Trees with maximum $p$-reinforcement number*

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A B S T R A C T
Let $G = (V, E)$ be a graph and $p$ a positive integer. The $p$-domination number $\gamma_p(G)$ is the minimum cardinality of a set $D \subseteq V$ with $|N_G(x) \cap D| \geq p$ for all $x \in V \setminus D$. The $p$-reinforcement number $r_p(G)$ is the smallest number of edges whose addition to $G$ results in a graph $G'$ with $\gamma_p(G') < \gamma_p(G)$. It is showed by Lu et al. (2013) that $r_p(T) \leq p + 1$ for any tree $T$ and $p \geq 2$. This paper characterizes all trees attaining this upper bound when $p \geq 3$.

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1. Introduction

For graph-theoretical terminology and notation not defined here we follow [19]. Let $G = (V, E) = (V(G), E(G))$ be a simple graph and $x \in V$. The neighborhood and degree of $x$ are $N_G(x) = \{y \in V : xy \in E\}$ and $d_G(x) = |N_G(x)|$, respectively. If $d_G(x) = 1$, then $x$ is called a leaf and its unique neighbor is called a stem. The set of leaves of $G$ is denoted by $L(G)$. Let $p \geq 1$ be an integer and $X \subseteq V$ with $x \in X$. A vertex $y \in N_G(x)$ is called a $p$-private neighbor of $x$ with respect to $X$ if $y \in V - X$ and $|N_G(y) \cap X| = p$. We use $N_p(x, X, G)$ to denote the set of $p$-private neighbors of $x$ with respect to $X$ in $G$.

For $X \subseteq V$, the subgraph induced by $X$ (resp. $V - X$) is denoted by $G[X]$ (resp. $G - X$). The complement $G^c$ of $G$ is the simple graph with vertex-set $V$ and edge-set $E(G^c) = \{xy : xy \notin E\}$. For $B \subseteq E(G^c)$, $G + B$ denotes the graph obtained from $G$ by adding $B$. To simplify notation, for $x \in V$ and subgraph $H \subseteq G$, we write $G - x$ and $G - H$ for $G - \{x\}$ and $G - V(H)$, respectively.

Let $p \geq 1$ be an integer and $X \subseteq V$. For $Y \subseteq V$, $Xp$-dominates $Y$ in $G$ if for each $y \in Y$, either $y \in X$ or $|N_G(y) \cap X| \geq p$. We write $X >_p Y$ if $Xp$-dominates $Y$, and write $X \not>_p Y$ otherwise. In particular, if $X >_p V$ then $X$ is called a $p$-dominating set, abbreviated $DS_p$, of $G$. The $p$-domination number $\gamma_p(G)$ is the minimum cardinality of a $DS_p$ of $G$. A $DS_p$ with cardinality $\gamma_p(G)$ is called a $\gamma_p$-set of $G$. The $p$-reinforcement number $r_p(G)$ is the smallest number of edges in $G^c$ that have to be added to $G$ in order to reduce $\gamma_p(G)$, that is

$$r_p(G) = \min(|B| : B \subseteq E(G^c) \text{ with } \gamma_p(G + B) < \gamma_p(G)).$$

By convention,

$$r_p(G) = 0 \text{ if } \gamma_p(G) \leq p.$$  

Clearly, $\gamma_1$ and $r_1$ are the well-known domination $\gamma$ and reinforcement $r$, respectively.

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The concept of $p$-domination was introduced by Fink and Jacobson [10] in 1985 and has been well studied for recent decade (see, for example, [2, 4, 7–9, 11]). Chellali et al. [5] gave an excellent survey on this topic. The $p$-reinforcement number, introduced by Lu, Hu and Xu [17], is a parameter for measuring vulnerability of $p$-domination, is also a natural extension of the classical reinforcement number which was introduced by Kok and Mynhardt [15] and studied by a number of authors (see, for example, [6, 12–14, 20]). Motivated by the work of these authors, Lu, Hu and Xu [17] studied $p$-reinforcement, found a method to determine $r_p$ in terms of $y_p$ and showed that the decision problem on $r_p$ is NP-hard and established some upper bounds.

Surprisingly, for a tree $T$ of order $n$, the known upper bounds for $r_p(T)$ are of distinct forms according to $p = 1$ and $p \geq 2$. For $p = 1$, Blair et al. [1] gave a sharp upper bound $r_1(T) \leq \frac{n}{2}$. For $p \geq 2$, however, there is an upper bound for $r_p(T)$ which is independent of $n$.

**Theorem 1.1** (Lu, Hu and Xu [17]), $r_p(T) \leq p + 1$ for any tree $T$ and $p \geq 2$.

In this paper we characterize all extremal trees in Theorem 1.1 for $p \geq 3$ by a recursive construction. The rest of this paper is organized as follows. The main result of this paper is stated in Section 2. To prove the main result, we propose two needed parameters $\eta_p$ and $\mu_p$ in Section 3 and use them to establish some structural properties of a tree $T$ with $r_p(T) = p + 1$ for $p \geq 3$ in Section 4. In Section 5 we complete the proof of the main result. A conclusion is in Section 6.

## 2. Main result

Throughout this paper, we always suppose that $p \geq 3$ is an integer. In this section we will give a constructive characterization of trees with $p$-reinforcement number $p + 1$. First, we state two known results.

**Lemma 2.1.** Every $DS_p$ of a graph contains all vertices of degree less than $p$.

**Lemma 2.2** (Lu et al. [16]), Let $p \geq 2$ be an integer and $D$ be a $DS_p$ of a tree $T$. Then $D$ is the unique $\gamma_p$-set of $T$ if and only if for each $x \in D$ with $d_T(x) \geq p$, $|N_D(x) \cap D| \leq p - 2$ or $|N_D(x, D, T)| \geq 2$.

Let $t \geq p$ be an integer. The spider $S_t$ is a tree obtained from a star $K_{1,t}$ by attaching one leaf at each leaf of $K_{1,t}$. Two important trees $F_{p-1}$ and $F_{t,p-1}$ in our construction are shown in Fig. 1, where $F_{p-1}$ (resp. $F_{t,p-1}$) is obtained from a star $K_{1,2}$ (resp. a spider $S_t$) by attaching $p - 1$ leaves at each leaf of $K_{1,t}$ (resp. $S_t$).

In Fig. 1, we call $y$ the center of $F_{p-1}$ (resp. $F_{t,p-1}$). It is obvious that the set of black vertices in $F_{p-1}$ (resp. $F_{t,p-1}$ for $t \geq p$) is the unique $\gamma_p$-set of $F_{p-1}$ (resp. $F_{t,p-1}$). For a star $K_{1,m}$ ($m \geq 2$), the unique stem is also called the center of $K_{1,m}$.

The construction of the graph $G \oplus xy$ is to get the graph with vertex-set $V(G) \cup \{xy\}$ and edge-set $E(G) \cup \{xy\} \cup E(H)$, where $x \in V(G)$ and $y \in V(H)$.

**Definition 2.3.** Let $G$ be a tree with a unique $\gamma_p$-set $X$. A new tree $T$ is constructed from $G$ by the following operation $\sigma$.

\[ \sigma : T = G \oplus xy. \]

where $H \in \{K_{1,p-1}, K_{1,p}, F_{p-1}, F_{t,p-1}\}$, $y$ is the center of $H$ and $x$ must fulfill the following conditions:

1. $x \in X$ if $H = K_{1,p-1}$.
2. $x \notin X$ if $H = K_{1,p}$.
3. $x \in X$ and $|N_p(x, X, G)| \geq \min\{p + 1, |N_p(x) \cap X| + 2\}$ if $H = F_{p-1}$.
4. $x$ is an arbitrary vertex in $G$ if $H = F_{t,p-1}$.

Note that the tree $H \in \{K_{1,p-1}, F_{p-1}, F_{t,p-1}\}$ with $t \geq p$ has a unique $\gamma_p$-set, denoted by $\gamma_H$. By Definition 2.3, the following observation follows almost immediately from Lemmas 2.1 and 2.2.

**Observation 2.4.** Let $p \geq 3$ and $t \geq p$ be two integers and $G$ a tree different to $K_{1,p-1}$ with a unique $\gamma_p$-set $X$. Then the tree $T = G \oplus xy$ obtained from $G$ by operation $\sigma$ has a unique $\gamma_p$-set

\[ X \cup \{U(H) \text{ if } H = K_{1,p-1}, \gamma_H \text{ if } H \in \{K_{1,p}, F_{p-1}, F_{t,p-1}\}. \]
Since the star $K_{1,p}$ has a unique $\gamma_p$-set, by Observation 2.4, we can define a family $\mathcal{F}_p$ of trees as follows.

$\mathcal{F}_p = \{ T : T$ is obtained from the star $K_{1,p}$ via a finite series of operation $\otimes \}$. It must be pointed out that $K_{1,p} \not\in \mathcal{F}_p$. We now are ready to establish our main result whose proof is postponed to Section 5.

**Theorem 2.5.** For an integer $p \geq 3$ and a tree $T$, $r_p(T) = p + 1$ if and only if $T \in \mathcal{F}_p$.

3. Notations and lemmas

The notations $\eta_p$ and $\mu_p$ introduced by Lu, Hu and Xu [17] play important roles in the study of $p$-reinforcement. In this section, we present their definitions and fundamental results.

Let $G = (V,E)$ be a graph and $X \subseteq V$. For each vertex $x \in V$, define

$$\eta_p(x, X, G) = \begin{cases} p - |N_G(x) \cap X| & \text{if } X \not\supseteq_p x; \\ 0 & \text{if } X \supseteq_p x. \end{cases}$$

(3.1)

If $|X| \geq p$, then there is a subset $B_x \subseteq E(G')$ with $|B_x| = \eta_p(x, X, G)$ such that $X \supseteq_p x$ in $G + B_x$ and $S$ is a $DS_p$ of $G + (\cup_{x \in V} B_x)$, which implies that $r_p(G) \leq | \cup_{x \in V} B_x | = \sum_{x \in V} \eta_p(x, X, G)$ by the definition of $r_p$. Motivated by this inequality, Lu, Hu and Xu [17] define

$$\eta_p(S, X, G) = \sum_{x \in S} \eta_p(x, X, G) \quad \text{for } S \subseteq V,$$

(3.2)

and prove the following two lemmas.

**Lemma 3.1** (Lu, Hu and Xu [17]). Let $p$ be an integer and $G$ a graph. If $\gamma_p(G) > p$, then

$$r_p(G) = \min \{ \eta_p(V(G), X, G) : X \subseteq V(G) \text{ with } |X| < \gamma_p(G) \}.$$

Let $G$ be a graph and $X \subseteq V(G)$. If $|X| < \gamma_p(G)$ and $\eta_p(V(G), X, G) = r_p(G)$, then $X$ is called an $\eta_p$-set of $G$.

**Lemma 3.2** (Lu, Hu and Xu [17]). Let $p$ be an integer and $G$ be a graph. If $X$ is an $\eta_p$-set of $G$, then $|X| = \gamma_p(G) - 1$.

The following observation is trivial by (3.1) and (3.2).

**Observation 3.3.** Let $G$ be a graph and $S, X \subseteq V(G)$. Then

1. $\eta_p(S, X, G) \geq \eta_p(S_1, X, G)$ for any $S_1 \subseteq S$.
2. $\eta_p(S, X, G) \leq \eta_p(S, X_1, G)$ for any $X_1 \subseteq X$.
3. $\eta_p(S, X, H) \geq \eta_p(S, X, H)$ for any supergraph $H$ of $G$.

By the definitions of $\eta_p$ and $\oplus$, the following lemma follows from Observation 3.3 and Lemmas 3.2 and 3.1 immediately.

**Lemma 3.4.** Let $G_i$ be a graph with $x_i \in V(G_i)$ for $i = 1, 2$ and $H = G_1 \oplus x_1 x_2 G_2$.

1. For any $X \subseteq V(G_i)$ ($i = 1, 2$),

$$\eta_p(V(G_1), X_1, G_1) - \eta_p(V(G_2), X_1 \cup X_2, H) = \begin{cases} 1 & \text{if } X_1 \not\supseteq_p x_1 \text{ and } X_2 \not\supseteq_p x_2; \\ 0 & \text{otherwise}. \end{cases}$$

(2) If $\gamma_p(G_1) > p$ and $\gamma_p(H) \geq \gamma_p(G_1) + \gamma_p(G_2)$, then $r_p(H) \leq r_p(G_1)$.

Now we present the parameter $\mu_p$. Let $G = (V, E)$ be a graph and $X \subseteq V$. For $x \in X$, define

$$\mu_p(x, X, G) = |N_p(x, X, G)| + \max \{ 0, p - |N_G(x) \cap X| \}.$$  

(3.3)

**Lemma 3.5** (Lu, Hu and Xu [17]). For a graph $G$,

$$r_p(G) \leq \min \{ \mu_p(x, X, G) : X \text{ is a } \gamma_p \text{-set of } G \text{ and } x \in X \}.$$  

4. Properties for a tree $T$ with $r_p(T) = p + 1$

In this section, we use the parameters $\eta_p$ and $\mu_p$ to establish some lemmas of a tree $T$ with $r_p(T) = p + 1$, which will be applied in the proof of Theorem 2.5.

**Lemma 4.1.** Let $p \geq 3$ and $T$ a tree with $r_p(T) = p + 1$. If $D$ is a $\gamma_p$-set of $T$, then

1. $N_p(x, D, T) \neq \emptyset$ for each $x \in D$.
2. $D$ is the unique $\gamma_p$-set of $T$. 

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Proof. Let $x$ be any vertex in $D$. Then

$$|N_p(x, D, T)| = \mu_p(x, D, T) - \max\{0, p - |N_T(x) \cap D|\} \quad \text{(by (3.3))}$$

$$\geq \mu_p(x, D, T) - p$$

$$\geq r_p(T) - p \quad \text{(by Lemma 3.5)}$$

$$= 1$$

and so the conclusion (1) holds.

We now prove the conclusion (2). Since $D$ is a $\gamma_p$-set of $T$ and $r_p(T) = p + 1 > 0$, $|D| = \gamma_p(T) \geq p + 1$ by (1.1). If $d_T(x) < p$ for any $x \in D$, then $D$ is the unique $\gamma_p$-set by Lemma 2.1, and so the conclusion follows. Assume now that there is some $x \in D$ such that $d_T(x) \geq p$. By Lemma 3.5 and (3.3),

$$p + 1 = r_p(T) \leq \mu_p(x, D, T) = |N_p(x, D, T)| + \max\{0, p - |N_T(x) \cap D|\},$$

that is,

$$|N_p(x, D, T)| \geq p + 1 - \max\{0, p - |N_T(x) \cap D|\}.$$ 

If $|N_T(x) \cap D| \geq p - 1$, then $\max\{0, p - |N_T(x) \cap D|\} \leq 1$, and so $|N_p(x, D, T)| \geq p \geq 3$. This fact implies that $D$ satisfies the second condition in Lemma 2.2, from which $D$ is the unique $\gamma_p$-set of $T$. The lemma follows. ■

Let $p \geq 3$ and $T$ a tree with $r_p(T) = p + 1$. Through this paper, we use $\mathcal{W}_T$ to represent the unique $\gamma_p$-set of $T$. For any $xy \in E(T)$, let $T_x$ denote the component of $T - x$ containing $y$.

Lemma 4.2. Let $p \geq 3$ and $T$ a tree with $r_p(T) = p + 1$. For any $x \in \mathcal{W}_T$ and $y \in N_T(x)$,

1. If $y \notin N_p(x, \mathcal{W}_T, T)$, then $r_p(T_x) = p + 1$ and $\mathcal{W}_T \cap V(T_y)$.
2. If $y \in N_p(x, \mathcal{W}_T, T)$, then
   (a) either $T_y$ is a star $K_{1,p-1}$ with center $y$ or $r_p(T_y) = 1$ and $\mathcal{W}_T \cap V(T_y)$ is an $\eta_p$-set of $T_y$, and
   (b) $\eta_p(V(T_y), X, T_y) \geq p - 1$ for $X \subseteq V(T_y)$ with $x \in X$ and $|X| \leq |\mathcal{W}_T \cap V(T_y)|$.

Proof. Let $Y = \mathcal{W}_T \cap V(T_y)$ and $Z = \{z \in V(T_y) \mid Y : |N_T(z) \cap \mathcal{W}_T| = p\}$. Note that $x \in \mathcal{W}_T$ and $\mathcal{W}_T$ is the unique $\gamma_p$-set of $T$.

Since $p \geq 3$, $Y \neq \emptyset$ and so $Z \neq \emptyset$ by Lemma 4.1(1). For any $z \in Z$, since $\mathcal{W}_T \supseteq z$ and $x \in \mathcal{W}_T$, 

$$|N_T(z) \cap \mathcal{W}_T| = |N_T(z) \cap \mathcal{W}_T - x| \geq p - 1,$$

with equality if and only if $z = y$. Hence either $T_y$ is a star $K_{1,p-1}$ with center $y$ or $|V(T_y)| > p$. In the former case, the conclusion (b) in (2) is trivial by (3.1) and (3.2). Thus, to prove the lemma, we only need to consider the case of $|V(T_y)| > p$.

We claim that $\gamma_p(T_y) > p$. Suppose, to be contrary, that $\gamma_p(T_y) \leq p$. Furthermore, $\gamma_p(T_y) = p$ since $|V(T_y)| > p$. Note that $p \geq 3$ and $p$ vertices in a tree have at most one common neighbor. Since $T_y$ is a tree, $T_y = K_{1,p}$. Let $z$ be the center of $T_y$. Since $p \geq 3$ and $\mathcal{W}_T$ is a $DS_p$ of $T$, it follows from Lemmas 2.1 and 4.1(1) that $L(T_y) = Y$ and $z \in Z$. If $z = y$, then $|N_T(z) \cap \mathcal{W}_T| = |T_y \cup \{x\}| = p + 1$, which contradicts that $z \in Z$. If $z \neq y$, then $y \in \mathcal{W}_T$ and $N_T(y) = \{x, z\}$, furthermore $N_T(y) \cap \mathcal{W}_T = \{y\}$ and $N_p(y, \mathcal{W}_T, T) = 1$. By (3.3),

$$\mu_p(y, \mathcal{W}_T, T) = |N_p(y, \mathcal{W}_T, T)| + \max\{0, p - |N_T(y) \cap \mathcal{W}_T|\} = 1 + (p - 1) = p,$$

from which and Lemma 3.5 we obtain that $r_p(T) \leq \mu_p(y, \mathcal{W}_T, T) = p$, a contradiction. The claim holds.

Firstly, we prove (1). Let $T - T_y = T_z$. Then $T = T_x \cup T_y$. Since $x \in \mathcal{W}_T$ and $y \notin N_p(x, \mathcal{W}_T, T)$, $\mathcal{W}_T \cap V(T_z) \supseteq Y = \mathcal{W}_T \cap V(T_y)$. It follows that $\gamma_p(T_y) = |\mathcal{W}_T| = |\mathcal{W}_T \cap V(T_z) + |Y| = \gamma_p(T_y) + \gamma_p(T_z)$. Furthermore, $\gamma_p(T) = \gamma_p(T_y) + \gamma_p(T_z)$ since the union of a $\gamma$-set of $T_y$ and a $\gamma_p$-set of $T_z$ is a $DS_p$ of $T$. So $Y$ is a $\gamma_p$-set of $T_y$. By Theorem 1.1 and Lemma 3.4(2), 

$p + 1 \geq r_p(T_y) = r_p(T) = p + 1$ and so $\mathcal{W}_T = Y$ by Lemma 4.1(2).

Secondly, we prove the conclusion (a) of (2). Since $y \in N_p(x, \mathcal{W}_T, T)$ and $\mathcal{W}_T \supseteq V(T_y)$, $|N_T(y) \cap \mathcal{W}_T - x| = p - 1$ and $Y \supseteq V(T_y) - \{y\}$. Thus, by (3.2) and (3.1),

$$\eta_p(V(T_y), Y, T_y) = \eta_p(y, Y, T_y) + \sum_{z \in V(T_y) - \{y\}} \eta_p(z, Y, T_y) = 1.$$ (4.1)

We claim that $|Y| < \gamma_p(T_y)$. Assume, to the contrary, that $|Y| \geq \gamma_p(T_y)$. Let $Y' = \gamma_p$-set of $T_y$. Since $x \in \mathcal{W}_T$ and $\mathcal{W}_T$ is a unique $\gamma_p$-set of $T$, $\mathcal{W}_T \supseteq \gamma_p \supseteq V(T) - V(T_x)$. So $(\mathcal{W}_T - Y) \cup Y' \supseteq V(T)$ and

$$|\mathcal{W}_T - Y| \cup Y' = (|\mathcal{W}_T| - |Y|) + |Y'| = \gamma_p(T) - |Y| + \gamma_p(T_y) \leq \gamma_p(T).$$

This fact means that $(\mathcal{W}_T - Y) \cup Y'$ is a $\gamma_p$-set of $T$ different from $\mathcal{W}_T$, a contradiction. The claim holds. Therefore, by Lemma 3.1 and (4.1),

$$r_p(T_y) \leq \eta_p(V(T_y), Y, T_y) = 1.$$ 

Note that $r_p(T_y) \geq 1$ by (1.1) since $\gamma_p(T_y) > p$. Thus, $r_p(T_y) = \eta_p(V(T_y), Y, T_y) = 1$ and $Y = \mathcal{W}_T \cap V(T_y)$ is an $\eta_p$-set of $T_y$. 

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Finally, we prove the conclusion (b) of (2). Let $X \subseteq V(T)$ such that $y \in X$ and $|X| \leq |Y|$. It suffices to show that 

$$\eta_T(V(T), X, T_p) \geq p - 1.$$

For any $u \in N_T(y)$, let $T_u$ be the component of $T - y$ containing $u$. Since $y \in X - \eta_T$ and $|X| \leq |Y|$, 

$$\sum_{u \in N_T(y)} |X \cap V(T_u)| = |X - \{y\}| \leq |Y| - 1 = \sum_{u \in N_T(y)} |\eta_T \cap V(T_u)| - 1,$$

which implies that there is some $u \in N_T(y)$ such that $|X \cap V(T_u)| < |\eta_T \cap V(T_u)|$. Let

$$S = (\eta_T - V(T_u)) \cup (X \cap V(T_u)).$$

Then $|S| < |\eta_T| = \gamma_T(T)$. Since $|N_T(y) \cap S| \geq |N_T(y) \cap (\eta_T - V(T_u))| \geq |N_T(y) \cap \eta_T| - 1 = p - 1$, by (3.1),

$$\eta_T(y, S, T) \leq 1. \tag{4.2}$$

Since $\eta_T - V(T_u) \supseteq V(T - T_u - y)$, by (3.1) and (3.2),

$$\eta_T(V(T), S, T) = \eta_T(V(T), S, T) + \eta_T(y, S, T) \tag{4.3}$$

It follows from Lemma 3.1 that

$$p + 1 = r_T(T) \leq \eta_T(V(T), S, T)$$

(by Observation 3.3(2) and (4.2)) \leq \eta_T(V(T), S \cap V(T_u), T) + 1

(by Observation 3.3(3)) \leq \eta_T(V(T_u), S \cap V(T_u), T) + 1,

that is, $\eta_T(V(T_u), S \cap V(T_u), T) \geq p$. Note that $T_y = (T_y - T_u) \cup T_u$ and $X \cap V(T_u) = S \cap V(T_u)$. Therefore,

$$\eta_T(V(T_u), X, T) \geq \eta_T(V(T), X, T_p) \geq \eta_T(V(T), S \cap V(T_u), T) - 1 \geq p - 1$$

as required. The lemma follows. \hfill \blacksquare

**Remark 4.3.** With a similar argument, both Lemmas 4.1 and 4.2 are also true for $p = 2$.

**Lemma 4.4.** Let $p \geq 3$, $T$ a tree with $r_T(T) = p + 1$, and $x \in \eta_T$ such that $\mu_T(x, \eta_T, T) \geq p + 2$. For any $X \subseteq V(T - x)$ with $|X| < \gamma_T(T)$, $\eta_T(V(T), X, T) \geq p + 2$.

**Proof.** Let $X$ be a counterexample to the lemma with $|X \cap \eta_T|$ as large as possible. Since $|X| < \gamma_T(T)$, Lemma 3.1 implies that $\eta_T(V(T), X, T) \geq r_T(T) = p + 1$, furthermore, $\eta_T(V(T), X, T) = p + 1 = r_T(T)$ since $X$ is a counterexample to the lemma. Thus $X$ is an $\eta_T$-set of $T$ and $|X| = \gamma_T(T) - 1$ by Lemma 3.2.

Let $N_T(x, \eta_T, T) = \{x_1, \ldots, x_t\}$ and $N_T(x) = \{x_1, \ldots, x_i, x_{i+1}, \ldots, x_d\}$, where $d = d_T(x)$. For each $i$, let $T_i$ be the component of $T - x$ containing $x_i$. Since $x \in \eta_T - X$ and $|X| = \gamma_T(T) - 1$, 

$$\sum_{i=1}^{d} |X \cap V(T_i)| = |X| = \gamma_T(T) - 1 = |\eta_T| - 1 = \sum_{i=1}^{d} |\eta_T \cap V(T_i)| \tag{4.4}$$

and, by (3.2) and Lemma 3.4(1),

$$p + 1 = \eta_T(V(T), X, T) = \eta_T(x, X, T) + \sum_{i=1}^{d} \eta_T(V(T_i), X \cap V(T_i), T_i). \tag{4.5}$$

**Claim 1.** For $t + 1 \leq i \leq d$, $X \cap V(T_i) = \eta_T \cap V(T_i)$ if $|X \cap V(T_i)| = |\eta_T \cap V(T_i)|$.

**Proof.** Suppose, to be contrary, that $X \cap V(T_i) \neq \eta_T \cap V(T_i).$ Let $X' = (\eta_T \cap V(T_i)) \cup (X - V(T_i))$. Then $x \notin X'$, $|X'| = |X| < \gamma_T(T)$ and $|X' \cap \eta_T| > |X \cap \eta_T|$.

Since $x_i \notin N_T(x, \eta_T, T)$, $\eta_T \cap V(T_i)$ is the unique $\gamma_T$-set of $T_i$ by Lemma 4.2(1), and so $X \cap V(T_i) \neq p V(T_i)$ but $X' \neq p V(T_i)$ in $T$. Thus,

$$\eta_T(V(T_i), X \cap V(T_i), T) \geq 1, \tag{4.6}$$

$$\eta_T(V(T_i), X', T) = 0. \tag{4.7}$$
Note that $T = T_i \oplus x(T - T_i)$ and $X' \cap V(T - T_i) = X' - V(T_i) = X - V(T_i)$. By (3.2),
\[
\eta_p(V(T), X', T) = \eta_p(V(T_i), X', T) + \eta_p(V(T - T_i), X', T)
\]
(by (4.7) and Lemma 3.4(1)) \leq 0 + \eta_p(V(T - T_i), X' - V(T_i), T - T_i) \leq \eta_p(V(T - T_i), X - V(T_i), T - T_i)
\]
(by Lemma 3.4(1)) \leq \eta_p(V(T - T_i), X, T) + 1
\]
(by (4.6)) \leq \eta_p(V(T - T_i), X, T) + \eta_p(V(T_i), X \cap V(T_i), T)
\]
(since $x \not\in X$) = \eta_p(V(T - T_i), X, T) + \eta_p(V(T_i), X, T)
\]
(by Lemma 3.1) = \eta_p(V(T), X, T) \geq r_p(T) = p + 1,

which means that $X'$ is another counterexample to Lemma 4.4 with $|X' \cap \mathcal{F}_r| > |X \cap \mathcal{F}_r|$, a contradiction to the choice of $X$. \qed

**Claim 2.** $|X \cap V(T_i)| = |\mathcal{F}_r \cap V(T_i)|$ for $1 \leq i \leq d$.

**Proof.** Suppose not, (4.4) implies that there is some $i$ such that $|X \cap V(T_i)| < |\mathcal{F}_r \cap V(T_i)|$. Let
\[
X' = (\mathcal{F}_r - V(T_i)) \cup (X \cap V(T_i)).
\]
Then $|X'| < |\mathcal{F}_r| = \gamma_p(T)$. Since $\mathcal{F}_r >_p V(T)$ and $x \in \mathcal{F}_r - V(T) \subseteq X', X' >_p V(T - T_i)$ in $T$ and so $\eta_p(V(T - T_i), X', T) = 0$. Therefore,
\[
\eta_p(V(T_i), X \cap V(T_i), T) \geq \eta_p(V(T_i), X', T) \geq r_p(T) = p + 1,
\]
(by Lemma 3.4(1)) from which and (4.5), it follows that
\[
\eta_p(V(T), X', T) = p + 1,
\]
\[
\eta_p(V(T), X \cap V(T_i), T) = 0 \quad \text{for } j \neq i,
\]
\[
\eta_p(x, X, T) = 0.
\]
\[\tag{4.10}
\]
Note that $X'$ is an $\eta_p$-set of $T$ by (4.8) since $r_p(T) = p + 1$ and $|X'| < \gamma_p(T)$. By Lemma 3.2, $|X'| = \gamma_p(T) - 1 = |\mathcal{F}_r| - 1$ and so
\[
|X \cap V(T_i)| = |X'| - |\mathcal{F}_r - V(T_i)| = |\mathcal{F}_r| - 1 - |\mathcal{F}_r - V(T_i)| = |\mathcal{F}_r \cap V(T_i)| - 1.
\]
\[\tag{4.11}
\]
On the other hand, (4.9) implies that, for $j \neq i$, $X \cap V(T_j) >_p V(T_j)$ in $T_j$ and it follows from Lemmas 4.2 and 3.2 that
\[
|X \cap V(T_j)| \geq \gamma_p(T_j) = \begin{cases} |\mathcal{F}_r \cap V(T_j)| + 1 & \text{if } 1 \leq j \leq t; \\ |\mathcal{F}_r \cap V(T_j)| & \text{if } t + 1 \leq j \leq d. \end{cases}
\]
\[\tag{4.12}
\]
By Claim 1, $X \cap V(T_j) = \mathcal{F}_r \cap V(T_j)$ for $3 \leq j \leq d$ and so
\[
|\mathcal{F}_r(x) \cap X| = \sum_{j=1}^{d} \left| \mathcal{F}_r(x) \cap (X \cap V(T_j)) \right| + \sum_{j=3}^{d} \left| \mathcal{F}_r(x) \cap (\mathcal{F}_r \cap V(T_j)) \right| \leq (1 + 1) + |\mathcal{F}_r(x) \cap \mathcal{F}_r| = 2 < p,
\]
which means that $\eta_p(x, X, T) \geq 1$ by (3.1) since $x \not\in X$, a contradiction to (4.10). The claim follows. \qed

We now continue to prove the lemma. Let $I = \{i \mid 1 \leq i \leq t \text{ and } x_i \in X\}$. Since $p \geq 3$, it follows from Claim 2 and the conclusion (b) in Lemma 4.2(2) that
\[
\sum_{i \in I} \eta_p(V(T_i), X \cap V(T_i), T_i) \geq \sum_{i \in I} (p - 1) = |I|(p - 1).
\]
\[\tag{4.13}
\]
For $i \in \{1, \ldots, t\} \setminus I$, Claim 2 and the conclusion (a) in Lemma 4.2(2) together imply that $|X \cap V(T_i)| = |\mathcal{W}_i \cap V(T_i)| < \gamma_p(T_i)$, and so $X \cap V(T_i) \not\subset V(T_i)$. Thus,

$$\sum_{i \in \{1, \ldots, t\} \setminus I} \eta_p(V(T_i), X \cap V(T_i), T_i) \geq \sum_{i \in \{1, \ldots, t\} \setminus I} 1 = t - |I| = |\mathcal{W}_p(x, \mathcal{W}_T, T)| - |I|.$$  \hfill (4.14) 

On the other hand, Claims 1 and 2 together imply that $X \cap V(T_i) = \mathcal{W}_T \cap V(T_i)$ for $t + 1 \leq t \leq d$. Note that $x_i \not\in \mathcal{W}_T$ for $1 \leq i \leq t$. Thus

$$|N_T(x) \cap X| = \sum_{i=1}^{d} |N_T(x) \cap (X \cap V(T_i))|$$

$$= |I| + \sum_{i=t+1}^{d} |N_T(x) \cap (\mathcal{W}_T \cap V(T_i))| = |N_T(x) \cap \mathcal{W}_T| + |I|.$$ \hfill (4.15)

Since $x \not\in X$, $\eta_p(x, X, T) = \max\{0, p - |N_T(x) \cap X|\}$ by (3.1). Therefore,

$$p + 1 = \eta_p(x, X, T) + \sum_{i=1}^{d} \eta_p(V(T_i), X \cap V(T_i), T_i) \quad \text{(by (4.5))}$$

$$\geq \max\{0, p - |N_T(x) \cap X|\} + |N_T(x, \mathcal{W}_T, T)| + |I|(p - 2) \quad \text{(by (4.13)-(4.14))}$$

$$\geq \max\{0, p - |N_T(x) \cap \mathcal{W}_T|\} - |I| + |N_T(x, \mathcal{W}_T, T)| + |I|(p - 2) \quad \text{(by (4.15))}$$

$$\geq \mu_p(x, \mathcal{W}_T, T) \quad \text{(by (3.3), since } p \geq 3)$$

$$\geq p + 2.$$ 

a contradiction. The lemma follows. \hfill \blacksquare

**Remark 4.5.** Lemma 4.4 is not true for $p = 2$.

Consider the tree $T$ shown in Fig. 2, in which $\mathcal{W}_T$ consists of all large circles in $T$, $\gamma_2(T) = |\mathcal{W}_T| = 17$, $r_2(T) = 3$, $x \in \mathcal{W}_T$ and $\mu_2(x, \mathcal{W}_T, T) = 4$ by (3.3). Let $X$ be the set of black vertices in $T$. Then $|X| = 16 < \gamma_2(T)$, however, $\eta_2(V(T), X, T) = 3$ by (3.1) and (3.2).

5. Proof of Theorem 2.5

In this section, we will complete the proof of Theorem 2.5. For the convenience, let $H_1 = K_{1,p-1}$, $H_2 = K_{1,p}$, $H_3 = F_{p-1}$ and $H_4 = F_{t,p-1}$ with $t \geq p$. Let $\psi_i$ denote the operation $\psi$ if $H = H_i$ for $i \in \{1, 2, 3, 4\}$ in Definition 2.3.

Let $p \geq 3$ and $T$ a tree obtained from a star $K_{1,p}$ by $\psi_i$ for some $i \in \{1, 2, 3, 4\}$. By the condition of $\psi_i$, $i \neq 3$ and

$$T = \begin{cases} F_{p-1} & \text{if } i = 1; \\ S_{p,p} & \text{if } i = 2; \\ F_{t+1,p-1} & \text{if } i = 3; \\ F_{t+1,p-1} \text{ or } K_{1,p} \oplus_{xy} F_{p-1} & \text{if } i = 4, \end{cases}$$

where $t \geq p$, $x$ is the center of $K_{1,p}$, and $S_{p,p}$ is a tree obtained from a complete graph $K_2$ by attaching $p$ leaves at each vertex of $K_2$. By calculating $\eta_p$ in (3.1) and (3.2), $r_p(T) = p + 1$ by Lemma 3.1.

The sufficiency of Theorem 2.5 follows from the above fact and the following lemma by the definition of $\mathcal{S}_p$.

**Lemma 5.1.** Let $p \geq 3$ be an integer and $G$ a tree with $r_p(G) = p + 1$. If $T$ is obtained from $G$ by operation $\psi_i$ for $i \in \{1, 2, 3, 4\}$, then $r_p(T) = p + 1$.

**Proof.** Since $T$ is obtained from $G$ by operation $\psi_i$ for some $i \in \{1, 2, 3, 4\}$,

$$T = G \oplus_{xy} H_i,$$

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where $y$ is the center of $H_i$ and $x$ satisfies the conditions in Definition 2.3. Note that $r_p(G) = p + 1$ and $G$ has the unique $γ_p$-set $\mathcal{G}$ by Lemma 4.1(2). By Observation 2.4,

$$γ_p(T) = |\mathcal{G}| + \begin{cases} p - 1 & \text{if } i = 1; \\ γ_p(H_i) & \text{if } i \in \{2, 3, 4\}. \end{cases} \quad (5.1)$$

To complete the proof of the lemma, it suffices to show that $r_p(T) \geq p + 1$ by Theorem 1.1. Suppose, to be contrary, that $r_p(T) \leq p$. Let $S$ be an $η_p$-set of $T$ such that

1. $|S \cap V(G)|$ is as small as possible,
2. subject to (1), $|S \cap \mathcal{G}|$ is as large as possible.

Then $η_p(V(T), S, T) = r_p(T) \leq p$ and $|S| = γ_p(T) - 1$ by Lemma 3.2. We will deduce a contradiction by distinguishing the following two cases.

Case 1. $|S \cap V(G)| \geq |\mathcal{G}|$.

We claim that $S \cap V(G) = \mathcal{G}$. Suppose, to be contrary, that $S \cap V(G) \neq \mathcal{G}$. Let

$$S' = \mathcal{G} \cup \left\{ S \cap V(H_i) \cup \{y\} \right\} \text{ if } |S \cap V(G)| = |\mathcal{G}|;$$

$$S' = \mathcal{G} \cup \left\{ S \cap V(H_i) \right\} \cup \{y\} \text{ if } |S \cap V(G)| > |\mathcal{G}|.$$ 

Then $|S'| \leq |S| = γ_p(T) - 1$ and either $S' \cap V(G) < |S \cap V(G)|$ or $|S' \cap V(G)| = |S \cap V(G)|$ and $|S' \cap \mathcal{G}| > |S \cap \mathcal{G}|$. This contradicts the choice of $S$ if $S'$ is an $η_p$-set of $T$. Thus, to prove the claim, it suffices to show that $S'$ is an $η_p$-set of $T$. Since $S$ is an $η_p$-set of $T$, by the definition of $η_p$-set,

$$η_p(V(T), S', T) \geq η_p(V(T), S, T). \quad (5.2)$$

Note that $η_p(V(G), S', T) = 0$ since $S' \cap V(G) \neq \mathcal{G} \supseteq V(G)$. If $|S \cap V(G)| > |\mathcal{G}|$, then $S' \cap V(H_i) = S \cap V(H_i) \cup \{y\}$ and so $η_p(V(H_i), S', T) \leq η_p(V(H_i), S, T)$ by (3.1) and (3.2). Therefore, by (3.2),

$$η_p(V(T), S', T) = η_p(V(G), S', T) + η_p(V(H_i), S', T) \leq η_p(V(G), S, T) + η_p(V(H_i), S, T) = η_p(V(T), S, T). \quad (5.3)$$

If $|S \cap V(G)| = |\mathcal{G}|$, then $S \cap V(G) \neq \mathcal{G}$ and $\mathcal{G}$ is the unique $γ_p$-set of $G$. Note that $S \cap V(H_i) = S' \cap V(H_i)$ and $T = G \oplus y H_i$. Let $δ = 0$ if $y \in S$, and $δ = 1$ if $y \notin S$. By (3.1) and (3.2),

$$η_p(V(T), S, T) = η_p(V(G), S, T) + η_p(V(H_i), S, T) \geq δ + (η_p(V(H_i), S', T) - δ) = η_p(V(G), S', T) + η_p(V(H_i), S', T) = η_p(V(T), S', T). \quad (5.4)$$

Since $S$ is an $η_p$-set of $T$, $S'$ is also an $η_p$-set of $T$ by (5.2)–(5.4). The claim holds.

By the above claim, $|\mathcal{G}| + |S \cap V(H_i)| = |S| = γ_p(T) - 1$ and so, by (5.1),

$$|S \cap V(H_i)| = \begin{cases} p - 2 & \text{if } i = 1; \\ γ_p(H_i) - 1 & \text{if } i \in \{2, 3, 4\}. \end{cases}$$

Note that $H_i \in \{K_{1, p-1}, K_{1, p}, F_{p-1}, F_{p-1}\}$ with $t \geq p \geq 3$. By calculating directly $η_p$ by (3.1) and (3.2),

$$η_p(V(H_i), S \cap V(H_i), H_i) \geq \begin{cases} p + 2 & \text{if } i \neq 2 \text{ and } S \cap V(H_i) \neq y \cdot y; \\ p + 1 & \text{otherwise.} \end{cases} \quad (5.5)$$

Note that $T = G \oplus y H_i$. Since $S \cap V(G) = \mathcal{G}$, $η_p(V(G), S, T) = 0$ and, by Definition 2.3, $i \neq 2$ if $x \in S$. It follows from (3.2), Lemma 3.4 and (5.5) that

$$η_p(V(T), S, T) = η_p(V(H_i), S, T) + η_p(V(G), S, T)$$

$$\geq \begin{cases} η_p(V(H_i), S \cap V(H_i), H_i) - 1 & \text{if } x \in S \text{ and } S \cap V(H_i) \neq y \cdot y; \\ η_p(V(H_i), S \cap V(H_i), H_i) & \text{otherwise}. \end{cases} \geq p + 1.$$

Since $S$ is an $η_p$-set of $T$, $r_p(T) = η_p(V(T), S, T) \geq p + 1$ by Lemma 3.1, which contradicts the assumption $r_p(T) \leq p$. 

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Case 2. \(|S \cap V(G)| < |\mathcal{G}|\).

Note that \(T' = G \oplus x V(H_i)\), with \(r_p(T') \leq p\) and \(S\) is an \(\eta_p\)-set of \(T\). Since \(|S \cap V(G)| < |\mathcal{G}|\), Lemma 3.1 implies that \(\eta_p(V(G), S \cap V(G), G) \geq r_p(G) = p + 1\), and hence

\[
p \geq r_p(T) = \eta_p(V(T), S, T)
\]

(by (3.2)) \(= \eta_p(V(H_i), S, T)\)

(by Lemma 3.4(1)) \(\geq \eta_p(V(G), S \cap V(G), G) - \begin{cases} 1 & \text{if } S \cap V(G) \not\supseteq p x \text{ and } y \in S; \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\geq \begin{cases} p \quad & \text{if } S \cap V(G) \not\supseteq p x \text{ and } y \in S; \\ p + 1 & \text{otherwise.} \end{cases}
\]

from which we obtain that \(S \cap V(G) \not\supseteq p x \) (and so \(x \not\in S\), \(y \in S\).

\[
\eta_p(V(G), S \cap V(G), G) = p + 1, \quad \text{and } \eta_p(V(H_i), S, T) = 0.
\]

By (5.6), \(\eta_p(V(G), S \cap V(G), G) = r_p(G)\) and so \(S \cap V(G)\) is an \(\eta_p\)-set of \(G\). Note that \(|S| = \gamma_p(T) - 1\). By Lemma 3.2 and (5.1), \(|S \cap V(G)| = |\mathcal{G}| - 1\) and

\[
|S \cap V(H_i)| = \begin{cases} p - 1 & \text{if } i = 1; \\ \eta_p(H_i) & \text{if } i \in \{2, 3, 4\}. \end{cases}
\]

Note that \(H_i \in \{K_{1,p-1}, K_{1,p}, F_{p-1}, F_{1,p-1}\} (t \geq p \geq 3)\) with \(x \in S \cap V(H_i)\). Since \(x \not\in S\), \(S \cap V(H_i) \supseteq p V(H_i)\) by (5.7), which and (5.8) together imply that \(H_i = F_{p-1}\). By the definition of \(\sigma\) in Definition 2.3,

\[
|N_p(x, \mathcal{G}), G) \geq \min\{p + 1, |N_C(x) \cap \mathcal{G}| + 2\}.
\]

Thus either \(\mu_p(x, \mathcal{G}, G) \geq p + 2\) by (3.3) or \(|N_p(x, \mathcal{G}, G) = p + 1\) and \(|N_C(x) \cap \mathcal{G}| \geq p\).

In the former case, it follows from Lemma 4.4 that \(\eta_p(V(G), S \cap V(G), G) \geq p + 2\) since \(x \not\in S\) and \(S \cap V(G) = |\mathcal{G}| - 1\), which contradicts (5.6).

In the latter case, since \(S \cap V(G) \supseteq p x\) and \(|N_C(x) \cap \mathcal{G}| \geq p\), there is a vertex \(z \in N_C(x)\) that \(z \in \mathcal{G}\) but \(z \not\in S\). Let \(G_z\) denote the component of \(G - x\) containing \(z\). Note that \(z \not\in N_p(x, \mathcal{G}, G)\) and \(\eta_p(x, S \cap V(G), G) \geq 1\) since \(x \in \mathcal{G}\) and \(S \cap V(G) \supseteq p x\). Since \(|S \cap V(G_z)| < |\mathcal{G} \cap V(G_z)|\), then \(\eta_p(V(G_z), S \cap V(G_z), G_z) \geq r_p(G_z) = p + 1\) by Lemmas 3.1 and 4.2(1). Since \(x \not\in S\), Lemma 3.4(1) implies that \(\eta_p(V(G_z), S \cap V(G_z), G) = \eta_p(V(G_z), S \cap V(G_z), G_z)\). Hence, by (3.2),

\[
\eta_p(V(G), S \cap V(G), G) \geq \eta_p(x, S \cap V(G), G) + \eta_p(V(G_z), S \cap V(G_z), G_z) \\
\geq 1 + \eta_p(V(G_z), S \cap V(G_z), G_z) \\
\geq p + 2,
\]

which contradicts (5.6). If \(|S \cap V(G_z)| \geq |\mathcal{G} \cap V(G_z)|\), then let

\[
S' = (S - V(G_z)) \cup (\mathcal{G} \cap V(G_z)).
\]

Obviously, \(|S' \cap V(G)| \leq |S \cap V(G)|\) and \(|S' \cap \mathcal{G}| \geq |S \cap \mathcal{G}|\) since \(z \in \mathcal{G} - S\). To end the proof, it suffices to prove that \(S'\) is an \(\eta_p\)-set of \(T\) (this is a contradiction to the choice of \(S\)). Note that \(T = (T - G_z) \oplus x G_z\). Since \(S' - V(G_z) = S - V(G_z)\) and \(z \in S'\), it follows from (3.1) and (3.2) that

\[
\eta_p(V(T - G_z), S', T) \leq \eta_p(V(T - G_z), S, T).
\]

Since \(z \not\in N_p(x, \mathcal{G}, G)\), Lemma 4.2(1) implies that \(\mathcal{G} \cap V(G_z)\) is the unique \(\gamma_p\)-set of \(G_z\) and so \(S' \supseteq p V(G_z)\) in \(T\). By (3.2),

\[
\eta_p(V(T, S', T) = \eta_p(V(T - G_z), S', T) + \eta_p(V(G_z), S', T) \\
\leq \eta_p(V(T - G_z), S, T) + 0 \\
\leq \eta_p(V(T - G_z), S, T) + \eta_p(V(G_z), S, T) = \eta_p(V(T), S, T).
\]

On the other hand, since \(|S'| = |S - V(G_z)| + |\mathcal{G} \cap V(G_z)| \leq |S - V(G_z)| + |S \cap V(G_z)| = |S|\) and \(S\) is an \(\eta_p\)-set of \(T\), \(\eta_p(V(T), S', T) \leq \eta_p(V(T), S, T)\) by Lemma 3.1. Hence,

\[
\eta_p(V(T), S', T) = \eta_p(V(T), S, T),
\]

which means that \(S'\) is also an \(\eta_p\)-set of \(T\). The lemma holds.

It remains to establish the necessity of Theorem 2.5. We do so by proving Lemma 5.2.

Let \(T\) be a tree with the unique \(\gamma_p\)-set \(\mathcal{G}\). Define

\[
M_p(T) = \{x \in V(T) | \text{ there is some } y \in \mathcal{G} \text{ such that } x \in N_p(y, \mathcal{G}, T)\}
\]

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and \( m_p(T) = |M_p(T)| \). Since each vertex in \( M_p(T) \) is the common \( p \)-private neighbor of exactly \( p \) vertices of \( \mathcal{G} \) with respect to \( \mathcal{G} \),
\[
m_p(T) = |M_p(T)| = \frac{1}{p} \sum_{x \in \mathcal{G}} |N_p(x, \mathcal{G}, T)|.
\] (5.9)

Let \( T \) be a tree rooted at \( r \) and \( y \in V(T) \). We use \( C(y) \) and \( D(y) \) to denote the sets of children and descendants, respectively, of \( y \), and define \( D[y] = D(y) \cup \{y\} \). For \( x \in V(T) \), the notation \( d_T(x, y) \) represents the distance between \( x \) and \( y \) in \( T \). Define
\[
\ell(y) = \max\{d_T(x, y) \mid x \in D[y]\}.
\] (5.10)

**Lemma 5.2.** Let \( p \geq 3 \) be an integer and \( T \) a tree. If \( r_p(T) = p + 1 \), then \( T \in \mathcal{F}_p \).

**Proof.** Induction on \( m_p(T) \). Since \( p \geq 3 \) and \( r_p(T) = p + 1 \), **Lemma 4.1** implies that \( T \) has the unique \( \gamma_p \)-set \( \mathcal{G} \) and \( |N_p(v, \mathcal{G}, T)| \geq 1 \) for \( v \in \mathcal{G} \). Note that \( |\mathcal{G}| = \gamma_p(T) > p \) by (1.1) since \( r_p(T) = p + 1 > 0 \). By (5.9),
\[
m_p(T) = \frac{1}{p} \sum_{x \in \mathcal{G}} |N_p(v, \mathcal{G}, T)| \geq \frac{1}{p} \sum_{v \in \mathcal{G}} 1 = \frac{1}{p} |\mathcal{G}| > 1.
\]
Furthermore, \( m_p(T) \geq 2 \) since \( m_p(T) \) is an integer.

If \( m_p(T) = 2 \), then let \( m_p(T) = \{x, y\} \). Since \( |N_p(v, \mathcal{G}, T)| \geq 1 \) for \( v \in \mathcal{G} \), \( \mathcal{G} = (N_1(x) \cap \mathcal{G}) \cup (N_1(y) \cap \mathcal{G}) \cup \{x, y\} \). Suppose that \( T \) has a vertex \( z \) not in \( \mathcal{G} \cup \{x, y\} \). Since \( p \geq 3 \) and \( \mathcal{G} \), \( z \) has two neighbors in either \( N_1(x) \cap \mathcal{G} \) or \( N_1(y) \cap \mathcal{G} \), which means that \( T \) contains a cycle with length 4. This contradicts that \( T \) is a tree. So \( V(T) = \mathcal{G} \cup \{x, y\} = (N_1(x) \cap \mathcal{G}) \cup (N_1(y) \cap \mathcal{G}) \cup \{x, y\} \).

Note that \( x \) and \( y \) have at most one common neighbor in \( T \). Since \( |N_1(x) \cap \mathcal{G}| = p \) and \( |N_1(y) \cap \mathcal{G}| = p \),
\[
T = F_{p-1} \text{ or } S_{p,p}.
\]
Thus \( T \) is obtained from \( K_{1,p} \) by \( \phi_1 \) if \( T = F_{p-1} \), otherwise by \( \phi_2 \), and so \( T \in \mathcal{F}_p \). This establishes the base case.

Let \( m_p(T) \geq 3 \). Assume that, for any tree \( T' \) with \( r_p(T') = p + 1 \), if \( m_p(T') < m_p(T) \) then \( T' \in \mathcal{F}_p \).

We root \( T \) at a leaf \( r \). Since \( m_p(T) \geq 3 \), \( T \) is not a star and so \( \ell(r) \geq 3 \) by (5.10). By (5.10), \( 0 \leq \ell(y) \leq \ell(r) \) for \( y \in V(T) \). Let
\[
V_i = \{y \in V(T) \mid \ell(y) = i\} \quad \text{for } 0 \leq i \leq \ell(r).
\] (5.11)

Then \( V_0, V_1, \ldots, V_{\ell(r)} \) is a partition of \( V(T) \), and satisfies the following properties.
1. \( V_0 = L(T) - \{r\} \subseteq \mathcal{G} \). It is trivial by (5.11) and **Lemma 2.1**.
2. For \( y \in V_i \), \( i \in M_p(T) \); (ii) \( d_T(y) = p \) or \( p + 1 \); (iii) \( T \in \mathcal{F}_p \) if \( d_T(y) = p + 1 \).

**Proof of II.** By (5.11), \( D(y) \subseteq V_0 \) and \( y \) has exactly one neighbor (i.e., the father of \( y \)) not in \( V_0 \). By I and **Lemma 4.1**(1), \( y \in M_p(T) \) and so \( d_T(y) = p \) or \( p + 1 \). Both (i) and (ii) hold.

We now prove (iii). Let \( T' = T - D[y] \) and \( x \) be the father of \( y \). Since \( D(y) \subseteq V_0 \subseteq \mathcal{G} \) by I and \( d_T(y) = p + 1 \), \( x \not\in \mathcal{G} \) and \( T[D[y]] = K_{1,p} \) with center \( y \). So \( T' = T' \oplus_y K_{1,p} \). Since \( \mathcal{G} \) is a \( DS_p \) of \( T \) containing no \( \{x, y\} \), \( \mathcal{G} \cap V(T') \supseteq V(T') \) and \( D(y) > p \). Thus
\[
\gamma_p(T) = |\mathcal{G}| = |\mathcal{G} \cap V(T')| + |D(y)| \geq \gamma_p(T') + \gamma_p(K_{1,p}).
\]
Furthermore, \( \gamma_p(T') = \gamma_p(T') + \gamma_p(K_{1,p}) \) since the union between a \( \gamma_p \)-set of \( T' \) and a \( \gamma_p \)-set of \( K_{1,p} \) is also a \( DS_p \) of \( T' \), which implies that \( \mathcal{G} \cap V(T') \) is a \( \gamma_p \)-set of \( T' \). Since \( M_p(T) \geq 3 \), \( T' \) contains at least two vertices not in \( \mathcal{G} \) and, to \( p \)-dominate them, \( |\mathcal{G} \cap V(T')| \geq p + 1 \). By **Theorem 1.1** and **Lemma 3.4**(2),
\[
p + 1 \geq r_p(T) \geq r_p(T') + p = p + 1,
\]
which implies that \( r_p(T') = p + 1 \). So \( \mathcal{G} \cap \mathcal{G} \cap V(T') \) by **Lemma 4.1**(2), and then \( m_p(T') = m_p(T) - 1 \) since \( y \in M_p(T) \) by (i). Applying the induction on \( T' \), \( T' \in \mathcal{F}_p \). Since \( T = T' \oplus_y K_{1,p} \) and \( y \not\in \mathcal{G} \cap V(T') \), \( T \) is obtained from \( T' \) by \( \phi_2 \), and so \( T \in \mathcal{F}_p \) (iii) follows. \( \square \)

To the end, assume, by II, that
\[
v \in M_p(T) \quad \text{and} \quad d_T(v) = p \quad \text{for each} \quad v \in V_1.
\] (5.12)

Then the father of each vertex in \( V_1 \) belongs to \( \mathcal{G} \), and so
\[
V_2 \subseteq \mathcal{G}.
\] (5.13)

Let \( x \in V_2 \) and \( P = xwvu \) be a path in \( T[D[x]] \) such that \( d_T(w) \) is as large as possible. By I, (5.12) and (5.13), \( u \in V_0 \subseteq \mathcal{G} \), \( v \in V_1 \subseteq M_p(T) \) is a stem of \( T \) with \( d_T(x) = p \), and \( w \in V_2 \subseteq \mathcal{G} \). By **Lemma 3.5**, \( \mu_p(w, \mathcal{G}, T) \geq r_p(T) = p + 1 \).
Case 1. \( \mu_p(w, \mathcal{T}, T) \geq p + 2 \).

Let \( T' = T - D[w] \). Since \( x \) is a stem of \( T \) and \( |D[w]| = d_T(w) - 1 = p - 1, T[D[w]] = K_{1,p-1} \) with center \( v \) and so \( T = T' \cup x, T' \geq 3, T' \) contains at least two \( p \)-private neighbors with respect to \( \mathcal{T} \) and hence \( |V(T')| \geq p + 2 \), which implies that \( \gamma_p(T') \geq p + 1 \) since \( p \) vertices of the tree \( T' \) have at least one common neighbor in \( T' \).

We claim that \( r_p(T') = p + 1 \). It suffices to prove \( r_p(T') \geq p + 1 \) by Theorem 1.1. Since \( w \in V_2 \subseteq \mathcal{T}, \mathcal{T} \cap V(T') \neq \emptyset, T' \), and so \( \gamma_p(T') = |\mathcal{T}| = |\mathcal{T} \cap V(T')| + (p - 1) \geq \gamma_p(T') + p - 1 \). Let \( X' \) be an \( \eta_p \)-set of \( T' \) and \( X = X' \cup D(w) \). By Lemma 3.2,

\[
|X| = |X'| + |D(w)| = (\eta_p(T') - 1) + (p - 1) \leq \gamma_p(T),
\]

and then \( \eta_p(V(T), X, T) \geq \eta_p(T') = p + 1 \) by Lemma 3.1. Since \( v \notin X, \eta_p(V(T'), X, T' \geq 1 \) implies that \( \eta_p(V(T'), X, T') = \eta_p(V(T'), X, T) \). If \( w \in X' \), then \( \eta_p(V(K_{1,p-1}), X, T) = 0 \) and

\[
r_p(T') = \eta_p(V(T'), X, T') = \eta_p(V(T'), X, T) = \eta_p(V(T), X, T) - \eta_p(V(K_{1,p-1}), X, T) \geq p + 1.
\]

If \( w \notin X' \), then \( \eta_p(V(K_{1,p-1}), X, T) = 1 \) by (3.1) and (3.2), and \( \eta_p(V(T), X, T) \geq p + 2 \) by Lemma 4.4 since \( r_p(T) = p + 1 \) and \( \mu_p(w, \mathcal{T}, T) \geq p + 2 \). Thus

\[
r_p(T') = \eta_p(V(T'), X, T') = \eta_p(V(T'), X, T) \geq p + 1.
\]

The claim follows.

Since \( r_p(T') = p + 1 \), \( T' \) has the unique \( \gamma_p \)-set \( \mathcal{T} \) of \( T' \) by Lemma 4.1(1). We now show \( \mathcal{T} \neq \mathcal{T} \cup V(T') \). Suppose, to the contrary, that \( \mathcal{T} \neq \mathcal{T} \cup V(T') \). Note that \( \mathcal{T} \cup V(T') \neq V(T) = V(T') \) since \( w \in \mathcal{T} \). Then \( |\mathcal{T} \cup V(T')| \geq |\mathcal{T}| + 1 \). Since \( \mathcal{T} \cup V(T') \neq V(T') \), \( \mathcal{T} \cup V(T') \neq |\mathcal{T}| + 1 \) since \( \mathcal{T} \cup V(T') \neq V(T) \) and \( |\mathcal{T} \cup D[w]| = |\mathcal{T}| + p \leq |\mathcal{T} \cup V(T')| + (p - 1) = \gamma_p(T'), \mathcal{T} \cup D[w] \) is a \( \gamma_p \)-set of \( T' \) different to \( \mathcal{T} \). This contradicts that \( \mathcal{T} \cup V(T') \) is the unique \( \gamma_p \)-set of \( T' \). Hence \( \mathcal{T} \neq \mathcal{T} \cup V(T') \).

Since \( \mathcal{T} \neq \mathcal{T} \cup V(T') \), \( w \notin N_p(w, \mathcal{T}, T) \), \( m_p(T') = m_p(T) - 1 \). Applying the induction on \( T' \), \( T' \in \mathcal{T}_p \). Since \( w \notin \mathcal{T} \cup V(T') = \mathcal{T} \), \( T \) is obtained from \( T' \) by \( \mathcal{E}_1 \), and hence \( T \in \mathcal{T}_p \).

Case 2. \( \mu_p(w, \mathcal{T}, T) = p + 1 \).

By the definition of \( \mu_p \) in (3.3),

\[
|N_p(w, \mathcal{T}, T)| + \max\{0, p - |N_T(w) \cap \mathcal{T}|\} = p + 1.
\]

Since \( w \in V_2 \subseteq \mathcal{T} \) by (5.13), \( w \) is not a stem of \( T \) and so \( C(w) \subseteq V_1 \subseteq M_p(T) \) by (5.11) and II(i). Therefore,

\[
C(w) \subseteq N_p(w, \mathcal{T}, T),
\]

and, for \( v' \in C(w) \), the component of \( T - w \) containing \( v' \) is a star \( K_{1,p-1} \) with center \( v' \).

Case 2.1 \( x \in \mathcal{T} \).

Let \( T' = T - D[w] \). Since \( x \in \mathcal{T}, N_p(w, \mathcal{T}, T) = C(w) \) by (5.15) and \( N_T(w) \cap \mathcal{T} = \{x\} \). Thus \( |C(w)| = 2 \) by (5.14). So \( T[D[w]] = F_{p-1} \) with center \( w \) and \( T = T' \cup x, F_{p-1} \).

Since \( x \notin N_p(w, \mathcal{T}, T) \), \( r_p(T') = p + 1 \) and \( \mathcal{T} \cap V(T') \subseteq \mathcal{T} \cap V(T) \) by Lemma 4.2(1). Thus \( m_p(T') = m_p(T) - |C(w)| < m_p(T) \). Applying the induction on \( T' \), \( T' \in \mathcal{T}_p \). Hence if \( x \) satisfies the condition of \( \mathcal{E}_3 \), that is,

\[
|N_p(x, \mathcal{T}, T')| \geq \min\{p + 1, |N_T(x) \cap \mathcal{T}|- 2\},
\]

then \( T \) is obtained from \( T' \) by \( \mathcal{E}_3 \) and \( T \in \mathcal{T}_p \).

We now show (5.16). Since \( x \in \mathcal{T} \) and \( r_p(T) = p + 1, \mathcal{T} \) and Lemma 3.5 together imply that \( |N_p(x, \mathcal{T}, T)| + \max\{0, p - |N_T(x) \cap \mathcal{T}|\} = \mu_p(x, \mathcal{T}, T) \geq r_p(T) = p + 1 \), that is,

\[
|N_p(x, \mathcal{T}, T)| \geq \min\{p + 1, |N_T(x) \cap \mathcal{T}|+ 1\}.
\]

Since \( w \in N_T(x) \cap \mathcal{T} \) and \( \mathcal{T} = \mathcal{T} \cap V(T') \), \( |N_p(x, \mathcal{T}, T)| = |N_p(x, \mathcal{T}, T)| \) and \( |N_T(x) \cap \mathcal{T}| = |N_T(x) \cap \mathcal{T}| - 1 \). Therefore, (5.16) follows from (5.17).

Case 2.2 \( x \notin \mathcal{T} \).

Let \( T' = T - D[x] \) and \( T_0 = T[D[x]] \). Then \( T = T' \cup x, T_0 \), where \( y \) is the father of \( x \).

We claim that \( T_0 = T_{p-1} \) with center \( x \), where \( t = |C(x)| \geq p \). Note that \( N_T(w) = C(w) \cup \{x\} \). Since \( x \notin \mathcal{T} \) and \( C(w) \subseteq N_p(w, \mathcal{T}, T) \) by (5.15), \( N_T(w) \cap \mathcal{T} = \emptyset \) and so \( |N_p(w, \mathcal{T}, T)| = 1 \) by (5.14). Therefore, \( C(w) \subseteq N_p(w, \mathcal{T}, T) = \{v\} \) and

\[
x \notin \mathcal{T} \cup N_p(w, \mathcal{T}, T).
\]

By (5.18), \( |N_T(x) \cap \mathcal{T}| \geq p + 1 \) and so \( t = |C(x)| = |N_T(x)| - 1 \geq p \). Let \( w' \in C(x) \). By the choice of \( P_4 = xwvu, d_T(w') \leq d_T(w) = 2 \leq p \). By Lemmas 2.1 and 4.1(1), \( w' \notin \mathcal{T} \) and \( N_p(w', \mathcal{T}, T) \neq \emptyset \). It follows that \( d_T(w') = 2 \) since \( x \in N_T(w') \) is not a \( p \)-private neighbor with respect to \( \mathcal{T} \). Let \( N_p(w', \mathcal{T}, T) = \{v'\} \). Then \( w' \in V_1 \) and \( d_T(w') = p \) by (5.12). By the arbitrariness of \( w' \), \( T_0 = T[D[x]] = T_{p-1} \) with center \( x \). The claim holds.
Since \( \varphi_T \cap V(T') >_p V(T') \) and \( \varphi_T \cap V(T_0) >_p V(T_0) \) by (5.18), \( \gamma_p(T) = |\varphi_T| = |\varphi_T \cap V(T')| + |\varphi_T \cap V(T_0)| \geq \gamma_p(T') + \gamma_p(T_0) \).
Furthermore, \( \gamma_p(T) = \gamma_p(T') + \gamma_p(T_0) \) because the union between a \( \gamma_p \)-set of \( T' \) and a \( \gamma_p \)-set of \( T_0 \) is a \( DS_p \) of \( T \). So \( \varphi_T \cap V(T') \) (resp., \( \varphi_T \cap V(T_0) \)) is a \( \gamma_p \)-set of \( T' \) (resp., \( T_0 \)).

Note that \( N_p(z, \varphi_T, T) \neq \emptyset \) for any \( z \in \varphi_T \) by Lemma 4.1(1). (5.18) implies that \( T' \) has at least one \( p \)-private neighbor with respect to \( \varphi_T \), and so \( \gamma_p(T') = |\varphi_T \cap V(T')| > p \).
If \( \gamma_p(T') = p \), then \( T' = K_{1,p} \). Thus \( T \) is obtained from \( K_{1,p} \) by \( \vartheta_4 \) and \( T \in \mathcal{F}_p \).
If \( \gamma_p(T') > p + 1 \), then \( r_p(T') \geq r_p(T) = p + 1 \) by Lemma 3.4(2), furthermore, \( r_p(T') = p + 1 \) by Theorem 1.1.
Since \( \varphi_T \cap V(T') \) is a \( \gamma_p \)-set of \( T' \), it follows from Lemma 4.1(2) that \( \varphi_T' = \varphi_T \cap V(T') \), and hence \( m_p(T') = m_p(T) - m_p(T_0) < m_p(T) \).

Applying the induction on \( T' \), \( T' \in \mathcal{F}_p \). Thus \( T \) is obtained from \( T' \) by \( \vartheta_4 \) and \( T \in \mathcal{F}_p \). □

6. Conclusion

We characterize all trees with \( p \)-reinforcement number \( p + 1 \) for \( p \geq 3 \) by a recursive construction. Our proof strongly depends on Lemma 4.4. However, Lemma 4.4 is not true for \( p = 2 \) (see Remark 4.5). When \( p = 2 \), Theorem 1.1 implies that \( r_2(T) \leq 3 \) for any tree \( T \). Very recently, Lu, Song and Yang [18] have presented a sufficient and necessary condition for a tree to have the 2-reinforcement number 3.

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