Fault diagnosability of arrangement graphs

Shuming Zhou [a,b,*], Jun-Ming Xu [c]

[a] School of Mathematics and Computer Science, Fujian Normal University, Fuzhou, Fujian 350007, PR China
[b] Key Laboratory of Network Security and Cryptology, Fujian Normal University, Fuzhou, Fujian 350007, PR China
[c] School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China

ARTICLE INFO

ABSTRACT

The growing size of the multiprocessor system increases its vulnerability to component failures. It is crucial to locate and to replace the faulty processors to maintain a system's high reliability. The fault diagnosis is the process of identifying faulty processors in a system through testing. In this paper, we show that the largest connected component of the survival graph contains almost all of the remaining vertices in the (n,k)-arrangement graph $A_{n,k}$ when the number of moved faulty vertices is up to twice or three times the traditional connectivity. Using this fault resiliency, we establish the conditional diagnosability of $A_{n,k}$ under the comparison model, and prove that the conditional diagnosability of $A_{n,k}$ is $(3k-2)(n-k)-3$ for $k \geqslant 4$, $n \geqslant k+2$; and the conditional diagnosability of $A_{n,n-1}$ is $3n-7$ for $n \geqslant 5$.

© 2013 Published by Elsevier Inc.

1. Introduction

Distributed processor architectures offer the potential advantage of high speed, provided that they are highly fault-tolerant and reliable, and have good communication between remote processors. An important component of such a distributed system is its network topology, which defines the inter-processor communication architecture. Fault-tolerance is especially important for interconnection networks, since computers may fail, creating faults in the network. To be reliable, the rest of the network should stay connected. Obviously, this can only be guaranteed if the number of faults is smaller than the minimum degree in the network. When the number of faults is larger than the minimum degree, some extensions of connectivity are necessary, since the graph may become disconnected. Some generalizations of connectivity were introduced and examined for various classes of graphs in [6], including super connectedness and tightly super connectedness, where only one singleton can appear in the remaining network, and restricted connectivity and super connectivity, where a remaining component must have a certain minimum size. As the faults increase in the graph, it is desirable that the largest component of the surviving network stays connected, with a few processors separated from the rest, since then the network will continue to be able to function. Many interconnection networks have been examined in this aspect, when the number of faults is roughly twice the minimum degree, see [8,22]. One can even go further and ask what happens when more vertices are deleted. This has been examined for the hypercube in [33–35] and for certain Cayley graphs generated by transpositions in [9], and it has been shown that the surviving network has a large component containing almost all vertices.

The process of identifying faulty processors in a system by analyzing the outcomes of available inter-processor tests is called system-level diagnosis. In 1967, Preparata et al. [28] established a foundation of system diagnosis and an original diagnostic model, called the PMC model. Its target is to identify the exact set of all faulty vertices before their repair or

replacement. All tests are performed between two adjacent processors, and it was assumed that a test result is reliable (respectively, unreliable) if the processor that initiates the test is fault-free (respectively, faulty). Recently, Fan et al. [18] have derived the diagnosability of DCC linear congruential graphs under the precise and pessimistic strategies based on the PMC diagnostic model. Zhu et al. [39] determined the conditional diagnosability of folded hypercubes under the PMC model. The comparison-based diagnosis models, first proposed by Malek [27] and Chwa and Hakimi [13], have been considered to be a practical approach for fault diagnosis in the multiprocessor systems. In these models, the same job is assigned to a pair of processors in the system and their outputs are compared by a central observer. This central observer performs diagnosis using the outcomes of these comparisons. Maeng and Malek [26] extended Malek's comparison approach to allow the comparisons carried out by the processors themselves. Sengupta and Dahbura[29] developed this comparison approach such that the comparisons have no central unit involved.

Lin et al. [24] introduced the conditional diagnosis under the comparison model. By evaluating the size of connected components, they obtained that the conditional diagnosability of the star graph S_n under the comparison model is $3n - 7$, which is about three times larger than that the classical diagnosability of star graphs. In the same method, Hsu et al. [20] have recently proved that the conditional diagnosability of the hypercube Q_n is $3n - 5$. This idea was attributed to Lai et al. [23] who are the first to use a conditional diagnosis strategy. They obtained that the conditional diagnosability of the hypercube Q_n is $4n - 7$ under the PMC model. Furthermore, Hsu et al. [20] exposed the difference between these two conditional diagnosis models.

The arrangement graph, proposed as a generalization of the star graph in an attempt to solve the scalability problem of the star graph topology, while preserving its attractive features, has been extensively studied [2,4,5,12,14–16,19,21,25,30]. Based on the fault tolerance of the arrangement graph, in this paper, we establish its conditional diagnosability under the comparison diagnosis model. The rest of this paper is organized as follows. Section 2 introduces some definitions, notations and the structure of the arrangement graph. Section 3 is devoted to the fault resiliency of $A_{n,k}$, and Section 4 concentrates on the conditional diagnosability of the arrangement graph. Section 5 concludes the paper.

2. Arrangement graphs

For notation and terminology not defined here we follow [31]. Specifically, we use a graph $G = G(V, E)$ to represent an interconnection network, where a vertex $u \in V$ represents a processor and an edge $(u, v) \in E$ represents a link between vertices u and v. If at least one end of an edge is faulty, the edge is said to be faulty; otherwise, the edge is said to be fault-free. Let S be a subset of $V(G)$. The subgraph of G induced by S, denoted by $G[S]$, is the graph with the vertex-set S and the edge-set $\{(u, v) \mid (u, v) \in E(G), u, v \in S\}$. For a vertex u in G, $N(u)$ denotes the set of all neighbors of u, i.e., $N(u) = \{v \mid (u, v) \in E\}$. Let S be a subgraph of G or a subset of $V(G)$, and let $N(S) = \cup_{u \in S} N(u) \backslash S$. We use K_n to denote the complete graph of order n, and $d(u, v)$ to denote the distance between u and v, the length of a shortest path between u and v in G. The diameter of G is defined as the maximum distance between any two vertices in G.

For any subset $F \subset V$, the notation $G - F$ denotes a graph obtained by removing all vertices in F from G and deleting those edges with at least one end-vertex in F, simultaneously. If $G - F$ is disconnected, F is called a *separating set*. A separating set F is called a *k-separating set* if $|F| = k$. The maximal connected subgraphs of $G - F$ are called *components*. The *connectivity* $\kappa(G)$ of G is defined as the minimum k for which G has a k-separating set; otherwise $\kappa(G)$ is defined as $n - 1$ if $G = K_n$. A graph G is called to be *k-connected* if $\kappa(G) \geqslant k$. A k-separating set is called to be *minimum* if $k = \kappa(G)$.

The interconnection network has been an important research area for parallel and distributed computer systems. Network reliability is one of the major factors in designing the topology of an interconnection network. The well-known hypercube is the first major class of interconnection networks.

As another topology of an interconnection network, Akers and Krishnamurthy [1] proposed the star graph S_n, which has smaller degree, diameter, and average distance than the comparable hypercube, while reserving symmetry properties and desirable fault-tolerant characteristics. As a result, the star graph has been recognized as an alternative to the hypercube. However, the star graph is less flexible in adjusting its sizes. With the restriction on the number of vertices, there is a large gap between $n!$ and $(n + 1)!$ for expanding an S_n to S_{n+1}. To relax the restriction of the numbers of vertices $n!$ in S_n, The arrangement graph was proposed by Day and Tripathi [15] as a generalization of the star graph S_n. It is more flexible in its size than S_n.

Definition 2.1. Given two positive integers n and k with $n > k$, let $\langle n \rangle$ denote the set $\{1, 2, \ldots, n\}$, and let $P_{n,k}$ be a set of arrangements of k elements in $\langle n \rangle$. The (n, k)-*arrangement graph*, denoted by $A_{n,k}$, has vertex-set $V(A_{n,k}) = P_{n,k}$ and edge-set $E(A_{n,k}) = \{(p, q) \mid p \text{ and } q \text{ differ in exactly one position}\}$.

The graph shown in Fig. 1 is a $(4, 2)$-arrangement graph $A_{4,2}$.

Clearly, $A_{n,k}$ is a $k(n - k)$-regular graph with $\frac{n!}{(n-k)!}$ vertices. It was showed by Day and Tripathi [15] that $A_{n,k}$ is vertex-symmetric and edge-symmetric and has the diameter of $\lfloor \frac{3k}{2} \rfloor$. Day and Tripathi [14] showed the connectivity $\kappa(A_{n,k}) = k(n - k)$.

Moreover, $A_{n,1}$ is isomorphic to the complete graph K_n, and $A_{n,n-1}$ is isomorphic to the n-dimensional star graph S_n. Chiang and Chen [12] showed that $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n.

For two distinct i and j in $\langle n \rangle$, let $V_{n,k}^{j:i}$ be the set of all vertices in $A_{n,k}$ with the jth position being i, that is,
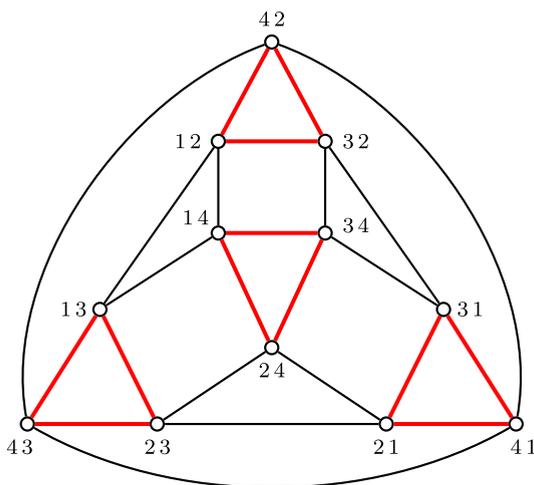
Fig. 1. The structure of $A_{4,2}$.

$$V_{n,k}^{j:i} = \{p|\ p = p_1 \cdots p_j \cdots p_k \in P_{n,k} \text{ and } p_j = i\}.$$

For a fixed position $j \in \langle n \rangle$, $\left\{V_{n,k}^{j:i}|\ 1 \leqslant i \leqslant n\right\}$ forms a partition of $V(A_{n,k})$. Let $A_{n,k}^{j:i}$ denote the subgraph of $A_{n,k}$ induced by $V_{n,k}^{j:i}$. Then for each $j \in \langle n \rangle$, $A_{n,k}^{j:i}$ is isomorphic to $A_{n-1,k-1}$. For example, a partition of $A_{4,2}$ is shown in Fig. 1, where red triangles are $A_{4,2}^{2:i}$'s with $i \in \langle 4 \rangle$, isomorphic to $A_{3,1} = K_3$.

Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. It is easy to check that each $A_{n,k}^{j:i}$ is a subgraph of $A_{n,k}$, and we say that $A_{n,k}$ is decomposed into n subgraphs $A_{n,k}^{j:i}$'s according to the jth position. For simplicity, by the symmetry of $A_{n,k}$ we shall take j as the last position k, and use $A_{n,k}^i$ to denote $A_{n,k}^{k:i}$.

Let $E(i,j)$ be the set of edges between $A_{n,k}^i$ and $A_{n,k}^j$, that is,

$$E(i,j) = \left\{(p,q) \in E(A_{n,k})|\ p \in V\left(A_{n,k}^i\right) \text{ and } q \in V\left(A_{n,k}^j\right)\right\}.$$

Clearly, $E(i,j)$ is a perfect matching (a set of edges in which any two edges have no common end-vertex) between $A_{n,k}^i$ and $A_{n,k}^j$, and

$$|E(i,j)| = \frac{(n-2)!}{(n-k-1)!}. \tag{2.1}$$

Let I be a subset of $\langle n \rangle$, and let H be a subset of $V\left(A_{n,k}^I\right)$ or a subgraph of $A_{n,k}^I$, where $A_{n,k}^I = \left\{A_{n,k}^i:\ i \in I\right\}$. Use $N^I(H)$ to denote the set of neighbors of H in $A_{n,k}^I$. Particularly, use $N^{\bar{i}}(H)$ and $N^i(H)$ as an abbreviation of $N^{\langle n \rangle \setminus i}(H)$ and $N^i(H)$, respectively, and call vertices in $N^{\bar{i}}(H)$ and $N^i(H)$ the *outer neighbors* and *inner neighbors* of H, respectively. Obviously, every vertex u of $A_{n,k}^i$ has $n - k$ outer neighbors, and two arbitrary outer neighbors of u are distributed in distinct subgraphs. We write u for $\{u\}$. It follows from the definitions that, for every $i \in \langle n \rangle$,

$$\begin{aligned}
&|N^i(u)| = (k-1)(n-k) \text{ and } |N^{\bar{i}}(u)| = n - k, \\
&N^{\bar{i}}(u) \cap N^{\bar{i}}(v) = \emptyset \text{ if } u, v \in A_{n,k}^i \text{ and } u \neq v,
\end{aligned} \tag{2.2}$$

and for any two distinct vertices $x \in A_{n,k}^i$ and $y \in A_{n,k}^j$ with $i \neq j$, and $I = \{i,j\}$,

$$|N^{\bar{I}}(x) \cap N^{\bar{I}}(y)| = 0 \text{ if } x \text{ and } y \text{ are not adjacent}. \tag{2.3}$$

We say that one vertex u is adjacent to some subgraph $A_{n,k}^j$ if u has an outer neighbor in $A_{n,k}^j$. Let

$$V_i = \{u_1 u_2 \cdots u_{i-1} x u_{i+1} \cdots u_k|\ x \in \langle n \rangle \setminus \{u_1, u_2, \cdots, u_{i-1}, u_{i+1}, \cdots, u_k\}\}$$

Then, when $n \geqslant k + 2$, the graph induced by V_i is a complete graph of order $n - k + 1$ and a subgraph of $A_{n,k}^{u_k}$, which implies that any two adjacent vertices have exactly $(n - k - 1)$ common neighbors. Thus, by the edge-transitivity of $A_{n,k}$, for any edge e,

$$|N(e)| = 2k(n-k) - (n-k-1) - 2 = (2k-1)(n-k) - 1. \tag{2.4}$$

In addition, the following property of $A_{n,k}$ is useful, which can be checked by the definition of $A_{n,k}$. For any two distinct vertices u and v in $A_{n,k}$,

$$|N(u) \cap N(v)| = \begin{cases} 0, & \text{if } d(u,v) \geqslant 3; \\ 2, & \text{if } d(u,v) = 2 \text{ and } n \geqslant k+2; \\ 1, & \text{if } d(u,v) = 2 \text{ and } n = k+1; \\ n-k-1, & \text{if } d(u,v) = 1. \end{cases} \tag{2.5}$$

Other properties of the arrangement graph has received considerable attention in the literature. First, Day and Tripathi [16] showed the existence of pancyclicity, that is $A_{n,k}$ contains cycles of all lengths. Hsieh et al. [19] investigated the existence of hamiltonian cycle in $A_{n,k}$ with faulty vertices, Lo and Chen [25] studied hamiltonian connectedness of $A_{n,k}$ with faulty edges. Hsu et al. [21] further obtained an optimal result that the graph $A_{n,k}$ ($n \geqslant k+2$) is $(k(n-k)-2)$-hamiltonian and $(k(n-k)-3)$-hamiltonian connected in $G-F$ for any $F \subset V(G) \cup E(G)$ with $|F| \leqslant f$. Teng et al. [30] have recently shown that $A_{n,k}$ is panpositionable hamiltonian and panconnected if $k > 1$ and $n \geqslant k+2$. In addition, Bai et al. [2] proposed a distributed algorithm with optimal time complexity and without message redundancy for one-to-all broadcasting in one-port communication model on the fault-free arrangement graphs, and also developed a fault tolerant broadcasting algorithm with less than $k(n-k)$ faulty edges. Chen et al. [4,5] presented efficient one/all-to-all broadcasting algorithms on the arrangement graphs by constructing $n-k$ spanning trees, where the height of each tree is $2n-1$.

3. Fault tolerance of the arrangement graph

The connectivity $\kappa(G)$ of a graph G is an important parameter to measure the fault tolerance of the network, while it has an obvious deficiency in that it tacitly assume that all elements in any subset of G can potentially fail at the same time. To compensate for this shortcoming, it would seem natural to generalize the classical connectivity by introducing some conditions or restrictions on the separating set S and/or the components of $G-S$.

The connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal results in a disconnected or a trivial (one vertex) graph. A k-regular k-connected graph is *super k-connected* if any one of its minimum separating sets is a set of the neighbors of some vertex. If, in addition, the deletion of a minimum separating set results in a graph with two components (one of which has only one vertex), then the graph is *tightly super k-connected*. For example, the complete bipartite graph $K_{n,n}$ is n-super connected but not tightly n-super connected. The notions of super connectedness and tightly super connectedness were first introduced in [3,6], respectively.

Esfahanian [17] first introduced the concepts of the restricted separating set and the restricted connectivity of a graph G. A set S of vertices is a *restricted separating set* if $G-S$ is disconnected and $N(x)$ is not completely contained in S for any vertex x in G. The *restricted connectivity* of G, denoted by $\kappa_r(G)$, is the minimum cardinality of a restricted vertex-cut.

Considering it is not easy to examine whether a separating set is restricted, Xu et al. [32] formally proposed the super connectivity, a weaker concept than the restricted connectivity. A separating set S of G is *super* if $G-S$ contains no isolated vertices. The *super connectivity* of G, denoted by $\kappa_s(G)$, is the minimum cardinality of a super separating set. Clearly, $\kappa(G) \leqslant \kappa_s(G) \leqslant \kappa_r(G)$ if $\kappa_r(G)$ exists.

It follows from definitions that the restricted connectivity or super connectivity can provide a more accurate measurement than the connectivity for fault tolerance of a large-scale interconnection network.

Usually, if the surviving graph $G-S$ contains a large connected component C when $G-S$ is not connected, the component C may be used as the functional subsystem, without incurring severe performance degradation. Thus, in evaluating a distributed system, it is indispensable to estimate the size of the maximal connected components of the underlying graph when the structure begins to lose processors.

Yang et al. [33–35] proved that the hypercube Q_n with f faulty processors has a component of size $2^n - f - 1$ if $f \leqslant 2n-3$, and size $2^n - f - 2$ if $f \leqslant 3n-6$. Yang et al. [36,37] also obtained that a similar result for the star graph S_n. Cheng et al. [7,10] gave a more detail result for S_n. The removal of any separating set of at most $2n-4$ vertices from S_n results in exact two components, one of them is a single vertex or edge. Cheng and Lipták [9] generalized this result for S_n with linearly many faults. Cheng et al. [11] presented a similar result for the 2-tree-generated networks with linearly many faults. In this section, we detail on the fault resilience of the arrangement graph $A_{n,k}$.

Throughout this paper, the notation F denotes a set of vertices in $A_{n,k}$. If F is regarded as a set of faulty vertices, then a subgraph H of $A_{n,k}$ is called *fault-free* if $V(H) \cap F = \emptyset$. Let

$$F_i = A_{n,k}^i \cap F \text{ and } f_i = |F_i| \text{ for } 1 \leqslant i \leqslant n. \tag{3.1}$$

We first discuss the tightly super connectedness. Since $A_{n,1}$ is isomorphic to a complete graph K_n, it is super connected but not tightly super connected. When $n = 4$, it is easy to check that $A_{4,2}$ is not tightly super connected since it has a separating set F with $|F| = 4$ such that two components of $A_{4,2} - F$ are both 4-cycles (see Fig. 1). Thus, in the following discussion, we assume $k \geqslant 3$.

Theorem 3.1. *For $k \geqslant 3$, $A_{n,k}$ is tightly super $k(n-k)$-connected.*

Proof. Let F be a minimum separating set in $A_{n,k}$. Then, using the notations defined in (3.1), we have that

$$|F| = \sum_{i=1}^{n} f_i = \kappa(A_{n,k}) = k(n-k).$$

By the definition of tightly super connectivity, we need to show that $A_{n,k} - F$ has exactly two components, one of them is a single vertex. We gain our ends by proving the following claims.   □

Claim 3.1.1. $f_i \geqslant (k-1)(n-k)$ for some $i \in \langle n \rangle$.

Proof. Suppose to the contrary that $f_i < (k-1)(n-k)$ for any $i \in \langle n \rangle$. Then $A_{n,k}^i - F_i$ is connected since $A_{n,k}^i$ is $(k-1)(n-k)$-connected. We will deduce a contradiction by showing that $A_{n,k} - F$ is connected. To this end, we only need to show that $A_{n,k}^i$ and $A_{n,k}^j$ can be connected in $A_{n,k} - F$ for any two distinct $i,j \in \langle n \rangle$.

In fact, by (2.1), when either $k \geqslant 4$ or $k = 3$ and $n \geqslant 6$, we have

$$|E(i,j)| = \frac{(n-2)!}{(n-k-1)!} > k(n-k) = |F|,$$

which implies that there exists a fault-free edge $e \in E(i,j)$. It follows that $A_{n,k}^i$ and $A_{n,k}^j$ can be connected in $A_{n,k} - F$ by the fault-free edge e.

When $k = 3$ and $n \in \{4,5\}$, we have

$$|E(i,j)| = \frac{(n-2)!}{(n-k-1)!} = \begin{cases} 2 < 3 = |F| & \text{if } n = 4; \\ 6 = |F| & \text{if } n = 5. \end{cases} \tag{3.2}$$

Without loss of generality, assume that there are no fault-free edges in $E(i,j)$ (otherwise $A_{n,3}^i$ and $A_{n,3}^j$ can be connected in $A_{n,3} - F$ by some fault-free edge in $E(i,j)$). By (3.2), there exist a fault-free edge e_1 in $E(i,x)$ and a fault-free edge e_2 in $E(x,j)$ for any $x \notin \{i,j\}$. Thus, $A_{n,3}^i$ and $A_{n,3}^j$ can be connected in $A_{n,3} - F$ by $A_{n,3}^x$ and the fault-free edges e_1 and e_2.   □

Claim 3.1.2. If there is some $i \in \langle n \rangle$ such that $|F - F_i| < (k-1)(n-k)$, then $A_{n,k} - \left(V\left(A_{n,k}^i\right) \cup (F - F_i) \right)$ is connected.

Proof. By the hypothesis, for any $j \in \langle n \rangle$ with $j \neq i$, we have $f_j < (k-1)(n-k)$, which implies that $A_{n,k}^j$ is connected since $A_{n,k}^j$ is $(k-1)(n-k)$-connected. Since for any two distinct $j,t \in \langle n \rangle \backslash \{i\}$,

$$|E(j,t)| = \frac{(n-2)!}{(n-k-1)!} > (k-1)(n-k) > |F - F_i|,$$

there exists a fault-free edge e in $E(j,t)$. Thus $A_{n,k}^j$ and $A_{n,k}^t$ can be connected in $A_{n,k} - F$ by the fault-free edge e. By the arbitrariness of j and t, $A_{n,k} - \left(V\left(A_{n,k}^i\right) \cup (F - F_i) \right)$ is connected.   □

Claim 3.1.3. $f_i \leqslant (k-1)(n-k)$ for any $i \in \langle n \rangle$.

Proof. If there is some $i \in \langle n \rangle$ such that $f_i > (k-1)(n-k)$, then

$$|F - F_i| < k(n-k) - (k-1)(n-k) = n-k < (k-1)(n-k).$$

By Claim 3.1.2, $A_{n,k} - \left(V\left(A_{n,k}^i\right) \cup (F - F_i) \right)$ is connected. Since every vertex in $A_{n,k}^i - F_i$ has exactly $n - k$ outer neighbors in $A_{n,k} - A_{n,k}^i$ and $|F - F_i| < n - k$, and at least one of the $n - k$ outer neighbors is fault-free, $A_{n,k} - F$ is still connected, a contradiction.   □

We now show our theorem. By Claims 3.1.1 and 3.1.3, there exists some $i \in \langle n \rangle$ such that $f_i = (k-1)(n-k)$. Thus, for $k \geqslant 3$,

$$|F - F_i| = k(n-k) - (k-1)(n-k) = n-k < (k-1)(n-k).$$

By Claim 3.1.2, $A_{n,k} - \left(A_{n,k}^i \cup (F - F_i) \right)$ is connected, which implies $A_{n,k} - A_{n,k}^i$ is $(n-k+1)$-connected.

Suppose that $A_{n,k}^i - F_i$ is connected. Since $k \geqslant 3$, $A_{n,k}^i$ is not a complete graph, and so $A_{n,k}^i - F_i$ has at least two vertices. Since every vertex in $A_{n,k}^i - F_i$ has exactly $n - k$ outer neighbors in $A_{n,k} - A_{n,k}^i$ and $|F - F_i| = n - k < 2(n-k)$, at least one of these outer neighbors is fault-free, and so $A_{n,k} - F$ is still connected, a contradiction. Therefore, $A_{n,k}^i - F_i$ is disconnected.

Let H^i be a minimum component of $A_{n,k}^i - F_i$. Since F is a minimum separating set in $A_{n,k}$ and $F_i \subset F$, H^i must be contained in some component H in $A_{n,k} - F$. Note that every vertex in H^i has exactly $n - k$ outer neighbors in $A_{n,k} - A_{n,k}^i$, each of them is in different $A_{n,k}^j$ with $j \neq i$, and $A_{n,k} - A_{n,k}^i$ is $(n-k+1)$-connected. To separate H from $A_{n,k} - F$ by using $n - k$ vertices in $F - F_i$, H

must be a single vertex, say x, and $F - F_i$ must be the $(n - k)$ outer neighbors of x in $A_{n,k} - A_{n,k}^i$. Thus, $H = H^i = \{x\}$ and $F = N(x)$. Since $A_{n,k} - \left(A_{n,k}^i \cup (F - F_i)\right)$ is connected and every vertex in $A_{n,k}^i - (F_i \cup \{x\})$ has $n - k$ fault-free outer neighbors in $A_{n,k} - A_{n,k}^i$, $A_{n,k} - (F \cup \{x\})$ is connected.

Thus, when $n \geqslant 4$ and $k \geqslant 3$, $A_{n,k}$ is tightly super $k(n - k)$-connected. The theorem follows.

Since $A_{n,n-1}$ is isomorphic to the star graph S_n and $A_{n,n-2}$ is isomorphic to the alternating group graph AG_n, by Theorem 3.1, we have the following corollaries immediately.

Corollary 3.2 (*Cheng and Lipman [7]*). *The star graph S_n is tightly super $(n - 1)$-connected for $n \geqslant 4$.*

Corollary 3.3. *The alternating group network AG_n is tightly super $(2n - 4)$-connected for $n \geqslant 5$.*

In the rest of this section, we will investigate the fault tolerance of $A_{n,k}$ when we remove a set F of vertices, where $|F|$ is roughly twice or three times of the traditional connectivity.

Let

$$I = \{i \in \langle n \rangle \mid f_i \geqslant (k - 1)(n - k)\},$$
$$A_{n,k}^I = \bigcup_{i \in I} A_{n,k}^i, F_I = \bigcup_{i \in I} F_i,$$

and let

$$J = \langle n \rangle \setminus I, \quad A_{n,k}^J = \bigcup_{j \in J} A_{n,k}^j, \quad F_J = \bigcup_{j \in J} F_j.$$

Lemma 3.4. *Let F be a set of faulty vertices in $A_{n,k}$ with $|F| \leqslant (3k - 2)(n - k) - 3$ and $k \geqslant 3$. Then $A_{n,k}^J - F_J$ is connected.*

Proof. If $|J| = 0$ then there is nothing to do, and so assume $|J| \geqslant 1$. By the hypothesis, for any $j \in J$, $f_j \leqslant (k - 1)(n - k) - 1$, that is, $A_{n,k}^j - F_j$ is connected since $A_{n,k}^j$ is $(k - 1)(n - k)$-connected. Thus, if $|J| = 1$ then the lemma holds. Assume $|J| \geqslant 2$ below. To prove the lemma, we only need to show that $A_{n,k}^i$ and $A_{n,k}^j$ are connected in $A_{n,k}^J - F_J$ for any two distinct $i, j \in J$. By (2.1), we have that

$$
|E(i,j)| \quad = (n - 2)(n - 3) \cdots (n - k)
\begin{cases}
> 2((k - 1)(n - k) - 1) & \text{if } k \geqslant 4 \text{ or } k = 3 \text{ and } n \geqslant 6; \\
= 2(2n - 7) & \text{if } k = 3 \text{ and } n \in \{4, 5\}.
\end{cases}
\tag{3.3}
$$

Thus, if there is a fault-free edge e in $E(i,j)$, then $A_{n,k}^i - F_i$ and $A_{n,k}^j - F_j$ can be connected by the fault-free edge e in $E(i,j)$. If there are no fault-free edges in $E(i,j)$ then, by (3.3), $k = 3$, $n \in \{4, 5\}$ and $f_i = f_j = 2n - 7$. In this case, $|F| = 7n - 24$, $|J| \geqslant 3$ and, for any three distinct $i, j, x \in J$,

$$
\begin{aligned}
|F| - (f_i + f_j) &\leqslant (7n - 24) - 2(2n - 7) \\
&= 3n - 10 \\
&= \begin{cases}
5 < |E(i,x)| = |E(x,j)| & \text{if } n = 5; \\
2 = |E(i,x)| = |E(x,j)| & \text{if } n = 4.
\end{cases}
\end{aligned}
\tag{3.4}
$$

If $n = 5$ then, by (3.4), there are a fault-free edge e_1 in $E(i,x)$ and a fault-free edge e_2 in $E(x,j)$. Then $A_{5,3}^i$ and $A_{5,3}^j$ can be connected in $A_{5,3} - F$ by $A_{5,3}^x$ and the fault-free edges e_1 and e_2.

If $n = 4$, then $f_i = f_j = 1$, and every vertex in $A_{4,3}^x$ has only one outer neighbor for each $x \in \{1, 2, 3, 4\}$. Thus, by (3.4), there are a fault-free edge e_1 in $E(i,x)$ and a fault-free edge e_2 in $E(x,j)$. Then $A_{4,3}^i$ and $A_{4,3}^j$ can be connected in $A_{4,3} - F$ by $A_{4,3}^x$ and the fault-free edges e_1 and e_2.

The lemma follows. □

Corollary 3.5. *Let F be a separating set of $A_{n,k}$ and $k \geqslant 3$. Then*

$$
1 \leqslant |I| \leqslant
\begin{cases}
2 & \text{if } |F| \leqslant (2k - 1)(n - k) - 1; \\
3 & \text{if } |F| \leqslant (3k - 2)(n - k) - 3.
\end{cases}
\tag{3.5}
$$

Corollary 3.6. *Let F be a separating set of $A_{n,k}$ with $|F| \leqslant (3k - 2)(n - k) - 3$ and $k \geqslant 3$. If H is a union of components of $A_{n,k} - F$ that contain no vertices in $A_{n,k}^J - F$, then*

$$
N^I(H) \subseteq F_I \quad \text{and} \quad N^{\bar{I}}(H) \subseteq F \setminus F_I.
\tag{3.6}
$$

Lemma 3.7. *Let F be a separating set of $A_{n,k}$ with $|F| \leqslant (3k-2)(n-k) - 3$ and $k \geqslant 3$. If there is some $i \in \langle n \rangle$ such that $|F| - f_i \leqslant 2(n-k) - 1$, then $A_{n,k} - F$ has exactly two components, one of which is a single vertex.*

Proof. By the hypothesis, for any $j \in \langle n \rangle \backslash \{i\}$,

$$f_j \leqslant |F| - f_i \leqslant 2(n-k) - 1.$$

Since $|I| \geqslant 1$ by Corol1ary 3.5, we have $I = \{i\}$. Since $A_{n,k} - F$ is disconnected, and $A_{n,k} - \left(A_{n,k}^i \cup F \right)$ is connected by Lemma 3.4, there is a component of $A_{n,k} - F$ that contains no vertices in $A_{n,k}^j - F_j$. Let H be a union of such components of $A_{n,k} - F$. By Corollary 3.6, $N^{\bar{i}}(H) \subseteq F \backslash F_i$. By (2.2) we have that

$$|V(H)|(n-k) \leqslant |F| - f_i \leqslant 2(n-k) - 1,$$

which yields $|V(H)| \leqslant 1$, that is, H is a single vertex, say u. By the choice of H, other components of $A_{n,k} - F$ must contain vertices in $A_{n,k}^j - F_j$. Since $A_{n,k}^j - F_j$ is connected, $A_{n,k} - (F \cup \{u\})$ is connected. It follows that $A_{n,k} - F$ has exactly two components, one of which is a single vertex.

The lemma follows. □

Lemma 3.8. *Let F be a separating set of $A_{n,k}$ with $|F| \leqslant (3k-2)(n-k) - 3$ and $k \geqslant 3$, and let H be a subgraph of $A_{n,k}^i - F_i$ for some $i \in \langle n \rangle$. If $N_{A_{n,k}^i}(H) \subseteq F_i$, then $|V(H)| \leqslant 2$.*

Proof. Let $h = |V(H)|$. We want to prove $h \leqslant 2$. Suppose to the contrary that $h \geqslant 3$. Take a subset $T \subseteq V(H)$ with $|T| = 3$. Let $T' = V(H - T)$. By the hypothesis, $N_{A_{n,k}^i}(T) \backslash T' \subseteq F_i$. Note that $A_{n,k}^i$ is $(k-1)(n-k)$-regular.

When $n = k + 1$, by (2.5), any two vertices of T have at most one common neighbor in $A_{n,k}$. It follows that

$$|N_{A_{n,k}^i}(T)| \geqslant 3(k-1)(n-k) - 4.$$

When $n \geqslant k + 2$, we denote $T = \{x, y, z\}$, and discuss as follows.

If $H[T]$ has no edges, then every pair of vertices in T has at most two common neighbors by (2.5), and so

$$|N_{A_{n,k}^i}(T)| \geqslant 3(k-1)(n-k) - 6.$$

If $H[T]$ has only one edge, say $e = (x, y)$, then x and y have $n-k-1$ common neighbors, z and x (resp. y) have at most two common neighbors by (2.5). It follows that

$$|N_{A_{n,k}^i}(T)| \geqslant 3(k-1)(n-k) - (n-k-1) - 6.$$

Similarly, by (2.5), we can obtain that if $H[T]$ has two edges then

$$|N_{A_{n,k}^i}(T)| \geqslant 3(k-1)(n-k) - 2(n-k-2) - 5;$$

if $H[T]$ has three edges,

$$|N_{A_{n,k}^i}(T)| \geqslant 3(k-1)(n-k) - 2(n-k-2) - 6.$$

Summing all cases, we have that

$$
\begin{aligned}
f_i &\geqslant |N_{A_{n,k}^i}(T) \backslash T'| \\
&\geqslant |N_{A_{n,k}^i}(T)| - (h-3) \\
&\geqslant 3(k-1)(n-k) - 6 - 2(n-k-2) - (h-3) \\
&= (3k-5)(n-k) - h + 1,
\end{aligned}
$$

that is,

$$f_i \geqslant (3k-5)(n-k) - h + 1. \tag{3.7}$$

Since $N^{\bar{i}}(H) \subseteq F - F_i$ by Corollary 3.6, $|F| - f_i \geqslant h(n-k)$, from which we have that

$$
\begin{aligned}
f_i &\leqslant |F| - h(n-k) \\
&\leqslant (3k-2)(n-k) - 3 - h(n-k) \\
&= (3k-2-h)(n-k) - 3,
\end{aligned}
$$

that is,

$$f_i \leqslant (3k-2-h)(n-k) - 3. \tag{3.8}$$

Combining (3.7) with (3.8), we can deduce that $(h - 3)(n - k) \leqslant h - 4$, a contradiction. Thus, we have $h \leqslant 2$. The lemma follows. □

Theorem 3.9. *Let F be a set of faulty vertices in $A_{n,k}$ with $|F| \leqslant (2k - 1)(n - k) - 1$ and $k \geqslant 3$. If $A_{n,k} - F$ is disconnected, then it has exactly two components, one of which is a single vertex or a single edge.*

Proof. Since $A_{n,k} - F$ is disconnected, F is a separating set of $A_{n,k}$.

Suppose that there exists some $i \in \langle n \rangle$ such that $f_i \geqslant (2k - 3)(n - k)$. Since $|F| \leqslant (2k - 1)(n - k) - 1 \leqslant (3k - 2)(n - k) - 3$ and

$$|F| - f_i \leqslant ((2k - 1)(n - k) - 1) - (2k - 3)(n - k) = 2(n - k) - 1,$$

$A_{n,k} - F$ has exactly two components, one of which is a single vertex by Lemma 3.7.

We now assume that $f_i \leqslant (2k - 3)(n - k) - 1$ for any $i \in \langle n \rangle$. Then

$$\begin{aligned} |V(A_{n,k}^i - F_i)| = \quad & (n - 1)(n - 2) \cdots (n - k) - f_i \\ \geqslant \quad & (n - 1)(n - 2) \cdots (n - k) - ((2k - 3)(n - k) - 1) \\ \geqslant \quad & 2. \end{aligned}$$

Since

$$|F| \leqslant (2k - 1)(n - k) - 1 \leqslant (3k - 2)(n - k) - 3,$$

by Lemma 3.4 $A_{n,k}^J - F_J$ is connected. Let H be a union of components of $A_{n,k} - F$ that contain no vertices in $A_{n,k}^J - F_J$. Thus, H is in $A_{n,k}^J$. By the choice of H, other components of $A_{n,k} - F$ must contain vertices in $A_{n,k}^J - F_J$. Since $A_{n,k}^J - F_J$ is connected, $A_{n,k} - (F \cup V(H))$ is connected. Thus, to complete the proof of the theorem, we only need to show that H is either a single vertex or a single edge. Consider two cases according to $|I| = 1$ or $|I| = 2$ by Corollary 3.5.

Case 1. $|I| = 1$, and let $I = \{i\}$.

Let $h = |V(H)|$. Then $h \leqslant 2$ by Lemma 3.8. If $h = 1$, then H is a single vertex.

If $h = 2$, we want to prove that H is a single edge. Suppose to the contrary that H consists of two isolated vertices, say u and v. Then u and v are not adjacent, $N(u) \cup N(v) \subseteq F$. By (2.5), we deduce a contradiction as follows.

$$\begin{aligned} |F| \geqslant \quad & |N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ = \quad & 2k(n - k) - |N(u) \cap N(v)| \\ > \quad & (2k - 1)(n - k) - 1 \geqslant |F|. \end{aligned}$$

Thus, H is a single edge.

Case 2. $|I| = 2$, and let $I = \{i, j\}$.

Under our hypothesis, by (2.2 and 3.6), we have that

$$\begin{aligned} n - k - 1 \leqslant |N^{\bar{I}}(H)| \leqslant \quad & |F \setminus (F_i \cup F_j)| \\ \leqslant \quad & (2k - 1)(n - k) - 1 - 2((k - 1)(n - k)) \\ = \quad & n - k - 1. \end{aligned}$$

Thus, $|N^{\bar{I}}(H)| = n - k - 1$.

Thus, by (2.2 and 2.2), there is exactly one vertex in $\left(A_{n,k}^i - F_i \right) \cap V(H)$ such that exact one of its outer neighbors is in $A_{n,k}^j$ and others are in $F \setminus (F_i \cup F_j)$. Similarly, there is exactly one vertex in $(A_{n,k}^j - F_j) \cap V(H)$ such that exact one of its outer neighbors is in $A_{n,k}^i$ and others are in $F \setminus (F_i \cup F_j)$. Thus, H is a single edge.

The proof of the theorem is complete. □

Since $A_{n,n-1}$ is isomorphic to the star graph S_n and $A_{n,n-2}$ is isomorphic to the alternating group graph AG_n, by Theorem 3.9, we have the following corollaries immediately.

Corollary 3.10 (*Cheng and Lipman [7]*). *Let F be a set of faulty vertices in the star graph S_n with $|F| \leqslant 2n - 4$ and $n \geqslant 4$. If $S_n - F$ is disconnected, then it has exactly two components, one of which is either a single vertex, or a single edge.*

Corollary 3.11. *Let F be a set of faulty vertices in the alternating group graph AG_n with $|F| \leqslant 4n - 11$ and $n \geqslant 5$. If $AG_n - F$ is disconnected, then it has exactly two components, one of which is either a single vertex, or a single edge.*

We now discuss the fault tolerance of $A_{n,k}$ with more faulty vertices up to $(3k - 2)(n - k) - 4$ when $n \geqslant k + 2$ and $(3k - 2)(n - k) - 3$ when $n = k + 1$, where the latter we write $3n - 8$ for $(3k - 2)(n - k) - 3$.

Theorem 3.12. *Let F be a set of faulty vertices in $A_{n,k}$ ($k \geqslant 4$) with $|F| \leqslant (3k - 2)(n - k) - 4$ when $n \geqslant k + 2$ and $|F| \leqslant 3n - 8$ when n = k + 1. If $A_{n,k} - F$ is disconnected, then it either has two components, one of which is an isolated vertex or an isolated edge, or has three components, two of which are isolated vertices.*

Proof. Since $A_{n,k} - F$ is disconnected, F is a separating set of $A_{n,k}$.

If there exists some $i \in \langle n \rangle$ such that

$$|F| - f_i \leqslant 2(n - k) - 1,$$

by Lemma 3.7, $A_{n,k} - F$ has exactly two components, one of which is a single vertex, and so the theorem holds. We now assume that, for any $i \in \langle n \rangle$

$$f_i \leqslant \begin{cases} (3k - 4)(n - k) - 4 & \text{for } n \geqslant k + 2; \\ (3k - 4)(n - k) - 3 & \text{for } n = k + 1. \end{cases}$$

Then $|V(A_{n,k}^i - F_i)| \geqslant 2$.

Let H be a union of components of $A_{n,k} - F$ that contain no vertices in $A_{n,k}^J - F_J$, and let $h = |V(H)|$. By Lemma 3.4, $A_{n,k}^J - F_J$ is connected. Thus, H is in $A_{n,k}^I$. By the choice of H, other components of $A_{n,k} - F$ must contain vertices in $A_{n,k}^J - F_J$. Since $A_{n,k}^J - F_J$ is connected, $A_{n,k} - (F \cup V(H))$ is connected. Thus, to complete the proof of the theorem, we only need to show that $h \leqslant 2$.

By Corollary 3.5, $1 \leqslant |I| \leqslant 3$. If $|I| = 3$, under our hypothesis, we have that

$$|F \setminus F_I| \leqslant (3k - 2)(n - k) - 4 - 3((k - 1)(n - k)) = n - k - 4. \tag{3.9}$$

Thus, by 2.2, 3.3 and 3.9, we can deduce a contradiction as follows.

$$n - k - 2 \leqslant |F \setminus F_I| \leqslant n - k - 4.$$

Thus, $1 \leqslant |I| \leqslant 2$. If $|I| = 1$, then $h \leqslant 2$ by Lemma 3.8. We only need to consider the case of $|I| = 2$. Let $I = \{i, j\}$, and let h_i and h_j be the numbers of vertices of H that lie in $A_{n,k}^i$ and $A_{n,k}^j$, respectively. Then $h_i \leqslant 2$ and $h_j \leqslant 2$ by Lemma 3.8. Without loss of generality, assume $f_i \geqslant f_j$.

Note that $A_{n,k}^i$ is isomorphic to $A_{n-1,k-1}$. If $f_i \leqslant (2k - 3)(n - k) - 2$ then, applying Theorem 3.9 to $A_{n,k}^i$, we have $h_i \leqslant 1$ since F_i cannot isolate an edge from $A_{n,k}^i$. Thus, $h = h_i + h_j \leqslant 2$. So, in the following discussion, we assume that

$$f_i \geqslant (2k - 3)(n - k) - 1. \tag{3.10}$$

If $f_j \geqslant k(n - k)$, when $|F| \leqslant (3k - 2)(n - k) - 3$, we have that

$$\begin{aligned} |F \setminus F_I| &\leqslant (3k - 2)(n - k) - 3 - k(n - k) - (2k - 3)(n - k) + 1 \\ &= n - k - 2. \end{aligned}$$

Note that, for every vertex of $V(H) \cap V(A_{n,k}^i)$, it has at most one outer neighbor in $A_{n,k}^j$ and others in $F \setminus F_I$. By (2.2) and (3.3), we have that

$$h_i(n - k - 1) \leqslant |F \setminus F_I| \leqslant n - k - 2,$$

which implies $h_i = 0$, and so $h = h_i + h_j \leqslant 2$.

Thus, under the condition (3.10), the remainder of the proof is to consider the case that

$$(k - 1)(n - k) \leqslant f_j \leqslant k(n - k) - 1. \tag{3.11}$$

We first note that, when $f_j \leqslant k(n - k) - 1$,

$$f_j \leqslant k(n - k) - 1 \leqslant (2k - 3)(n - k) - 2.$$

Thus, F_j isolates at most one vertex in $A_{n,k}^j$ by Theorem 3.9, that is, $h_j \leqslant 1$. If $h_j = 0$, then $h \leqslant 2$, and so the theorem holds. Assume $h_j = 1$ below.

By the condition (3.10) and the condition (3.11), we have that

$$\begin{aligned} f_i + f_j &\geqslant (2k - 3)(n - k) - 1 + (k - 1)(n - k) \\ &= (3k - 4)(n - k) - 1. \end{aligned} \tag{3.12}$$

Thus, by (3.12), when $|F| \leqslant (3k - 2)(n - k) - 4$ and $n \geqslant k + 2$,

$$|F \setminus F_I| = |F| - f_i - f_j \leqslant 2(n - k) - 3, \tag{3.13}$$

and when $|F| \leqslant 3n - 8$ and n = k + 1,

$$|F \setminus F_I| = |F| - f_i - f_j = 0. \tag{3.14}$$

Suppose to the contrary that $h_i = 2$. Let

$$V(H) \cap V(A_{n,k}^i) = \{x, y\} \text{ and } V(H) \cap V(A_{n,k}^j) = \{z\}.$$

Then at least one of x and y is not adjacent to z by (2.2). Without loss of generality, let x be not adjacent to z. Then, by (2.3),

$$|N^{\bar{I}}(x) \cap N^{\bar{I}}(z)| = 0. \tag{3.15}$$

By (2.2), we have

$$|N^{\bar{I}}(x) \cap N^{\bar{I}}(y)| = 0, \tag{3.16}$$

and

$$|N^{\bar{I}}(x) \cap N^{\bar{I}}(y) \cap N^{\bar{I}}(z)| = 0, \tag{3.17}$$

When $n \geqslant k + 2$, considering outer neighbors of y and z, by (2.3) and (2.5), we have that

$$|N^{\bar{I}}(y) \cap N^{\bar{I}}(z)| = \begin{cases} 0 & \text{if } (y, z) \notin E(A_{n,k}); \\ n - k - 1 & \text{if } (y, z) \in E(A_{n,k}). \end{cases} \tag{3.18}$$

By 3.6, (3.15)–(3.18), we have that

$$\begin{aligned} |F \setminus F_I| &\geqslant \left| N^{\bar{I}}(H) \right| \\ &= \sum_{u \in \{x,y,z\}} |N^{\bar{I}}(u)| - \sum_{u \neq v \in \{x,y,z\}} |N^{\bar{I}}(u) \cap N^{\bar{I}}(v)| \\ &\qquad + |N^{\bar{I}}(x) \cap N^{\bar{I}}(y) \cap N^{\bar{I}}(z)| \\ &\geqslant \begin{cases} 2(n - k - 1) & \text{if } (y, z) \in E(A_{n,k}); \\ 3(n - k - 1) & \text{if } (y, z) \neq E(A_{n,k}), \end{cases} \end{aligned}$$

which contradicts (3.13). Thus, $h_i \leqslant 1$ and so $h = h_i + h_j \leqslant 2$.

When $n = k + 1$, (3.14) implies that $|F| = 3n - 8$, $f_j = n - 2$ and $f_i = 2n - 6$. In other words, $F_j = N_{A_{n,k}^j}(z)$ and $F_i = N_{A_{n,k}^i}(x, y)$, the latter implies that x and y are adjacent. Since $n = k + 1$, the only outer neighbor of x, say u, and the only outer neighbor of y, say v, must be in $F_j \cup \{z\}$. Similarly, the only outer neighbor of z, say w, must be in $F_i \cup \{x, y\}$. Since x is not adjacent to z, $u \in F_j$. If $v = z$ then $|N(x) \cap N(z)| = 2$, which contradicts (2.5). Assume $v \in F_j$ below. If $w \in N(x)$, then $|N(x) \cap N(z)| = 2$; if $w \in N(y)$, then $|N(y) \cap N(z)| = 2$. No matter which case, it contradicts (2.5).

The proof of the theorem is complete. $\square$

The Theorem 3.12 is optimal in the following sense. When $n \geqslant k + 2$ and $k \geqslant 4$, we select such three vertices $x, y \in V(A_{n,k}^i)$ and $z \in V(A_{n,k}^j)$ that $(y, z) \in E(i, j)$ and $(x, y) \in E\left(A_{n,k}^i\right)$, see Fig. 2. Set $F = N(x, y, z)$. By (2.5), we have that

$$|N(x) \cap N(y)| = |N(y) \cap N(z)| = n - k - 1, |N(x) \cap N(z)| = 2.$$
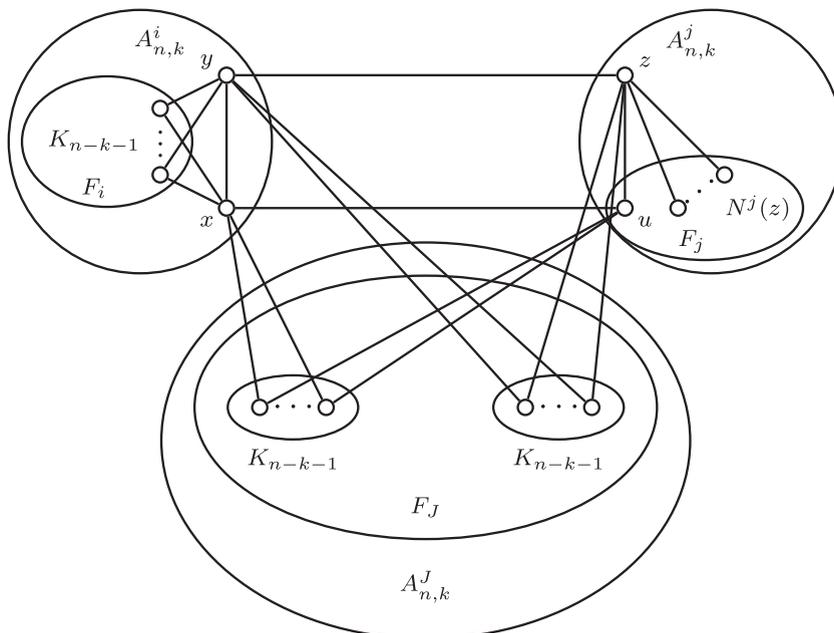
Fig. 2. The distribution of fault set F in $A_{n,k}$ with $k \geqslant 4$ and $n \geqslant k + 2$.

Then

$$|F| = 3k(n - k) - 2(n - k - 1) - 5 = (3k - 2)(n - k) - 3.$$

$A_{n,k} - F$ is connected and contains a path of length three.

Since $A_{n,n-1}$ is isomorphic to the star graph S_n and $A_{n,n-2}$ is isomorphic to the alternating group graph AG_n, by Theorem 3.12, we have the following corollaries immediately.

Corollary 3.13 (*Cheng and Lipták [10]*). *Let F be a set of faulty vertices in the star graph S_n with $|F| \leqslant 3n - 8$ and $n \geqslant 5$. If $S_n - F$ is disconnected, then it either has two components, one of which is an isolated vertex or an edge, or has three components, two of which are isolated vertices.*

Corollary 3.14. *Let F be a set of faulty vertices in the alternating group network AG_n with $|F| \leqslant 6n - 20$ and $n \geqslant 6$. If $AG_n - F$ is disconnected, then it either has two components, one of which is an isolated vertex or an edge, or has three components, two of which are isolated vertices.*

4. Diagnosability of arrangement graph

The comparison diagnosis strategy of a graph $G = (V, E)$ can be modeled as a multi-graph $M = (V, C)$, where C is a set of labeled edges. If the processors u and v can be compared by the processor w, there exists a labeled edge (u, v) in C, denoted by $(u, v)_w$. We call w the comparator of u and v. Since different comparators can compare the same pair of processors, M is a multi-graph. Denote the comparison result as $\sigma((u, v)_w)$ such that $\sigma((u, v)_w) = 0$ if the outputs of u and v agree, and $\sigma((u, v)_w) = 1$ if the outputs disagree. If the comparator w is fault-free and $\sigma((u, v)_w) = 0$, the processors u and v are fault-free; while $\sigma((u, v)_w) = 1$, at least one of the three processors u, v and w is faulty. The collection of the comparison results defined as a function $\sigma: C \rightarrow \{0, 1\}$, is called the *syndrome* of the diagnosis. If the comparator w is faulty, the comparison result is unreliable. A faulty comparator can lead to unreliable results, so a set of faulty vertices may produce different syndromes. A subset $F \subsetneq V$ is said to be *compatible* with a syndrome σ if σ can arise from the circumstance that all vertices in F are faulty and all vertices in $V - F$ are fault-free. A system G is said to be *diagnosable* if, for every syndrome σ, there is a unique $F \subset V$ that is compatible with σ. A system is said to be *t-diagnosable* if the system is diagnosable as long as the number of faulty vertices does not exceed t. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, write as $t(G)$. Let $\sigma_F = \{\sigma | \sigma$ is compatible with $F\}$. Two distinct subsets F_1 and F_2 of $V(G)$ are said to be *indistinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_1} \neq \phi$, and distinguishable otherwise [20,24,29]. There are several different ways to verify whether a system is *t-diagnosable* under the comparison approach. The following lemma obtained by Sengupta and Dahbura [29] gives necessary and sufficient conditions to ensure distinguishability.

Lemma 4.1 (*Sengupta and Dahbura [29]*). *Let G be a graph, F_1 and F_2 be two distinct subsets of vertices in G. The pair (F_1, F_2) is distinguishable if and only if at least one of the following conditions is satisfied.*

(1) *There are two distinct vertices u and $w \in V(G - F_1 \cup F_2)$ and a vertex $v \in F_1 \Delta F_2$ such that $(u, v)_w \in C$, where $F_1 \triangle F_2 = (F_1 \backslash F_2) \cup (F_2 \backslash F_1)$.*
(2) *There are two distinct vertices u and $v \in F_1 \backslash F_2$ (or $F_2 \backslash F_1$) and a vertex $w \in V(G - F_1 \cup F_2)$ such that $(u, v)_w \in C$.*

Lin et al. [24] introduced the so-called conditional diagnosability of a system under the situation that no set of faulty vertices can contain all neighbors of any vertex in the system. A fault-set $F \subset V(G)$ is called a conditional fault-set if $G - F$ has no isolated vertex. A system $G(V, E)$ is said to be conditionally *t-diagnosable* if F_1 and F_2 are distinguishable for each pair (F_1, F_2) of distinct conditional fault-sets in G with $|F_1| \leqslant t$ and $|F_2| \leqslant t$. The *conditional diagnosability* of G, denoted by $t_c(G)$, is defined as the maximum value of t for which G is conditionally *t-diagnosable*. Clearly, $t_c(G) \geqslant t(G)$. Zhou and Xiao [38] obtained the conditional diagnosability of the alternating group networks based on the fault tolerance of this network structure. This section will focus on the conditional diagnosability of arrangement graphs.

Theorem 4.2. $t_c(A_{n,k}) \leqslant (3k - 2)(n - k) - 3$ for $k \geqslant 4$, $n \geqslant k + 2$; $t_c(A_{n,k}) \leqslant 3n - 7$ for $k \geqslant 4$, $n = k + 1$.

Proof. When $n \geqslant k + 2$, we select four vertices x, y, z, $u \in V(A_{n,k})$, such that (x, u), $(y, z) \in E(i, j)$, and $(x, y) \in E(A_{n,k}^i)$, then $(u, z) \in E(A_{n,k}^j)$. Set $A = N[x, y, z]$, $F_1 = A - \{y, z\}$, and $F_2 = A - \{x, y\}$. We get

$$|F_1| = |F_2| = (3k - 2)(n - k) - 2, \text{ and } |F_1 - F_2| = |F_2 - F_1| = 1.$$

It is easy to check that F_1 and F_2 are two conditional faulty sets, and F_1 and F_2 are indistinguishable. Thus, we have

$$t_c(A_{n,k}) \leqslant (3k - 2)(n - k) - 3.$$

When $n = k + 1$, we select three vertices x, y, $z \in V(A_{n,k})$, such that (x,y), $(y,z) \in E(A_{n,k})$. By (2.5), any two of x, y, z have no common neighbor. Set

$$A = N[x, y, z], F_1 = A - \{y, z\}, \text{ and } F_2 = A - \{y, z\}.$$

We get $|F_1| = |F_2| = 3n - 6$, and $|F_1 - F_2| = |F_2 - F_1| = 1$. It is easy to check that F_1 and F_2 are two conditional faulty sets, and F_1 and F_2 are indistinguishable. Thus, we have $t_c(A_{n,k}) \leqslant 3n - 7$.   □

Lemma 4.3. *Let F_1 and F_2 be any two distinct conditional fault-sets of $A_{n,k}$ with $|F_1| \leqslant (3k - 2)(n - k) - 3$, $|F_2| \leqslant (3k - 2)(n - k) - 3$ for $k \geqslant 4$, $n \geqslant k + 2$; or $|F_1| \leqslant 3n - 7$, $|F_2| \leqslant 3n - 7$ for $k \geqslant 4$, $n = k + 1$. Denote by H the maximum component of $A_{n,k} - F_1 \cap F_2$. Then, for every vertex $u \in F_1 \Delta F_2$, $u \in H$.*

Proof. Without loss of generality, we assume that $u \in F_1 - F_2$. Since F_2 is a conditional fault-set, there is a vertex $v \in (A_{n,k} - F_2) - \{u\}$ such that $(u,v) \in E(A_{n,k})$. Suppose that u is not a vertex of H. Then v is not in H, so u and v are in one small component of $A_{n,k} - F_1 \cap F_2$. Since F_1 and F_2 are distinct, we have

$$|F_1 \cap F_2| \leqslant (3k - 2)(n - k) - 4 \text{ for } n \geqslant k + 2;$$

or

$$|F_1 \cap F_2| \leqslant 3n - 8 \text{ for } n = k + 1.$$

Hence $\{u, v\}$ forms a component K_2 in $A_{n,k} - F_1 \cap F_2$ by Theorem 3.12, i.e., the vertex u is the unique neighbor of v in $A_{n,k} - F_1 \cap F_2$. This is a contradiction since F_1 is a conditional fault-set, but all the neighbors of v are faulty in $A_{n,k} - F_1$.   □

Lemma 4.4 (*Lin [24]*). *Let G be a graph with $\delta(G) \geqslant 2$, and let F_1 and F_2 be any two distinct conditional fault-sets of G with $F_1 \subset F_2$. Then (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.*

Lemma 4.5. *Let F_1 and F_2 be any two distinct conditional fault-sets of $A_{n,k}$. If $|F_1| = (3k - 2)(n - k) - 3$ and $|F_2| = (3k - 2)(n - k) - 3$ $k \geqslant 4$, $n \geqslant k + 2$; or $|F_1| \leqslant 3n - 7$, $|F_2| \leqslant 3n - 7$ for $k \geqslant 4$, $n = k + 1$. Then (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.*

Proof. By Lemma 4.4 (F_1, F_2) is a distinguishable conditional pair if $F_1 \subset F_2$ or $F_2 \subset F_1$. Now, we assume that $|F_1 - F_2| \geqslant 1$, and $|F_2 - F_1| \geqslant 1$. Let $S = F_1 \cap F_2$. Then we have $|S| \leqslant (3k - 2)(n - k) - 4$ for $k \geqslant 4$, $n \geqslant k + 2$; or $|S| \leqslant 3n - 8$ for $k \geqslant 4$, $n = k + 1$. Let H be the largest connected component of $A_{n,k} - F_1 \cup F_2$. By Lemma 4.3, every vertex in $F_1 \Delta F_2$ is in H. We claim that H has a vertex u outside $F_1 \cup F_2$ that has no neighbor in H. Since every vertex has degree $k(n - k)$, the vertices in S can have at most $k(n - k)|S|$ neighbors in H. There are at most $|F_1| + |F_2| - |S|$ vertices in $F_1 \cup F_2$ and at most two vertices of $A_{n,k} - S$ may not belong to H by Theorem 3.12. Thus, we have:

$$\frac{n!}{(n-k)!} \quad -k(n - k)|S| - (|F_1| + |F_2| - |S|) - 2$$
$$\geqslant \frac{n!}{(n-k)!} - (k(n - k) + 1) \times ((3k - 2)(n - k) - 4) - 4$$
$$\geqslant 4 \text{ for } k \geqslant 4, n \geqslant k + 2;$$

and

$$\frac{n!}{(n-k)!} - \quad k(n - k)|S| - (|F_1| + |F_2| - |S|) - 2$$
$$\geqslant \quad n! - n \times (3n - 8) - 2$$
$$\geqslant \quad n! - 3n^2 + 8n - 2$$
$$\geqslant \quad 4 \text{ for } k \geqslant 4, n = k + 1.$$

Thus, there must be some vertex of H outside $F_1 \cup F_2$, which has no neighbors in S. Let u be such a vertex.

If u has no neighbor in $F_1 \cup F_2$, then we can find a path of length at least two within H to a vertex v in $F_1 \cup F_2$. We may assume that v is the first vertex of $F_1 \Delta F_2$ on this path, and let q and w be the two vertices on this path immediately before v (we may have $u = q$), so q and w are not in $F_1 \cup F_2$. The existence of the edges (q, w) and (w, v) ensures that (F_1, F_2) is a distinguishable conditional pair of $A_{n,k}$ by Lemma 4.1. Now we assume that u has a neighbor in $F_1 \Delta F_2$. Since the degree of u is at least 3, and u has no neighbor in S, there are three possibilities:

(1) u has two neighbors in $F_1 \backslash F_2$; or
(2) u has two neighbors in $F_2 \backslash F_1$; or
(3) u has at least one neighbor outside $F_2 \cup F_1$. In each sub-case above, Lemma 4.1 implies that (F_1, F_2) is a distinguishable conditional pair of $A_{n,k}$ under the comparison diagnosis model, and so the proof is complete.   □

Theorem 4.2 tells us that $t_c(A_{n,k}) \leqslant (3k-2)(n-k)-3$ for $k \geqslant 4$, $n \geqslant k+2$; $t_c(A_{n,k}) \leqslant 3n-7$ for $k \geqslant 4$, $n = k+1$. Lemma 4.5 shows that $t_c(A_{n,k}) \geqslant (3k-2)(n-k)-3$ for $k \geqslant 4$, $n \geqslant k+2$; $t_c(A_{n,k}) \geqslant 3n-7$ for $k \geqslant 4$, $n = k+1$. Thus, we have the following results.

Theorem 4.6. $t_c(A_{n,k}) = (3k-2)(n-k)-3$ for $k \geqslant 4$, $n \geqslant k+2$; $t_c(A_{n,k}) = 3n-7$ for $k \geqslant 4$, $n = k+1$.

Since $A_{n,n-1}$ is isomorphic to the star graph S_n and $A_{n,n-2}$ is isomorphic to the alternating group graph AG_n, by Theorem 3.12, we have the following corollaries immediately.

Corollary 4.7 (Lin et al. [24]). *The conditional diagnosability of the star graph S_n under the comparison model is $t_c(S_n) = 3n - 7$ for $n \geqslant 5$.*

Corollary 4.8. *The conditional diagnosability of the alternating group graph AG_n under the comparison model is $t_c(AG_n) = 6n - 19$ for $n \geqslant 6$.*

5. Conclusion

The issue of identifying faulty processors is important for the design of multiprocessor interconnected systems, which are implementable with VLSI. The process of identifying all the faulty processors is the system-level diagnosis. Based on the fault resiliency of arrangement graphs, the paper establishes the fault diagnosabilities of the arrangement graphs under the comparison model. The diagnosability of $A_{n,k}$ under the comparison model is only $k(n-k)$, while the conditional diagnosability of $A_{n,k}$ is $(3k-2)(n-2)-3$, which is about three times the traditional diagnosability under the comparison model. The distribution of fault-set F in $A_{n,k}$ with $k \geqslant 4$ and $n \geqslant k+2$ showed in Fig. 2 implies that the result is optimal. The fault resiliency of the arrangement graphs may also reveal its super connectivity and conditional connectivity of high order. This method can be also applied to other complex network structure, such as (n,k)-star graphs.

Acknowledgments

References

[1] S.B. Akers, B. Krishnamurthy, A group theoretic model for symmetric interconnection networks, Transactions on Computers 38 (4) (1989) 555–566.
[2] L.Q. Bai, H. Ebara, H. Nakano, H. Maeda, Fault-tolerant broadcasting on arrangement graph, The Computer Journal 41 (3) (1998) 171–184.
[3] D. Bauer, F. Boesch, C. Suffel, R. Tindell, Connectivity extremal problems and the design of reliable probabilistic networks, in: Y. Alavi, G. Chartrand (Eds.), The Theory and Application of Graphs, Wiley, New York, 1981, pp. 89–98.
[4] Y.S. Chen, T.Y. Juang, Y.Y. Shen, Congestion-free embedding of $2(n-k)$ spanning trees in an arrangement graph, Journal of Systems Architecture 47 (1) (2001) 73–86.
[5] Y.S. Chen, T.J. Juang, E.-H. Tseng, Efficient broadcasting in an arrangement graph using multiple spanning trees, IEICE Transactions on Fundamentals E-83-A (1) (2000) 139–149.
[6] E. Cheng, M.J. Lipman, H.A. Park, Super connectivity of star graphs, alternating group graphs and split-stars, Ars Combinatoria 59 (2001) 107–116.
[7] E. Cheng, M.J. Lipman, Increasing the connectivity of the star graphs, Networks 40 (3) (2002) 165–169.
[8] E. Cheng, L. Lipták, Fault resiliency of Cayley graphs generated by transpositions, International Journal of Foundations of Computer Science 18 (2007) 1005–1022.
[9] E. Cheng, L. Lipták, Linearly many faults in Cayley graphs generated by transposition trees, Information Science 177 (2007) 4877–4882.
[10] E. Cheng, L. Lipták, Structural properties of Cayley graphs generated by transposition trees, Congressus Numerantium 180 (2006) 81–96.
[11] E. Cheng, L. Lipták, F. Sala, Linearly many faults in 2-tree-generated networks, Networks 55 (2) (2010) 90–98.
[12] W.K. Chiang, R.J. Chen, On the arrangement graph, Information Processing Letters 66 (4) (1998) 215–219.
[13] K.Y. Chwa, S.L. Hakimi, On fault identification in diagnosable system, IEEE Transactions on Computers C-30 (6) (1981) 414–422.
[14] K. Day, A. Tripathi, Characterization of Node Disjoint Paths in Arrangement Graphs, Technical Report TR 91-43, Computer Science Department, University of Minnesota, 1991.
[15] K. Day, A. Tripathi, Arrangement graphs: a class of generalized star graphs, Information Processing Letters 42 (5) (1992) 235–241.
[16] K. Day, A. Tripathi, Embedding of cycles in arrangement graphs, IEEE Transactions on Computers 42 (8) (1993) 1002–1006.
[17] A.H. Esfahanian, Generalized measures of fault tolerance with application to n-cube networks, IEEE Transactions on Computers 38 (1989) 1586–1591.
[18] J. Fan, J. Yang, G.D. Zhou, L. Zhao, W.Z. Zhang, Diagnosable evaluation of DCC linear congruential graphs under the PMC diagnostic model, Information Sciences 179 (2009) 1785–1791.
[19] S.Y. Hsieh, G.H. Chen, C.W. Ho, Fault-free hamiltonian cycles in faulty arrangement graphs, IEEE Transactions on Parallel and Distributed Systems 10 (3) (1999) 223–237.
[20] G.H. Hsu, C.F. Chiang, L.M. Shih, L.H. Hsu, J.J.M. Tan, Conditional diagnosability of hypercubes under the comparison diagnosis model, Journal of Systems Architecture 55 (2) (2009) 140–146.
[21] H.C. Hsu, T.K. Li, J.M. Tan, L.H. Hsu, Fault hamiltonicity and fault hamiltonian connectivity of the arrangement graphs, IEEE Transactions on Computers 53 (1) (2004) 39–53.
[22] T.L. Kung, C.K. Lin, T. Liang, L.Y. Hsu, J.J.M. Tan, Fault diameter of hypercubes with hybrid node and link faults, Journal of Interconnection Networks 10 (3) (2009) 233–242.

[23] P.L. Lai, J.J.M. Tan, C.P. Chang, L.H. Hsu, Conditional diagnosability measure for large multiprocessors systems, IEEE Transactions on Computers 54 (2005) 165–175.
[24] C.K. Lin, J.J.M. Tan, L.H. Hsu, E. Cheng, Lipták, Conditional diagnosability of Cayley graphs generalized by transposition tree under the comparison diagnosis model, Journal of Interconnection networks 9 (2008) 83–97.
[25] R.S. Lo, G.H. Chen, Embedding hamiltonian path in faulty arrangement graphs with the backtracking method, IEEE Transactions on Parallel and Distributed Systems 12 (2) (2001) 209–222.
[26] J. Maeng, M. Malek, A comparison connection assignment for self-diagnosis of multiprocessor systems, in: Proceedings of 11th International Fault-Tolerant Computing, 1981, pp. 173–175.
[27] M. Malek, A comparison connection assignment for diagnosis of multiprocessor systems, in: Proceedings of the 7th International Symposium on Computer Architecture, 1980, pp. 31–35.
[28] F.P. Preparata, G. Metze, R.T. Chien, On the connection assignment problem of diagnosable systems, IEEE Transactions on Computers 16 (1967) 848–854.
[29] A. Sengupta, A. Dahbura, On self-diagnosable multiprocessor systems: diagnosis by the comparison approach, IEEE Transaction on Computers 41 (1992) 1386–1396.
[30] Y.H. Teng, J.J.M. Tan, L.H. Hsu, Panpositionable hamiltonicity and panconnectivity of the arrangement graphs, Applied Mathematics and Computation 198 (2008) 414–432.
[31] J.-M. Xu, Toplogical Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.
[32] J.-M. Xu, J.-W. Wang, W.-W. Wang, Super and restricted connectivity of some interconnection networks, Ars Combinatoria 94 (2010) 25–32.
[33] X.F. Yang, D.J. Evans, B. Chen, G.M. Megson, H. Lai, On the maximal connected component of hypercube with faulty vertices, International Journal of Computer Mathematics 81 (2004) 515–525.
[34] X.F. Yang, D.J. Evans, G.M. Megson, On the maximal connected component of hypercube with faulty vertices II, International Journal of Computer Mathematics 81 (2004) 1175–1185.
[35] X.F. Yang, D.J. Evans, G.M. Megson, On the maximal connected component of a hypercube with faulty vertices III, International Journal of Computer Mathematics 83 (2006) 27–37.
[36] X.F. Yang, G.M. Megson, Y.Y. Tang, Y.K. Xing, Largest connected component of a star graph with faulty vertices, International Journal of Computer Mathematics 85 (12) (2008) 1771–1778.
[37] X.F. Yang, G.M. Megson, X.F. Liao, J.Q. Cao, Generalized matching networks and their properties, International Journal of Parallel, Emergent and Distributed Systems 22 (3) (2007) 185–192.
[38] S.M. Zhou, W.J. Xiao, Conditional diagnosability of alternating group networks, Information Processing Letters 110 (10) (2010) 403–409.
[39] Q. Zhu, S.-Y. Liu, M. Xu, On conditional diagnosability of the folded hypercubes, Information Sciences 178 (4) (2008) 1069–1077.