J Comb Optim
DOI 10.1007/s10878-013-9597-9

On the p-reinforcement and the complexity

You Lu · Fu-Tao Hu · Jun-Ming Xu

Abstract Let $G = (V, E)$ be a graph and p be a positive integer. A subset $S \subseteq V$ is called a p-dominating set if each vertex not in S has at least p neighbors in S. The p-domination number $\gamma_p(G)$ is the size of a smallest p-dominating set of G. The p-reinforcement number $r_p(G)$ is the smallest number of edges whose addition to G results in a graph G' with $\gamma_p(G') < \gamma_p(G)$. In this paper, we give an original study on the p-reinforcement, determine $r_p(G)$ for some graphs such as paths, cycles and complete t-partite graphs, and establish some upper bounds on $r_p(G)$. In particular, we show that the decision problem on $r_p(G)$ is NP-hard for a general graph G and a fixed integer $p \geq 2$.

Keywords Domination · p-Domination · p-Reinforcement · NP-hard

1 Induction

For notation and graph-theoretical terminology not defined here we follow Xu (2003). Specifically, let $G = (V, E)$ be an undirected graph without loops and multi-edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set, where $E \neq \emptyset$.

Y. Lu
Department of Applied Mathematics, Northwestern Polytechnical University,
Xi'an 710072, Shanxi, People's Republic of China
e-mail: luyou@nwpu.edu.cn

F.-T. Hu · J.-M. Xu (✉)
Department of Mathematics, Wentsun Wu Key Laboratory of CAS, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China
e-mail: xujm@ustc.edu.cn

F.-T. Hu
e-mail: hufu@mail.ustc.edu.cn

Published online: 23 February 2013

For $x \in V$, the *open neighborhood*, the *closed neighborhood* and the *degree* of x are denoted by $N_G(x) = \{y \in V : xy \in E\}$, $N_G[x] = N_G(x) \cup \{x\}$ and $deg_G(x) = |N_G(x)|$, respectively. $\delta(G) = \min\{deg_G(x) : x \in V\}$ and $\Delta(G) = \max\{deg_G(x) : x \in V\}$ are the minimum degree and the maximum degree of G, respectively. For any $X \subseteq V$, let $N_G[X] = \cup_{x \in X} N_G[x]$.

For a subset $D \subseteq V$, let $\overline{D} = V \setminus D$. The notation G^c denotes the complement of G, that is , G^c is the graph with vertex-set $V(G)$ and edge-set $\{xy : xy \notin E(G)$ for any $x, y \in V(G)\}$. For $B \subseteq E(G^c)$, we use $G + B$ to denote the graph with vertex-set V and edge-set $E \cup B$. For convenience, we denote $G + \{xy\}$ by $G + xy$ for an $xy \in E(G^c)$.

A nonempty subset $D \subseteq V$ is called a *dominating set* of G if $|N_G(x) \cap D| \geq 1$ for each $x \in \overline{D}$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of all dominating sets in G. The domination is a classical concept in graph theory. The early literature on the domination with related topics is, in detail, surveyed in the two books by Haynes et al. (1998a,b).

In 1985, Fink and Jacobson introduced the concept of a generalization domination in a graph. Let p be a positive integer. A subset $D \subseteq V$ is a *p-dominating set* of G if $|N_G(x) \cap D| \geq p$ for each $x \in \overline{D}$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of all p-dominating sets in G. A p-dominating set with cardinality $\gamma_p(G)$ is called a γ_p-set of G. For $S, T \subseteq V$, the set S can p-dominate T in G if $|N_G(x) \cap S| \geq p$ for every $x \in T \setminus S$. Clearly, the 1-dominating set is the classical dominating set, and so $\gamma_1(G) = \gamma(G)$. The p-domination is investigated by many authors (see, for example, Blidia and Chellali 2005; Blidia et al. 2006; Chellali et al. 2012; Caro and Roditty 1990; Favaron 1985). Very recently, Chellali et al. (2012) have given an excellent survey on this topics. The following are two simple observations.

Observation 1.1 *If G is a graph with $|V(G)| \geq p$, then $\gamma_p(G) \geq p$.*

Observation 1.2 *Every p-dominating set of a graph contains all vertices of degree at most $p - 1$.*

Clearly, addition of some extra edges to a graph could result in decrease of its domination number. In 1990, Kok and Mynhardt first investigated this problem and proposed the concept of the reinforcement number. The *reinforcement number* $r(G)$ of a graph G is defined as the smallest number of edges whose addition to G results in a graph G' with $\gamma(G') < \gamma(G)$. By convention $r(G) = 0$ if $\gamma(G) = 1$.

The reinforcement number has received much research attention (see, for example, Blair et al. 2008; Dunbar et al. 1998; Huang et al. 2009), and its many variations have also been well described and studied in graph theory, including total reinforcement Henning et al. (2011); Sridharan et al. (2007), independence reinforcement Zhang et al. (2003), fractional reinforcement Chen et al. (2003); Domke and Laskar (1997) and so on. In particular, Blair et al. (2008), Hu and Xu (2012), independently, showed that the problem determining $r(G)$ for a general graph G is NP-hard.

Motivated by the work of Kok and Mynhardt (1990), in this paper, we introduce the p-reinforcement number, which is a natural extension of the reinforcement number. The *p-reinforcement number* $r_p(G)$ of a graph G is the smallest number of edges of

G^c that have to be added to G in order to reduce $\gamma_p(G)$, that is

$$r_p(G) = \min\{|B| : B \subseteq E(G^c) \text{ with } \gamma_p(G+B) < \gamma_p(G)\}.$$

It is clear that $r_1(G) = r(G)$. By Observation 1.1, we can also make a convention, $r_p(G) = 0$ if $\gamma_p(G) \le p$. Thus $r_p(G)$ is well-defined for any graph G and integer $p \ge 1$. In this paper, we always assume $\gamma_p(G) > p$ when we consider the p-reinforcement number for a graph G.

The rest of this paper is organized as follows. In Sect. 2 we present an equivalent parameter for calculating the p-reinforcement number of a graph. As its applications, we determine the values of the p-reinforcement numbers for special classes of graphs such as paths, cycles and complete t-partite graphs in Sect. 3, and show that the decision problem on p-reinforcement is NP-hard for a general graph and a fixed integer $p \ge 2$ in Sect. 4. Finally, we establish some upper bounds for the p-reinforcement number of a graph G by terms of other parameters of G in Sect. 5.

2 Preliminary

Let G be a graph with $\gamma(G) > 1$ and $B \subseteq E(G^c)$ with $|B| = r(G)$ such that $\gamma(G+B) < \gamma(G)$. Let X be a γ-set of $G+B$. Then $|B| \ge |V(G) \setminus N_G[X]|$. On the other hand, given any set $X \subseteq V(G)$, we can always choose a subset $B \subseteq E(G^c)$ with $|B| = |V(G) \setminus N_G[X]|$ such that X dominates $G+B$. It is a simple observation that, to calculate $r(G)$, Kok and Mynhardt (1990) proposed the following parameter

$$\eta(G) = \min\{|V(G) \setminus N_G[X]| : \ X \subseteq V(G), |X| < \gamma(G)\}, \tag{2.1}$$

and showed $r(G) = \eta(G)$. We can refine this technique to deal with the p-reinforcement number $r_p(G)$.

Let G be a graph with $\gamma_p(G) > p$. For any $X \subseteq V(G)$, let

$$X^* = \{x \in \overline{X} : |N_G(x) \cap X| < p\}. \tag{2.2}$$

Let $B \subseteq E(G^c)$ with $|B| = r_p(G)$ such that $\gamma_p(G+B) < \gamma_p(G)$, and let X be a γ_p-set of $G+B$. Then

$$|B| \ge \sum_{x \in X^*} (p - |N_G(x) \cap X|).$$

On the other hand, given any set $X \subseteq V(G)$ with $|X| \ge p$, we can always choose a subset $B \subseteq E(G^c)$ with

$$|B| = \sum_{x \in X^*} (p - |N_G(x) \cap X|)$$

such that X can p-dominate $G + B$. Motivated by this observation, we introduce the following notations. For a subset $X \subseteq V(G)$,

$$\eta_p(x, X, G) = \begin{cases} p - |N_G(x) \cap X| & \text{if } x \in X^* \\ 0 & \text{otherwise} \end{cases} \quad \text{for } x \in V(G), \tag{2.3}$$

$$\eta_p(S, X, G) = \sum_{x \in S} \eta_p(x, X, G) \quad \text{for } S \subseteq V(G), \text{ and} \tag{2.4}$$

$$\eta_p(G) = \min\{\eta_p(V(G), X, G) : |X| < \gamma_p(G)\}. \tag{2.5}$$

A subset $X \subseteq V(G)$ is called an η_p*-set* of G if $\eta_p(G) = \eta_p(V(G), X, G)$. Clearly, for any two subsets $S', S \subseteq V(G)$ and two subsets $X', X \subseteq V(G)$,

$$\begin{aligned} \eta_p(S', X, G) &\le \eta_p(S, X, G) \quad \text{if } S' \subseteq S, \\ \eta_p(S, X, G) &\le \eta_p(S, X', G) \quad \text{if } |X'| \le |X|. \end{aligned}$$

Thus, we have the following simple observation.

Observation 2.1 *If X is an η_p-set of a graph G, then $|X| = \gamma_p(G) - 1$.*

The following result shows that computing $r_p(G)$ can be referred to computing $\eta_p(G)$ for a graph G with $\gamma_p(G) \ge p + 1$.

Theorem 2.2 *For any graph G and positive integer p, $r_p(G) = \eta_p(G)$ if $\gamma_p(G) > p$.*

Proof Let X be an η_p-set of G. Then $|X| = \gamma_p(G) - 1$ by Observation 2.1. Let $Y = \{y \in V(G) : \eta_p(y, X, G) > 0\}$. Then $Y = X^*$ is contained in $\overline{X}$, where X^* is defined in (2.2). Thus, $\eta_p(G) = \eta_p(X^*, X, G)$. We construct a new graph G' from G, for each $y \in X^*$, by adding $\eta_p(y, X, G)$ edges of G^c to G joining y to $\eta_p(y, X, G)$ vertices in X. Clearly, X is a p-dominating set of G', that is, $\gamma_p(G') \le |X|$. Let $B = E(G') - E(G)$. Then

$$\gamma_p(G) = |X| + 1 > |X| \ge \gamma_p(G') = \gamma_p(G + B),$$

which implies $r_p(G) \le |B|$. It follows that

$$r_p(G) \le |B| = \sum_{y \in X^*} \eta_p(y, X, G) = \eta_p(X^*, X, G) = \eta_p(G). \tag{2.6}$$

On the other hand, let B be a subset of $E(G^c)$ such that $|B| = r_p(G)$ and $\gamma_p(G + B) = \gamma_p(G) - 1$. Let $G' = G + B$ and X' be a γ_p-set of G'. For every $xy \in B$, X' cannot p-dominate the graph $G' - xy$ by the minimality of B. This fact means that only one of x and y is in X'. Without loss of generality, assume $y \in \overline{X'}$. Since X' cannot p-dominate y in $G' - xy$ and so in G, $|N_G(y) \cap X'| < p$. Let Z be all end-vertices of edges in B and $Y = \overline{X'} \cap Z$.

Since X' is a γ_p-set of G', $|N_{G'}(u) \cap X'| \geq p$ for any $u \in \overline{X'}$. In other words, any $u \in \overline{X'}$ with $|N_G(u) \cap X'| < p$ must be in Y. It follows that

$$\sum_{u \in \overline{X'}} \eta_p(u, X', G) = \sum_{y \in Y} (p - |N_G(y) \cap X'|) = |B|. \tag{2.7}$$

By (2.7), we immediately have that

$$\eta_p(G) \leq \eta_p(V(G), X', G) = \sum_{u \in \overline{X'}} \eta_p(u, X', G) = |B| = r_p(G).$$

Combining this with (2.6), we obtain $r_p(G) = \eta_p(G)$, and so the theorem follows. □

Note that when $p = 1$, X^* defined in (2.2) is $V(G) \setminus N_G[X]$. This fact means that $\eta(G)$ defined in (2.1) is a special case of $p = 1$ in (2.5), that is, $\eta_1(G) = \eta(G)$. Thus, the following corollary holds immediately.

Corollary 2.1 (Kok and Mynhardt 1990) *$r(G) = \eta(G)$ if $\gamma(G) > 1$.*

Using Observation 1.2 and Theorem 2.2, the following corollary is obvious.

Corollary 2.2 *Let $p \geq 1$ be an integer and G be a graph with $\gamma_p(G) > p$. If $\Delta(G) < p$, then*

$$r_p(G) = p - \Delta(G).$$

3 Some exact values

In this section we will use Theorem 2.2 to calculate the p-reinforcement numbers for some classes of graphs.

We first determine the p-reinforcement numbers for paths and cycles. Let P_n and C_n denote, respectively, a path and a cycle with n vertices. When $p = 1$, Kok and Mynhardt (1990) proved that $r(P_n) = r(C_n) = i$ if $n = 3k + i \geq 4$, where $i \in \{1, 2, 3\}$. We will give the exact values of $r_p(P_n)$ and $r_p(C_n)$ for $p \geq 2$. The following observation is simple but useful.

Observation 3.1 *For integer $p \geq 2$,*

$$\gamma_p(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1 & \text{if } \ p = 2 \\ n & \text{if } \ p \geq 3 \end{cases} \text{ and } \gamma_p(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } \ p = 2 \\ n & \text{if } \ p \geq 3. \end{cases}$$

Theorem 3.2 *Let $p \geq 2$ be an integer. If $\gamma_p(P_n) > p$ then*

$$r_p(P_n) = \begin{cases} 2 & \text{if } p = 2 \text{ and } n \text{ is odd} \\ 1 & \text{if } p = 2 \text{ and } n \text{ is even} \\ p - 2 & \text{if } p \geq 3. \end{cases}$$

Proof Let $P_n = x_1x_2\cdots x_n$ and X be an η_p-set of P_n. By Theorem 2.2 and $\gamma_p(P_n) > p$, $r_p(P_n) = \eta_p(P_n) = \eta_p(V(P_n), X, P_n) \geq 1$. For $p \geq 3$, it is easy to see that $r_p(P_n) = p - 2$ by Corollary 2.2. Assume that $p = 2$ below.

If n is even, then by Observation 3.1, $\gamma_2(P_n) - \gamma_2(C_n) = 1$, which implies that $r_2(P_n) \leq 1$. Furthermore, $r_2(P_n) = 1$.

If n is odd, then $\gamma_2(P_n) = \frac{n+1}{2}$ by Observation 3.1, and so $n \geq 5$ since $\gamma_2(P_n) > 2$. Let

$$X' = \bigcup_{i=1}^{\frac{n-1}{2}} \{x_{2i}\}.$$

Clearly, $|X'| = \frac{n-1}{2} = \gamma_2(P_n) - 1$. So

$$\eta_2(V(P_n), X, P_n) \leq \eta_2(V(P_n), X', P_n) = \eta_2(x_1, X', P_n) + \eta_2(x_n, X', P_n) = 2.$$

Suppose that $\eta_2(V(P_n), X, P_n) = 1$. Then X can 2-dominate either $V(P_n) \setminus \{x_1\}$ or $V(P_n) \setminus \{x_n\}$. In both cases, we have

$$|X| \geq \gamma_2(P_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \frac{n-1}{2} + 1,$$

which contradicts with $|X| = \frac{n-1}{2}$. Hence $r_2(P_n) = \eta_2(V(P_n), X, P_n) = 2$. □

Theorem 3.3 *Let $p \geq 2$ be an integer. If $\gamma_p(C_n) > p$ then*

$$r_p(C_n) = \begin{cases} 2 & \text{if } p = 2 \text{ and } n \text{ is odd} \\ 4 & \text{if } p = 2 \text{ and } n \text{ is even} \\ p - 2 & \text{if } p \geq 3. \end{cases}$$

Proof Let $C_n = x_1x_2\cdots x_nx_1$. If $p \geq 3$ then the result holds obviously by Corollary 2.2. In the following, we only need to calculate the values of $r_p(C_n)$ for $p = 2$. Let X be an η_2-set of C_n. Then $r_2(C_n) = \eta_2(C_n) = \eta_2(V(C_n), X, C_n)$ by Theorem 2.2. Note that $n \geq 5$ since $\gamma_2(C_n) = \lceil \frac{n}{2} \rceil > 2$.

If n is odd, then let

$$X' = \bigcup_{i=1}^{\frac{n-1}{2}} \{x_{2i-1}\}.$$

Clearly, $|X'| = \frac{n-1}{2} = \gamma_2(C_n) - 1$ by Observation 3.1, and $\eta_2(V(C_n), X', C_n) = \eta_2(x_{n-1}, X', C_n) + \eta_2(x_n, X', C_n) = 2$. So

$$r_2(C_n) = \eta_2(V(C_n), X, C_n) \leq \eta_2(V(C_n), X', C_n) = 2.$$

Since X is not a 2-dominating set of C_n, there must be two adjacent vertices, denoted by x_i and x_{i+1}, of C_n not in X. This fact means that $\eta_2(x_i, X, C_n) \geq 1$ and $\eta_2(x_{i+1}, X, C_n) \geq 1$. So

$$r_2(C_n) = \eta_2(V(C_n), X, P_n) \geq \eta_2(x_i, X, C_n) + \eta_2(x_{i+1}, X, C_n) \geq 2.$$

Hence $r_2(C_n) = 2$.

If n is even, then $n \geq 6$. Deleting X and all vertices 2-dominated by X from C_n, we can obtain a result graph, denoted by H, each of whose components is a path with length at least 2. Denote all components of H by $H_1, \cdots, H_h$, where $h \geq 1$. In the case that $h = 1$ and the length of H_1 is equal to one, X can 2-dominate a subgraph of C_n that is isomorphic to P_{n-2}. By Observation 3.1,

$$|X| \geq \gamma_2(P_{n-2}) = \lfloor \frac{n-2}{2} \rfloor + 1 = \frac{n}{2},$$

which contradicts that $|X| = \gamma_2(C_n) - 1 = \lceil \frac{n}{2} \rceil - 1 = \frac{n}{2} - 1$. In other cases, we can find that

$$r_2(C_n) = \eta_2(V(C_n), X, C_n) \geq 4.$$

Let

$$X'' = \bigcup_{i=1}^{\frac{n}{2}-1} \{x_{2i-1}\}.$$

It is easy to check that $|X''| = \frac{n}{2} - 1 = \gamma_2(C_n) - 1$ and $\eta_2(V(C_n), X'', C_n) = 4$. So

$$r_2(C_n) = \eta_2(V(C_n), X, C_n) \leq \eta_2(V(C_n), X'', C_n) = 4.$$

Hence $r_2(C_n) = 4$ and so the theorem is true. □

Next we consider the p-reinforcement number for a complete t-partite graph $K_{n_1,\cdots,n_t}$. To state our results, we need some symbols. For any subset $X = \{n_{i_1}, \dots, n_{i_r}\}$ of $\{n_1, \cdots, n_t\}$, define

$$|X| = r \text{ and } f(X) = \sum_{j=1}^{r} n_{i_j}.$$

For convenience, let $|X| = 0$ and $f(X) = 0$ if $X = \emptyset$. let

$$\mathscr{X} = \{X : X \text{ is a subset of } \{n_1, \cdots, n_t\} \text{ with } f(X) \geq \gamma_p(G)\}$$

and, for every $X \in \mathscr{X}$, define

$$f^*(X) = \max\{f(Y) : Y \text{ is a subset of } X \text{ with } |Y| = |X| - 1 \text{ and } f(Y) < p\}.$$

Theorem 3.4 *For any integer $p \geq 1$ and a complete t-partite graph $G = K_{n_1,\cdots,n_t}$ with $t \geq 2$ and $\gamma_p(G) > p$,*

$$r_p(G) = \min\{(p - f^*(X))(f(X) - \gamma_p(G) + 1) : X \in \mathscr{X}\}.$$

Proof Let $N = \{n_1, \cdots, n_t\}$ and $V(G) = V_1 \cup \cdots \cup V_t$ be the vertex-set of G such that $|V_i| = n_i$ for each $i = 1, \cdots, t$. Let

$$m = \min\{(p - f^*(X))(f(X) - \gamma_p(G) + 1) : X \in \mathscr{X}\}.$$

We first prove that $r_p(G) \leq m$. Let $X \subseteq \mathscr{X}$ (without loss of generality, assume $X = \{n_1, \cdots, n_k, n_{k+1}\}$ for some $0 \leq k \leq t-1$) such that

$$f^*(X) = n_1 + \cdots + n_k \text{ and } (p - f^*(X))(f(X) - \gamma_p(G) + 1) = m.$$

By $X \subseteq \mathscr{X}$, we know that $n_{k+1} = f(X) - f^*(X) \geq \gamma_p(G) - f^*(X)$. So we can pick a vertex-subset V'_{k+1} from V_{k+1} such that $|V'_{k+1}| = \gamma_p(G) - f^*(X) - 1$. Let

$$D = V_1 \cup \cdots \cup V_k \cup V'_{k+1}.$$

Clearly, $|D| = \gamma_p(G) - 1$. Since $\gamma_p(G) > p$, $|D| \geq p$ and so D can p-dominate $\cup_{i=k+2}^{t} V_i$. Hence by the definition of $\eta_p(V(G), D, G)$,

$$\begin{aligned}
\eta_p(V(G), D, G) &= \eta_p(V(G) \setminus D, D, G) \\
&= \sum_{v \in V_{k+1} \setminus V'_{k+1}} \eta_p(v, D, G) + \sum_{i=k+2}^{t} \eta_p(V_i, D, G) \\
&= |V_{k+1} \setminus V'_{k+1}|(p - f^*(X)) + 0 \\
&= (p - f^*(X))[n_{k+1} - (\gamma_p(G) - f^*(X) - 1)] \\
&= (p - f^*(X))(f(X) - \gamma_p(G) + 1) \\
&= m.
\end{aligned}$$

By Theorem 2.2, we have $r_p(G) = \eta_p(G) \leq \eta_p(V(G), D, G) = m$.

On the other hand, we will show that $r_p(G) \geq m$. For any subset M of N, we use $I(M)$ to denote the subindex-sets of all elements in M, that is,

$$I(M) = \{i : n_i \in M\}.$$

Let S be an η_p-set of G and let

$$\begin{aligned}
Y &= \{n_i : \quad |V_i \cap S| = |V_i| \quad \text{for } 1 \leq i \leq t, \text{ and} \\
A &= \{n_i : \quad 0 < |V_i \cap S| < |V_i| \quad \text{for } 1 \leq i \leq t.
\end{aligned}$$

Thus

$$f(Y \cup A) = f(Y) + f(A) = \sum_{i \in I(Y)} |V_i| + \sum_{i \in I(A)} |V_i| \geq |S| = \gamma_p(G) - 1 \quad (3.1)$$

by Observation 2.1. Since $\cup_{i \in I(Y)} V_i$ ($\subseteq S$) cannot p-dominate G,

$$f(Y) = \sum_{i \in I(Y)} n_i = |\cup_{i \in I(Y)} V_i| < p. \quad (3.2)$$

Hence, by (3.1) and $\gamma_p(G) > p$,

$$f(A) \geq \gamma_p(G) - 1 - f(Y) > \gamma_p(G) - p - 1 \geq 0,$$

which implies that $|A| \geq 1$.

Claim $|A| = 1$.

Proof of Claim Suppose that $|A| \geq 2$. Then we can choose i and j from $I(A)$ such that $i \neq j$. By the definition of A, we have $0 < |V_i \cap S| < |V_i|$ and $0 < |V_j \cap S| < |V_j|$. Therefore, we can pick two vertices x and y from $V_i \cap S$ and $V_j \setminus S$, respectively. Let

$$S' = (S \setminus \{x\}) \cup \{y\}.$$

Obviously, $|S'| = |S| = \gamma_p(G) - 1$, $|V_i \cap S'| = |V_i \cap S| - 1$ and $|V_j \cap S'| = |V_j \cap S| + 1$.

Note that G is a complete t-partite graph. For any $v \in V(G)$, we can easily find the value of $\eta_p(v, S', G) - \eta_p(v, S, G)$ by the definitions of $\eta_p(v, S', G)$ and $\eta_p(v, S, G)$ as follows:

$$\eta_p(v, S', G) - \eta_p(v, S, G) = \begin{cases} (p - |S| + |V_i \cap S| - 1) - 0 & \text{if } v = x \\ -1 & \text{if } v \in V_i \setminus S \\ 0 - (p - |S| + |V_j \cap S|) & \text{if } v = y \\ 1 & \text{if } v \in (V_j \setminus S) \setminus \{y\} \\ 0 & \text{otherwise.} \end{cases}$$

Since S is an η_p-set of G and $|S'| = |S|$, we have

$$\begin{aligned} 0 &\leq \eta_p(V(G), S', G) - \eta_p(V(G), S, G) \\ &= \sum_{v \in V(G)} (\eta_p(v, S', G) - \eta_p(v, S, G)) \\ &= (p - |S| + |V_i \cap S| - 1) - |V_i \setminus S| - (p - |S| + |V_j \cap S|) + |(V_j \setminus S) \setminus \{y\}| \\ &= (|V_i \cap S| - |V_i \setminus S|) - (|V_j \cap S| - |V_j \setminus S|) - 2. \end{aligned}$$

This means that

$$(|V_i \cap S| - |V_i \setminus S|) \geq (|V_j \cap S| - |V_j \setminus S|) + 2.$$

However, by the symmetry of V_i and V_j, we can also obtain

$$(|V_j \cap S| - |V_j \setminus S|) \geq (|V_i \cap S| - |V_i \setminus S|) + 2$$

by applying the similar discussion. This is a contradiction, and so the claim holds. □

By **Claim**, we can assume that $I(A) = \{h\}$. From the definitions of Y and A, we have $|Y \cup A| = |Y| + 1$ and

$$f(Y \cup A) = \sum_{i \in I(Y)} |V_i| + |V_h| \geq \sum_{i \in I(Y)} |V_i| + (|V_h \cap S| + 1) = |S| + 1 = \gamma_p(G).$$

It follows that $Y \cup A \in \mathscr{X}$. Thus, by (3.2) and the definition of $f^*(Y \cup A)$, we have $f(Y) \leq f^*(Y \cup A)$. Since $\gamma_p(G) > p$, $|S| = \gamma_p(G) - 1 \geq p$, and so S p-dominates $V(G) \setminus (\cup_{i \in I(Y \cup A)} V_i)$. Therefore, by Theorem 2.2,

$$\begin{aligned}
r_p(G) = \eta_p(G) = \eta_p(V(G), S, G) &= \eta_p(V(G) \setminus S, S, G) \\
&= \sum_{v \in V_h \setminus S} \eta_p(v, S, G) \\
&= (p - f(Y))|V_h \setminus S| \\
&= (p - f(Y))[|V_h| - (|S| - f(Y))] \\
&= (p - f(Y))(f(Y \cup A) - \gamma_p(G) + 1) \\
&\geq (p - f^*(Y \cup A))(f(Y \cup A) - \gamma_p(G) + 1) \\
&\geq m.
\end{aligned}$$

This completes the proof of the theorem. □

For example, let $G = K_{2,2,10,17}$ and $p = 11$. Then $\gamma_{11}(G) = 12$, and so

$$\mathscr{X} = \{\{17\}, \{2, 10\}, \{2, 17\}, \{10, 17\}, \{2, 2, 10\}, \{2, 2, 17\}, \{2, 10, 17\}, \{2, 2, 10, 17\}\}.$$

By Theorem 3.4, for any $X \in \mathscr{X}$, we have that

$$f^*(X) = \begin{cases} 0 & \text{if } X = \{17\}, \{2, 10, 17\} \text{ or } \{2, 2, 10, 17\}; \\ 2 & \text{if } X = \{2, 17\}; \\ 4 & \text{if } X = \{2, 2, 10\} \text{ or } \{2, 2, 17\}; \\ 10 & \text{if } X = \{2, 10\} \text{ or } \{10, 17\}. \end{cases}$$

Hence

$$\begin{aligned}
r_{11}(G) &= \min\{(11 - f^*(X))(f(X) - \gamma_{11}(G) + 1) : X \in \mathscr{X}\} \\
&= \min\{(11 - f^*(X))(f(X) - 11) : X \in \mathscr{X}\} \\
&= (11 - f^*(\{2, 10\}))(f(\{2, 10\}) - 11) \\
&= 1.
\end{aligned}$$

4 Complexity

Blair et al. (2008) and Hu and Xu (2012), independently, showed that the 1-reinforcement problem is NP-hard. Thus, for any positive integer p, the p-reinforcement problem is also NP-hard since the 1-reinforcement is a sub-problem of the p-reinforcement problem.

For each fixed p, p-dominating set is polynomial-time computable (see Downey and Fellows (1995, 1997) for definitions and discussion). However, the p-reinforcement number problem is hard even for specific values of the parameters. In this section, we will consider the following decision problem.

***p*-Reinforcement**

Instance: A graph G, p (≥ 2) is a fixed integer.

Question: Is $r_p(G) \leq 1$?

We will prove that ***p*-Reinforcement** $(p \geq 2)$ is also NP-hard by describing a polynomial transformation from the following NP-hard problem (see Garey and Johnson 1979).

3-Satisfiability (3SAT)

Instance: A set $U = \{u_1, \ldots, u_n\}$ of variables and a collection $\mathscr{C} = \{C_1, \ldots, C_m\}$ of clauses over U such that $|C_i| = 3$ for $i = 1, 2, \ldots, m$. Furthermore, every literal is used in at least one clause.

Question: Is there a satisfying truth assignment for C?

Theorem 4.1 *For a fixed integer $p \geq 2$, **p-Reinforcement** is NP-hard.*

Proof Let $U = \{u_1, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, \ldots, C_m\}$ be an arbitrary instance I of **3SAT**. We will show the NP-hardness of ***p*-Reinforcement** by reducing **3SAT** to it in polynomial time. To this aim, we construct a graph G as follows:

a. For each variable $u_i \in U$, associate a graph H_i, where H_i can be obtained from a complete graph K_{2p+2} with vertex-set $\{u_i, \overline{u}_i\} \cup (\cup_{j=1}^{p}\{v_{i_j}, \overline{v}_{i_j}\})$ by deleting the edge-subset $\cup_{j=1}^{p-1}\{u_i\overline{v}_{i_j}, \overline{u}_i v_{i_j}\}$;
b. For each clause $C_j \in \mathscr{C}$, create a single vertex c_j and join c_j to the vertex u_i (resp. $\overline{u}_i$) in H_i if and only if the literal u_i (resp. $\overline{u}_i$) appears in clause C_j for any $i \in \{1, \ldots, n\}$;
c. Add a complete graph T $(\cong K_p)$ and join all of its vertices to each c_j.

For convenience, let $X_i = \cup_{j=1}^{p}\{v_{i_j}\}$ and $\overline{X}_i = \cup_{j=1}^{p}\{\overline{v}_{i_j}\}$. Then $V(H_i) = \{u_i, \overline{u}_i\} \cup X_i \cup \overline{X}_i$. Use H_0 to denote the induced subgraph by $\{c_1, \cdots, c_m\} \cup V(T)$.

It is clear that the construction of G can be accomplished in polynomial time. To complete the proof of the theorem, we only need to prove that $\mathscr{C}$ is satisfiable if and only if $r_p(G) = 1$. We first prove the following two claims.

Claim 1 Let D be a γ_p-set of G. Then $|D| = p(n+1)$, moreover, $|V(H_i) \cap D| = p$ and $|\{u_i, \overline{u}_i\} \cap D| \leq 1$ for each $i \in \{1, 2, \ldots, n\}$.

Proof of Claim 1 Suppose there is some $i \in \{1, 2, \cdots, n\}$ such that $|V(H_i) \cap D| < p$. Then there must be a vertex, say x, of $V(H_i) \setminus D$ such that $N_G(x) \subseteq V(H_i)$. And so $|N_G(x) \cap D| \leq |V(H_i) \cap D| < p$, which contradicts that D is a γ_p-set of G. Thus

$|V(H_i) \cap D| \geq p$ for each $i \in \{0, 1, \cdots, n\}$, and so

$$\gamma_p(G) = |D| = \sum_{i=0}^{n} |V(H_i) \cap D| \geq p(n+1). \tag{4.1}$$

On the other hand, let

$$D' = \bigcup_{i=1}^{n} [(X_i - \{v_{i_p}\}) \cup \{\overline{u}_i\}] \cup V(T).$$

Clearly, $|D'| = p(n+1)$ and D' is a p-dominating set of G. Hence by (4.1),

$$p(n+1) \leq \sum_{i=0}^{n} |V(H_i) \cap D| = \gamma_p(G) \leq |D'| = p(n+1),$$

which implies that $\gamma_p(G) = p(n+1)$ and $|V(H_i) \cap D| = p$ for each $0 \leq i \leq n$. Furthermore, if $|\{u_i, \overline{u}_i\} \cap D| = 2$ then $|(X_i \cup \overline{X}_i) \cap D| = p - 2$. So we can choose a vertex from $X_i \cup \overline{X}_i$ that is not p-dominated by D. This is impossible since D is a γ_p-set of G, and so $|\{u_i, \overline{u}_i\} \cap D| \leq 1$. The claim holds. □

Claim 2 If there is an edge $e = xy \in G^c$ such that $\gamma_p(G + e) < \gamma_p(G)$, then any γ_p-set D_e of $G + e$ satisfies the following properties.

(i) $|V(H_i) \cap D_e| = p$ and $|\{u_i, \overline{u}_i\} \cap D_e| \leq 1$ for each $i \in \{1, \cdots, n\}$;
(ii) $\{c_1, \cdots, c_m\} \cap D_e = \emptyset$, and so $|V(T) \cap D_e| = p - 1$;
(iii) One of x and y belongs to $V(T) \setminus D_e$ and the other belongs to $H \cap D_e$, where $H = \cup_{i=1}^{n} V(H_i)$.

Proof of Claim 2 Because D_e is a γ_p-set of $G + e$ and $\gamma_p(G + e) < \gamma_p(G)$, one of x and y is not in D_e but the other is in D_e. Without loss of generality, say $x \notin D_e$ and $y \in D_e$. It is clear that $|N_G(x) \cap D_e| = p - 1$. Since vertex x is the unique vertex not be p-dominated by D_e, we have

$$\eta_p(V(G), D_e, G) = \eta_p(x, D_e, G) = p - (p - 1) = 1. \tag{4.2}$$

Let

$$D = D_e \cup \{x\}.$$

Then D is a p-dominating set of G and $|D| = |D_e| + 1 = \gamma_p(G + e) + 1 \leq \gamma_p(G)$. That is, D is a γ_p-set of G. By Claim 1,

$$|V(H_i) \cap D| = p \text{ for each } i = 0, 1, \cdots, n, \tag{4.3}$$

and $|\{u_i, \overline{u}_i\} \cap D_e| \leq |\{u_i, \overline{u}_i\} \cap D| \leq 1$ for $1 \leq i \leq n$.

Suppose that there exists some $i \in \{1, \cdots, n\}$ such that $|V(H_i) \cap D_e| \neq p$. Then by (4.3), $x \in V(H_i)$ and $|V(H_i) \cap D_e| = p-1$. Thus every vertex in $(X_i \cup \overline{X}_i) \setminus (D_e \cup \{x\})$ is dominated by at most $p-1$ vertices of D_e. Hence by $|X_i \cup \overline{X}_i| = 2p$,

$$\begin{aligned}\eta_p(V(G), D_e, G) &\geq \eta_p(X_i \cup \overline{X}_i, D_e, G) \geq |(X_i \cup \overline{X}_i) \setminus D_e| - 1 \\ &\geq 2p - (p-1) - 1 > 1,\end{aligned}$$

which contradicts with (4.2). Hence (i) holds.

Suppose that there is some $j \in \{1, \cdots, m\}$ such that $c_j \in D_e$. By (i) and (4.3), $x \in V(H_0)$ and so $|V(H_0) \cap D_e| = |V(H_0) \cap D| - 1 = p - 1$. Hence $|V(T) \cap D_e| \leq p - 2$ by $V(H_0) = \{c_1, \cdots, c_m\} \cup V(T)$. Since each vertex of T $(\cong K_p)$ has exact $p-1$ neighbors in D_e,

$$\eta_p(V(G), D_e, G) \geq \eta_p(V(T), D_e, G) = |V(T) \setminus D_e| = p - |V(T) \cap D_e| \geq 2.$$

This contradicts with (4.2). Thus $\{c_1, \cdots, c_m\} \cap D_e = \emptyset$, and so $|V(T) \cap D_e| = |V(H_0) \cap D_e| = p - 1$. Hence (ii) holds.

By (ii), T has a unique vertex, say z, not in D_e. From $|N_G(z) \cap D_e| = |V(H_0) \cap D_e| = p - 1$, the vertex z is not p-dominated by D_e. However, x is the unique vertex not be p-dominated by D_e in G by (4.2). Thus $z = x$, and so $x = z \in V(T) \setminus D_e$. By the construction of G and $xy \in G^c$, it is clear that $y \in (\cup_{i=1}^n V(H_i)) \cap D_e$. Hence (iii) holds. □

We now show that $\mathscr{C}$ is satisfiable if and only if $r_p(G) = 1$.

If $\mathscr{C}$ is satisfiable, then $\mathscr{C}$ has a satisfying truth assignment $t : U \to \{T, F\}$. According to this satisfying assignment, we can choose a subset S from $V(G)$ as follows:

$$S = S_0 \cup S_1 \cup \cdots \cup S_n,$$

where S_0 consists of $p-1$ vertices of T and

$$S_i = \begin{cases} u_i \cup (\overline{X}_i - \{\overline{v}_{i_p}\}) & \text{if } t(u_i) = T \\ \overline{u}_i \cup (X_i - \{v_{i_p}\}) & \text{if } t(u_i) = F \end{cases} \text{ for each } i \in \{1, \cdots, n\}.$$

It can be verified easily that $|S| = p(n+1) - 1 = \gamma_p(G) - 1$ and $\cup_{i=1}^n V(H_i)$ can be p-dominated by S. Since t is a satisfying true assignment for $\mathscr{C}$, each clause $C_j \in \mathscr{C}$ contains at least one true literal. That is, the corresponding vertex c_j has at least one neighbor in $\{u_1, \bar{u}_1 \cdots, u_n, \bar{u}_n\} \cap S$ by the definitions of G and S, and so every $c_j \in \{c_1, \cdots, c_m\}$ has at least p neighbors in S since $S_0 \subseteq N_G(c_j)$. Note that the unique vertex in $V(T) \setminus S_0$ has exact $p-1$ neighbors in S. By Theorem 2.2 and $|S| = \gamma_p(G) - 1$,

$$r_p(G) = \eta_p(G) \leq \eta_p(V(G), S, G) = \eta_p(V(T) \setminus S_0, S, G) = p - (p-1) = 1.$$

Furthermore, we have $r_p(G) = 1$ since $\gamma_p(G) > p$ by Claim 1.

Conversely, assume $r_p(G) = 1$. That is, there exists an edge $e = xy$ in G^c such that $\gamma_p(G+e) < \gamma_p(G)$. Let D_e be a γ_p-set of $G+e$. Define $t : U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if vertex } u_i \in D_e \\ F & \text{if vertex } u_i \notin D_e \end{cases} \quad \text{for } i = 1, \cdots, n. \tag{4.4}$$

We will show that t is a satisfying truth assignment for $\mathscr{C}$. Let C_j be an arbitrary clause in $\mathscr{C}$. By (ii) and (iii) of Claim 2, the corresponding vertex c_j is not in D_e and $|N_G(c_j) \cap D_e| \geq p$ since $c_j \notin \{x, y\}$. Then there must be some $i \in \{1, \cdots, n\}$ such that

$$|\{u_i, \overline{u}_i\} \cap N_G(c_j) \cap D_e| = 1, \tag{4.5}$$

since T contains exact $p-1$ vertices of D_e by (i) and (ii) of Claim 2. If $u_i \in N_G(c_j) \cap D_e$, then $u_i \in C_j$ and $t(u_i) = T$ by the construction of G and (4.4). If $\overline{u}_i \in N_G(c_j) \cap D_e$, then the literal $\overline{u}_i$ belongs to C_j by the construction of G. Note that $u_i \notin D_e$ from $\overline{u}_i \in D_e$ and (i) of Claim 2. This means that $t(u_i) = F$ by (4.4). Hence $t(\overline{u}_i) = T$. The arbitrariness of C_j with $1 \leq j \leq m$ shows that all the clauses in $\mathscr{C}$ is satisfied by t. That is, $\mathscr{C}$ is satisfiable.

The theorem follows. □

5 Upper bounds

For a graph G and $p = 1$, Kok and Mynhardt (1990) provided an upper bound for $r(G)$ in terms of the smallest private neighborhood of a vertex in some γ-set of G. Let $X \subseteq V(G)$ and $x \in X$. The *private neighborhood* of x with respect to X is defined as the set

$$PN(x, X, G) = N_G[x] \setminus N_G[X \setminus \{x\}]. \tag{5.1}$$

Set

$$\mu(X, G) = \min\{|PN(x, X, G)| : \ x \in X\}$$

and

$$\mu(G) = \min\{\mu(X, G) : \ X \text{ is a } \gamma\text{-set of } G\}. \tag{5.2}$$

Using this parameter, Kok and Mynhardt (1990) showed that $r(G) \leq \mu(G)$ if $\gamma(G) \geq 2$ with equality if $\gamma(G) = 1$. We generalize this result to any positive integer p.

In order to state our results, we need some notations. Let $X \subseteq V(G)$ and $x \in X$. A vertex $y \in \overline{X}$ is called a *p-private neighbor* of x with respect to X if $xy \in E(G)$ and $|N_G(y) \cap X| = p$. The *p-private neighborhood* of x with respect to X is defined as

$$PN_p(x, X, G) = \{y : \ y \text{ is a } p-\text{private neighbor of } x \text{ with respect to } X\}. \tag{5.3}$$

Let

$$\mu_p(x, X, G) = |PN_p(x, X, G)| + \max\{0, p - |N_G(x) \cap X|\}, \tag{5.4}$$
$$\mu_p(X, G) = \min\{\mu_p(x, X, G) : x \in X\}, \text{ and} \tag{5.5}$$
$$\mu_p(G) = \min\{\mu_p(X, G) : \ X \text{ is a } \gamma_p - \text{set of } G\}. \tag{5.6}$$

Theorem 5.1 *For any graph G and positive integer p,*

$$r_p(G) \le \mu_p(G)$$

with equality if $r_p(G) = 1$.

Proof If $\gamma_p(G) \le p$, then $r_p(G) = 0 \le \mu_p(G)$ by our convention. Assume that $\gamma_p(G) \ge p + 1$ below. Let X be a γ_p-set of G and $x \in X$ such that

$$\mu_p(G) = \mu_p(X, G) = \mu_p(x, X, G).$$

Since $|X| = \gamma_p(G) \ge p + 1$, we can choose a vertex, say u_y, from $X \setminus N_G(y)$ for each $y \in PN_p(x, X, G)$, and a subset X' with $|X'| = \max\{0, p - |N_G(x) \cap X|\}$ from $X \setminus N_G[x]$.

Let

$$G' = G + \{yu_y : \ y \in PN_p(x, X, G)\} + \{xv : \ v \in X'\}.$$

Obviously, $X \setminus \{x\}$ is a p-dominating set of G', which implies that

$$r_p(G) \le |PN_p(x, X, G)| + |X'| = \mu_p(x, X, G) = \mu_p(G).$$

Assume $r_p(G) = 1$. Then $\gamma_p(G) \ge p + 1$ and there exists an edge $xy \in E(G^c)$ such that $\gamma_p(G + xy) = \gamma_p(G) - 1$. Let $G' = G + xy$ and X be a γ_p-set of G'. Without loss of generality, assume that $x \in X$ and $y \in \overline{X}$. Clearly, y is a p-private neighbor of x with respect to X in G and $X \cup \{y\}$ is a γ_p-set of G, which implies

$$PN_p(y, X \cup \{y\}, G) = \emptyset \text{ and } p - |N_G(y) \cap (X \cup \{y\})| = 1,$$

that is, $\mu_p(y, X \cup \{y\}, G) = 1$. It follows that

$$r_p(G) \le \mu_p(G) \le \mu_p(X \cup \{y\}, G) \le \mu_p(y, X \cup \{y\}, G) = 1.$$

Thus, $r_p(G) = \mu_p(G) = 1$. The theorem follows. □

Note that $|PN_p(x, X, G)| \le deg_G(x)$ for any $X \subseteq V(G)$ and $x \in X$. By Theorem 5.1, we obtain the following corollary immediately.

Corollary 5.1 *For any graph G with maximum degree* $\Delta(G)$ *and positive integer* $p, r_p(G) \le \Delta(G) + p$.

Corollary 5.2 *Let p be a positive integer and G be a graph with minimum degree $\delta(G)$. If $\delta(G) < p$, then $r_p(G) \leq \delta(G) + p$.*

Proof Let X be a γ_p-set of G and $x \in V(G)$ with degree $\delta(G)$. Since $deg_G(x) = \delta(G) < p$, $x \in X$ by Observation 1.2. Note that $|PN_p(x, X, G)| \leq deg_G(x) = \delta(G)$ and $p - |N_G(x) \cap X| \leq p$. By Theorem 5.1,

$$\begin{aligned} r_p(G) &\leq \mu_p(G) \\ &\leq \mu_p(x, X, G) \\ &= |PN_p(x, X, G)| + \max\{0, p - |N_G(x) \cap X|\} \\ &\leq \delta(G) + p. \end{aligned}$$

The corollary follows. □

Consider $p = 1$. Let $X \subseteq V(G)$ and $x \in X$. If x is not an isolated vertex of the induced subgraph $G[X]$, then $PN(x, X, G)$ defined in (5.1) does not contain x and $\max\{0, 1 - |N_G(x) \cap X|\} = 0$ in (5.4). Otherwise, $PN(x, X, G)$ contains x and $\max\{0, 1 - |N_G(x) \cap X|\} = 1$. Notice that $PN_1(x, X, G)$ defined in (5.3) does not contain x. Hence, by (5.5),

$$\mu_1(x, X, G) = PN_1(x, X, G) + \max\{0, 1 - |N_G(x) \cap X|\} = |PN(x, X, G)|.$$

This fact means that $\mu(G)$ defined in (5.2) is a special case of $p = 1$ in (5.6), that is, $\mu_1(G) = \mu(G)$. Thus, by Theorem 5.1, the following corollary holds immediately.

Corollary 5.3 (Kok and Mynhardt 1990) *For any graph G with $\gamma(G) \geq 2$, $r(G) \leq \mu(G)$, with quality if $r(G) = 1$.*

Acknowledgments The work was supported by NNSF of China (No.10711233) and the Fundamental Research Fund of NPU (No. JC201150)

References

Blidia M, Chellali M, Favaron O (2005) Independence and 2-domination in trees. Australas J Combin 33:317–327

Blidia M, Chellali M, Volkmann L (2006) Some bounds on the p-domination number in trees. Discret Math 306:2031–2037

Blair JRS, Goddard W, Hedetniemi ST, Horton S, Jones P, Kubicki G (2008) On domination and reinforcement numbers in trees. Discret Math 308:1165–1175

Chellali M, Favaron O, Hansberg A, Volkmann L (2012) k-domination and k-independence in graphs: a survey. Graphs Combin. 28(1):1-55

Caro Y, Roditty Y (1990) A note on the k-domination number of a graph. Int J Math Sci 13:205–206

Chen X, Sun L, Ma D (2003) Bondage and reinforcement number of γ_f for complete multipartite graph. J Beijing Inst Technol 12:89–91

Dunbar JE, Haynes TW, Teschner U, Volkmann L (1998) Bondage, insensitivity, and reinforcement. In: Haynes TW, Hedetniemi ST, Slater PJ (eds) Domination in graphs: advanced topics, pp 471–489. Monogr. Textbooks Pure Appl. Math., 209, Marcel Dekker, New York, .

Domke GS, Laskar RC (1997) The bondage and reinforcement numbers of γ_f for some graphs. Discret Math 167/168:249–259

Downey RG, Fellows MR (1995) Fixed-parameter tractability and completeness I: basic results. SIAM J Comput 24:873–921

Downey RG, Fellows MR (1997) Fixed-parameter tractability and completeness II: on completeness for $W[1]$. Theor Comput Sci 54(3):465–474

Favaron O (1985) On a conjecture of Fink and Jacobson concerning k-domination and k-dependence. J Combin Theory Ser B 39:101–102

Fink JF, Jacobson MS (1985) n-domination in graphs. In: Alavi Y, Schwenk AJ (eds) Graph theory with applications to algorithms and computer science. Wiley, New York, pp 283–300

Garey MR, Johnson DS (1979) Computers and intractability: a guide to the theory of NP-completeness. Freeman, San Francisco

Haynes TW, Hedetniemi ST, Slater PJ (1998a) Fundamentals of domination in graphs. Marcel Deliker, New York

Haynes TW, Hedetniemi ST, Slater PJ (1998b) Domination in graphs: advanced topics. Marcel Deliker, New York

Henning MA, Rad NJ, Raczek J (2011) A note on total reinforcement in graph. Discret Appl Math 159:1443–1446

Hu F-T, Xu J-M (2012) On the complexity of the bondage and reinforcement problems. J Complex. 28(2): 192-201

Huang J, Wang JW, Xu J-M (2009) Reinforcement number of digraphs. Discret Appl Math 157:1938–1946

Kok J, Mynhardt CM (1990) Reinforcement in graphs. Congr Numer 79:225–231

Sridharan N, Elias MD, Subramanian VSA (2007) Total reinforcement number of a graph. AKCE Int J Graph Comb 4(2):192–202

Xu J-M (2003) Theory and application of graphs. Kluwer Academic Publishers, Dordrecht

Zhang JH, Liu HL, Sun L (2003) Independence bondage and reinforcement number of some graphs. Trans Beijing Inst Technol 23:140–142