 \le j \le 2n-4\}.$$

Then

$$|\mathscr{P}| = 4n - 8. \tag{3.17}$$

Since these paths defined in (3.16) are edges disjoint, comparing (3.15) and (3.17), we have that there exists a path $P \in \mathscr{P}$ such that $E(P) \cap F' = \emptyset$. Then X can be connected to a vertex in L.

Case 1.2. $X\overline{X}_n \in F$.

If $X_n\overline{X}_{n-1} \notin F$, we are done. So assume that $X_n\overline{X}_{n-1} \in F$. Since there are no isolated edges in $AQ_n - F$, we have $E_R(X_n) - F_R \neq \emptyset$.

If $X_n\overline{X}_n \notin F_R$, we can obtain a path joining X via a path to a vertex in $L - X$ by the $4n-8$ paths $\mathscr{P}$ constructed in Case 1.1. Hence, we can assume $X_n\overline{X}_n \in F_R$ below. We will construct $4n-9$ edge disjoint paths joining X to some vertex in $L - X$ in the following.

Let $X_nW \in E_R(X_n) - F_R$. Then $W \neq \overline{X}_n$.

Let

$$\begin{aligned}
E_1' &= \{WX^i : \ WX^i \in E_R(W) \backslash \{X_nW\}\},\\
E_2' &= \{X_nY^j : \ X_nY^j \in E_R(X_n) \backslash \{X_nW, X_n\overline{X}_n\}\},\\
F^* &= F \setminus (E_L(X) \cup \{X\overline{X}_n, X_n\overline{X}_{n-1}, X_n\overline{X}_n\}).
\end{aligned}$$

Then $|E_1'| = 2n-4$, $|E_2'| = 2n-5$ and

$$|F^*| \le 6n - 10 - (2n-3) - 3 = 4n - 10. \tag{3.18}$$

Let

$$P_i^* = (X, X_n, W, X^i, X_n^i) \text{ and } Q_j^* = (X, X_n, Y^i, \overline{Y^j}_n)) \tag{3.19}$$

be a path joining X to some vertex in $L - X$, and let

$$\mathscr{P}^* = \{P_i^* : 1 \le i \le 2n-4\} \cup \{Q_j^* : 1 \le j \le 2n-5\}.$$

Then

$$|\mathscr{P}^*| = 4n - 9. \tag{3.20}$$

Since these paths defined in (3.19) are edges disjoint, comparing (3.18) and (3.20), there exists a path $P^* \in \mathscr{P}^*$ such that $E(P^*) \cap F^* = \emptyset$. This implies that vertex X can be connected to a vertex in $L - X$.

Step 2. Any vertex X of R can be connected to L in $AQ_n - F$.

Suppose that X is an arbitrary vertex in R, if $\{XX_n, X\overline{X}_n\} \nsubseteq F$, where X_n and $\overline{X}_n$ are both in L, we are done. Thus, assume that $\{XX_n, X\overline{X}_n\} \subseteq F$. Since there is neither isolated vertex nor isolated edge in $AQ_n - F$, the vertex X lies on a path T of length 2 in $R - F_R$. Assume $V(T) = \{X, Y, Z\}$. If $\{YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n\} \nsubseteq F$, we are done. Hence, assume that $\{YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n\} \subseteq F$ in the following.

Let $F^* = F \setminus \{XX_n, X\overline{X}_n, YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n\}$.

Then

$$|F^*| \leq 6n - 10 - 6 = 6n - 16. \tag{3.21}$$

Note that

$$|E_R(T)| \geq 3 * (2n - 5) = 6n - 15. \tag{3.22}$$

By Lemma 2.8, we have $|N_R(X) \cap N_R(Y) \cap N_R(Z)| \leq 1$. We will construct edge disjoint paths joining $V(T)$ to $V(L)$ according to the following two cases.

Case 2.1 $|N_R(X) \cap N_R(Y) \cap N_R(Z)| = 0$.

In this case, each vertex in $N_R(T)$ is incident to at most two edges in $E_R(T)$. Since every vertex is incident to exactly two crossing edges in AQ_n, we can construct $|E_R(T)|$ edge disjoint paths $\mathscr{P}$ joining $V(T)$ to $V(L)$ as follows.

For any vertex W^i in $N_R(T)$, if W^i is incident to exactly one edge AW^i in $E_R(T)$ where $A \in V(T)$, let $P_i = (A, W^i, W^i_n)$; if W^i is incident to exactly two edges AW^i and BW^i in $E_R(T)$ where $A, B \in V(T)$, let $P_i = (A, W^i, W^i_n)$ and $P'_i = (B, W^i, \overline{W^i}_n)$.

Comparing (3.21) and (3.22), there exists a path $P \in \mathscr{P}$ such that $E(P) \cap F^* = \emptyset$. This implies that vertex X can be connected to a vertex in L.

Case 2.2 $|N_R(X) \cap N_R(Y) \cap N_R(Z)| = 1$.

In this case, there is a vertex U in $N_R(T)$ incident to exactly three edges in $E_R(T)$ and other vertices in $N_R(T)$ is incident to at most two edges in $E_R(T)$.

Since $|F_R| \leq |F^*|$, comparing (3.21) and (3.22), there exists an edge $e \in E_R(T) \setminus F_R$. Without loss of generality, we may assume $e = XW$. If $\{WW_n, W\overline{W}_n\} \nsubseteq F$, where W_n and $\overline{W}_n$ are both in L, we are done. Thus, assume that $\{WW_n, W\overline{W}_n\} \subseteq F$.

Let $F' = F \setminus \{XX_n, X\overline{X}_n, YY_n, Y\overline{Y}_n, ZZ_n, Z\overline{Z}_n, WW_n, W\overline{W}_n\}$.

Then

$$|F'| \leq 6n - 10 - 8 = 6n - 18. \tag{3.23}$$

Since every vertex is incident to exactly two crossing edges in AQ_n, we can construct edge disjoint paths $\mathscr{P}$ joining $V(T)$ to $V(L)$ as follows.

For any vertex W^i in $N_R(T) \setminus \{W\}$, if W^i is incident to exactly one edge AW^i in $E_R(T)$ where $A \in V(T)$, let $P_i = (A, W^i, W^i_n)$; if W^i is incident to exactly two edges AW^i and

BW^i in $E_R(T)$ where $A, B \in V(T)$, let $P_i = (A, W^i, W^i_n)$ and $P'_i = (B, W^i, \overline{W^i}_n)$; if $W^i = U$, let $P_i = (Y, U, U_n)$ and $P'_i = (Z, U, \overline{U}_n)$.

Case 2.2.1 The vertex W is incident to one edge in $E_R(T)$. Hence, we construct at least $|E_R(T)| - 2 \geq 6n - 17$ edge disjoint paths $\mathscr{P}$ jointing $V(T)$ to $V(L)$. By (3.23), there exists a path $P \in \mathscr{P}$ such that $E(P) \cap F' = \emptyset$. This implies that vertex X can be connected to a vertex in L.

Case 2.2.2 The vertex W is incident to at least two edges in $E_R(T)$. Hence, we construct at least $|E_R(T)| - 3 \geq 6n - 18$ edge disjoint paths $\mathscr{P}$ jointing $V(T)$ to $V(L)$. If there exists a path $P \in \mathscr{P}$ such that $E(P) \cap F' = \emptyset$, we are done. Assume $E(P) \cap F' \neq \emptyset$ for every path in $\mathscr{P}$. By (3.23), the faulty edges F' are all in $\mathscr{P}$. Let $P = (X, W, W_j, (W_j)_n)$, where W_j is not in T and $WW_j \nsubseteq F$. Then, P is fault-free. This implies that vertex X can be connected to a vertex in L.

We proved that $AQ_n - F$ is connected, which means $\lambda_2(AQ_n) \geq 6n - 9$ for $n \geq 4$.

The theorem follows. □

4. Conclusions

In this paper, we explore two stronger measurement parameters for the reliability and the tolerance of networks called the 2-extra connectivity $\kappa_2(G)$ and the 2-extra edge-connectivity $\lambda_2(G)$ of a connected graph G, which not only compensate for some shortcomings but also generalize the classical connectivity $\kappa(G)$ and the classical edge-connectivity $\lambda(G)$, and so can provide more accurate measures for the reliability and the tolerance of a large-scale parallel processing system. The augmented cube AQ_n, as an important variant of the hypercube Q_n, has many desirable properties (for more results, see, for example, [3–5, 7–10, 14, 17, 18, 20–22, 24, 27, 28]). Here, we have showed that $\kappa_2(AQ_n) = 6n - 17$ for $n \geq 9$; and $\lambda_2(AQ_n) = 6n - 9$ for $n \geq 4$. In other words, for $n \geq 9$ (respectively, $n \geq 4$), at least $6n - 17$ vertices (respectively, $6n - 9$ edges) of AQ_n have to be removed to get a disconnected graph that contains no isolated vertices and isolated edges. Compared with previous results, our results enhance the fault tolerant ability of this kind of network theoretically.

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