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Total and paired domination numbers of toroidal meshes

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Abstract Let G be a graph without isolated vertices. The total domination number of G is the minimum number of vertices that can dominate all vertices in G, and the paired domination number of G is the minimum number of vertices in a dominating set whose induced subgraph contains a perfect matching. This paper determines the total domination number and the paired domination number of the toroidal meshes, i.e., the Cartesian product of two cycles C_n and C_m for any $n \geq 3$ and $m \in \{3, 4\}$, and gives some upper bounds for $n, m \geq 5$.

Keywords Total domination number · Paired domination number · Toroidal meshes · Cartesian product

1 Introduction

For notation and graph-theoretical terminology not defined here we follow Xu (2003). Specifically, let $G = (V, E)$ be an undirected graph without loops, multi-edges and isolated vertices, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set, which is a subset of $\{xy | xy$ is an unordered pair of $V\}$. A graph G is *nonempty* if $E(G) \neq \emptyset$. Two vertices x and y are *adjacent* if $xy \in E(G)$. For a vertex x, denote $N(x) = \{y : xy \in E(G)\}$ be the *neighborhood* of x. For a subset $D \subseteq V(G)$, we use

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$G[D]$ to denote the subgraph of G induced by D. We use C_n and P_n to denote a cycle and a path of order n, respectively, throughout this paper.

A subset $D \subseteq V(G)$ is called a *dominating set* if $N(x) \cap D \neq \emptyset$ for each vertex $x \in V(G) \setminus D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. A thorough study of domination appears in Haynes et al. (1998a, 1998b). A subset $D \subseteq V(G)$ of G is called a *total dominating set*, introduced by Cockayne et al. (1980), if $N(x) \cap D \neq \emptyset$ for each vertex $x \in V(G)$ and the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. The total domination in graphs has been extensively studied in the literature. A survey of selected recent results on this topic is given by Henning (2009).

A dominating set D of G is called to be *paired*, introduced by Haynes and Slater (1995, 1998), if the induced subgraph $G[D]$ contains a perfect matching. The *paired domination number* of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a paired dominating set of G. Clearly, $\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G)$ since a paired dominating set is also a total dominating set of G, and $\gamma_p(G)$ is even. Pfaff et al. (1983) and Haynes and Slater (1998) showed that the problems determining the total-domination and the paired-domination for general graphs are NP-complete. Some exact values of total-domination numbers (for example El-Zahar et al. 2008; Rall 2005) and paired-domination numbers (for example Brešar et al. 2005, 2007) for some special classes of graphs have been determined by several authors. In particular, $\gamma_t(P_n \times P_m)$ and $\gamma_p(P_n \times P_m)$ for $2 \leq m \leq 4$ are determined by Gravier (2002), and Proffitt et al. (2001), respectively.

Use $G_{n,m}$ to denote the toroidal meshes, i.e., the Cartesian product $C_n \times C_m$ of two cycles C_n and C_m. Klavžar and Seifter (1995) determined $\gamma(G_{n,m})$ for any $n \geq 3$ and $m \in \{3, 4, 5\}$. In this paper, we obtain the following results.

$$\gamma_t(G_{n,3}) = \left\lceil \frac{4n}{5} \right\rceil;$$

$$\gamma_p(G_{n,3}) = \begin{cases} \lceil \frac{4n}{5} \rceil & \text{if } n \equiv 0, 2, 4 \pmod 5, \\ \lceil \frac{4n}{5} \rceil + 1 & \text{if } n \equiv 1, 3 \pmod 5; \end{cases}$$

$$\gamma_t(G_{n,4}) = \gamma_p(G_{n,4}) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+1 & \text{if } n \equiv 1, 3 \pmod 4, \\ n+2 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

2 Preliminary results

In this section, we recall some definitions, notations and results used in the proofs of our main results. Throughout this paper, we assume that a cycle C_n has the vertex-set $V(C_n) = \{1, \ldots, n\}$.

Use $G_{n,m}$ to denote the toroidal meshes, i.e., the Cartesian product $C_n \times C_m$, which is a graph with vertex-set $V(G_{n,m}) = \{x_{ij} |\ 1 \leq i \leq n, 1 \leq j \leq m\}$ and two vertices x_{ij} and $x_{i'j'}$ being linked by an edge if and only if either $i = i' \in V(C_n)$ and $jj' \in E(C_m)$, or $j = j' \in V(C_m)$ and $ii' \in E(C_n)$.

Let $Y_i = \{x_{ij} |\ 1 \leq j \leq m\}$ for $1 \leq i \leq n$, called a set of *vertical vertices* in $G_{n,m}$.

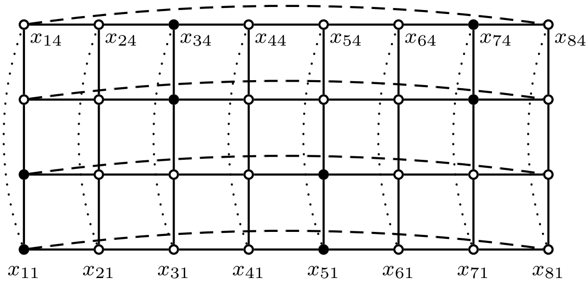

Fig. 1 The minimum total (paired) dominating set (*bold vertices*) of $G_{8,4}$

In Gavlas and Schultz (2002), defined an efficient total dominating set, which is such a total dominating set D of G that $|N(v) \cap D| = 1$ for every $v \in V(G)$. The related research results can be found in Dejter and Serra (2003), Gavlas and Schultz (2002), Huang and Xu (2008).

Lemma 2.1 (Gavlas and Schultz 2002) *If a graph G has an efficient total dominating set D, then the edge-set of the subgraph $G[D]$ forms a perfect matching, and so the cardinality of D is even, and $\{N(v) : v \in D\}$ partitions $V(G)$.*

Lemma 2.2 *Let G be a k-regular graph of order n. Then $\gamma_t(G) \geq \frac{n}{k}$, with equality if and only if G has an efficient total dominating set.*

Proof Since G is k-regular, each $v \in V(G)$ can dominate at most k vertices. Thus $\gamma_t(G) \geq \frac{n}{k}$. It is easy to observe that the equality holds if and only if there exists a total dominating set D such that $\{N(v) : v \in D\}$ partitions $V(G)$, equivalently, D is an efficient total dominating set. □

Lemma 2.3 $\gamma_t(G_{n,m}) = \gamma_p(G_{n,m}) = \frac{nm}{4}$ *for* $n, m \equiv 0 \pmod 4$.

Proof Let $D = \{x_{ij}, x_{i(j+1)}, x_{(i+2)(j+2)}, x_{(i+2)(j+3)} : i, j \equiv 1 \pmod 4\}$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Figure 1 is such a set D in $G_{8,4}$. It is easy to see that D is a paired dominating set of $G_{n,m}$ with cardinality $\frac{nm}{4}$. Thus, $\gamma_p(G_{n,m}) \leq \frac{nm}{4}$.

By Lemma 2.2, $\gamma_t(G_{n,m}) \geq \frac{nm}{4}$. Since $\gamma_t(G_{n,m}) \leq \gamma_p(G_{n,m})$, $\gamma_t(G_{n,m}) = \gamma_p(G_{n,m}) = \frac{nm}{4}$. □

3 Total and paired domination number of $G_{n,3}$

In this section, we determine the exact values of the total and the paired domination numbers of $G_{n,3}$, as stated the following theorem.

Theorem 3.1 *For any $n \geq 3$,*

$$\gamma_t(G_{n,3}) = \left\lceil \frac{4n}{5} \right\rceil$$

and

$$\gamma_p(G_{n,3}) = \begin{cases} \lceil \frac{4n}{5} \rceil, & \text{if } n \equiv 0, 2, 4 \pmod 5; \\ \lceil \frac{4n}{5} \rceil + 1, & \text{if } n \equiv 1, 3 \pmod 5). \end{cases}$$

Proof Let D be a minimum total dominating set of $G_{n,3}$. First, we may assume that $|Y_i \cap D| \le 2$ for any $1 \le i \le n$. Since the symmetry of $G_{n,3}$, we only consider the case $i \notin \{1, n\}$. Indeed, if $|Y_i \cap D| = 3$ for some $i \notin \{1, n\}$, then $x_{(i-1)1}$ and $x_{(i-1)3}$ can not belong to D at the same time since otherwise $(D \setminus Y_i) \cup \{x_{(i+1)1}, x_{(i+1)2}\}$ is also a total dominating set of $G_{n,3}$ but with cardinality less than D, also $x_{(i+1)1}$ and $x_{(i+1)3}$ can not belong to D at the same time. Therefore the set $D' = (D \setminus \{x_{i1}, x_{i3}\}) \cup \{x_{(i-1)2}, x_{(i+1)2}\}$ is also a total dominating set of $G_{n,3}$ with $|D'| = |D|$, and hence we can assume that $|Y_i \cap D| \le 2$.

Let α_k be the number of i's for which $|Y_i \cap D| = k$ for $1 \le i \le n$ and $0 \le k \le 2$. Then we have

$$\alpha_0 + \alpha_1 + \alpha_2 = n. \tag{3.1}$$

Assume $|Y_i \cap D| = 0$ for some $i \notin \{1, n\}$ (we only consider the case $i \notin \{1, n\}$ since the symmetry of $G_{n,3}$). At least one of $|Y_{i-1} \cap D|$ and $|Y_{i+1} \cap D|$ is 2 since the three vertices in Y_i should be dominated by D, which means that

$$2\alpha_2 - \alpha_0 \ge 0. \tag{3.2}$$

If $|Y_i \cap D| = 2$ for some i with $1 \le i \le n$, then the two vertices in $Y_i \cap D$ can dominate at most 7 vertices. Since any vertex $x \in D$ can dominate at most 4 vertices, we have

$$4\alpha_1 + 7\alpha_2 \ge 3n. \tag{3.3}$$

The sum of (3.1), (3.2) and (3.3) implies

$$5\alpha_1 + 10\alpha_2 \ge 4n,$$

and, hence,

$$\gamma_t(G_{n,3}) = |D| = \alpha_1 + 2\alpha_2 \ge \left\lceil \frac{4n}{5} \right\rceil. \tag{3.4}$$

To obtain the upper bounds of $\gamma_t(G_{n,3})$ and $\gamma_p(G_{n,3})$, we set

$$D = \left\{x_{i2} : i \equiv 1, 2 \pmod 5\right\} \cup \left\{x_{j1}, x_{j3} : j \equiv 4 \pmod 5\right\},$$

where $1 \le i \le n$. See Fig. 2, where D consists of bold vertices.

If $n \not\equiv 3 \pmod 5$, then D is a total dominating set and $\gamma_t(G_{n,3}) \le |D| = \lceil \frac{4n}{5} \rceil$.

If $n \equiv 3 \pmod 5$, then $D \cup \{x_{n2}\}$ is a total dominating set and $\gamma_t(G_{n,3}) \le |D| + 1 = \lceil \frac{4n}{5} \rceil$.

Combining these facts with (3.4), we have that $\gamma_t(G_{n,3}) = \lceil \frac{4n}{5} \rceil$.

If $n \equiv 0, 2, 4 \pmod 5$, then D is a paired dominating set and $\gamma_p(G_{n,3}) \le |D| = \lceil \frac{4n}{5} \rceil$.

If $n \equiv 1 \pmod 5$, then $D \cup \{x_{n1}\}$ is a paired dominating set and $\gamma_p(G_{n,3}) \le |D| + 1 = \lceil \frac{4n}{5} \rceil + 1$.

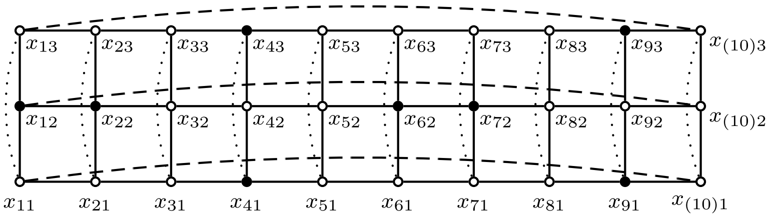

Fig. 2 The minimum paired dominating set (*bold vertices*) of $G_{10,3}$

If $n \equiv 3 \pmod 5$, then $D \cup \{x_{n1}, x_{n2}\}$ is a paired dominating set and $\gamma_p(G_{n,3}) \leq |D| + 2 = \lceil \frac{4n}{5} \rceil + 1$.

Since $\gamma_p(G_{n,3}) \geq \gamma_t(G_{n,3})$ and $\gamma_p(G_{n,3})$ is even, $\gamma_p(G_{n,3}) = \lceil \frac{4n}{5} \rceil$ if $n \equiv 0, 2, 4 \pmod 5$, and $\gamma_p(G_{n,3}) = \lceil \frac{4n}{5} \rceil + 1$ if $n \equiv 1, 3 \pmod 5$.

The theorem follows. □

4 Total and paired domination number of $G_{n,4}$

In this section, we determine the exact values of $\gamma_t(G_{n,4})$ and $\gamma_p(G_{n,4})$, the latter was obtained by Brešar et al. (2007).

Lemma 4.1 $\gamma_p(G_{n,4}) = \gamma_t(G_{n,4}) = n + 1$ *for* $n \equiv 1, 3 \pmod 4$.

Proof For $n \equiv 1 \pmod 4$, let

$$D = \left\{x_{i1}, x_{i2}, x_{(i+2)3}, x_{(i+2)4} : i \equiv 1 \pmod 4, i \neq n\right\} \cup \{x_{n1}, x_{n2}\}.$$

Then D is a paired dominating set of $G_{n,4}$ with cardinality $n + 1$. For $n \equiv 3 \pmod 4$, $D = \{x_{i1}, x_{i2}, x_{(i+2)3}, x_{(i+2)4} : i \equiv 1 \pmod 4\}$ is a paired dominating set of $G_{n,4}$ with cardinality $n + 1$. Thus, $\gamma_t(G_{n,4}) \leq \gamma_p(G_{n,4}) \leq n + 1$ for $n \equiv 1, 3 \pmod 4$.

By Lemma 2.2, $\gamma_t(G_{n,4}) \geq \frac{4n}{4} = n$. Now, we prove $\gamma_t(G_{n,4}) \geq n + 1$. Suppose to the contrary that $\gamma_t(G_{n,4}) = n$. By Lemma 2.2, $G_{n,4}$ has an efficient total dominating set D'. By Lemma 2.1, $|D'| = n$ is even, a contradiction. Therefore $\gamma_t(G_{n,4}) > n$, and hence $\gamma_p(G_{n,4}) = \gamma_t(G_{n,4}) = n + 1$. □

Lemma 4.2 $\gamma_t(G_{n,4}) \leq \gamma_p(G_{n,4}) \leq n + 2$ *for* $n \equiv 2 \pmod 4$.

Proof Let

$$D = \left\{x_{i1}, x_{i2}, x_{(i+2)3}, x_{(i+2)4} : i \equiv 1 \pmod 4, i \leq n - 2\right\}$$
$$\cup \{x_{(n-1)1}, x_{(n-1)2}, x_{n1}, x_{n2}\}.$$

Then D is a paired dominating set of $G_{n,4}$ with cardinality $n + 2$. Thus, $\gamma_t(G_{n,4}) \leq \gamma_p(G_{n,4}) \leq n + 2$. □

To prove $\gamma_t(G_{n,4}) \geq n+2$ for $n \equiv 2 \pmod 4$, we need the following notations and two lemmas. Let $H_i^j = Y_i \cup Y_{i+1} \cup \cdots \cup Y_{i+j-1}$, and let G_i^j be the graph obtained from $G_{n,4} - H_i^j$ by adding the edge-set $\{x_{(i-1)k}x_{(i+j)k} : 1 \leq k \leq 4\}$, where the subscripts are modulo n. Clearly, $G_i^j \cong G_{n-j,4}$.

Lemma 4.3 *Let D be a total dominating set of $G_{n,4}$. Then $|D \cap H_i^4| \geq 4$ for any i with $1 \leq i \leq n$. Moreover, if there exists some i with $1 \leq i \leq n$ such that $|N(v) \cap D| = 1$ for any vertex v in H_i^4, then $D' = D \setminus (D \cap H_i^4)$ is a total dominating set of G_i^4.*

Proof Without loss of generality, assume $i = 2$. It can be easy verified to dominate 8 vertices in $Y_3 \cup Y_4$, at least 4 vertices are needed, and hence $|D \cap H_2^4| \geq 4$.

We now show the second assertion. Suppose to the contrary that D' is not a total dominating set of G_2^4. Then there is a vertex u in $Y_1 \cup Y_6$ such that it is not dominated by D', that is, $N_{G_2^4}(u) \cap D' = \emptyset$. Without loss of generality assume $u = x_{11}$. Then $x_{21} \in D$ and $x_{61} \notin D$. Also $x_{41} \notin D$ since $|N(x_{31}) \cap D| = 1$.

Since x_{33} should be dominated by D and $|N(x_{33}) \cap D| = 1$, only one of x_{32}, x_{34}, x_{23}, and x_{43} belongs to D. If $x_{32} \in D$ or $x_{34} \in D$, then $|N(x_{31}) \cap D| \geq 2$, a contradiction. If $x_{23} \in D$, then $|N(x_{22}) \cap D| \geq 2$, a contradiction. Thus, $x_{43} \in D$. Since x_{51} should be dominated by D, $x_{52} \in D$ or $x_{54} \in D$. But then $|N(x_{53}) \cap D| \geq 2$, a contradiction. Thus, $D' = D \setminus (D \cap H_2^4)$ is a total dominating set of G_i^4. □

Lemma 4.4 *Let D be a total dominating set of $G_{n,4}$. If x_{ij} is dominated by two vertices $u, v \in D$, then there exists a vertex w in H_{i-1}^2 or H_i^2 such that $|N(w) \cap D| \geq 2$.*

Proof Without loss of generality, let $i = j = 2$. If $u, v \in Y_2$, then assume $u = x_{21}$, $v = x_{23}$ and, hence, $|N(x_{24}) \cap D| \geq 2$.

If one of u and v is in Y_2 and another is in $Y_1 \cup Y_3$, then without loss of generality assume $u = x_{21} \in Y_2$ and $v = x_{32} \in Y_3$. And then $|N(x_{31}) \cap D| \geq 2$.

If one of u and v is in Y_1 and another is in Y_3, then without loss of generality assume $u = x_{12} \in Y_2$ and $v = x_{32} \in Y_3$. Since x_{24} should be dominated by D, let $s \in N(x_{24}) \cap D$. It is clearly that $N(s) \cap N(u) \neq \emptyset$ or $N(s) \cap N(v) \neq \emptyset$, which implies that there exists a vertex $w \notin \{u, v\}$ in $H_1^2 \cup H_2^2$ such that $|N(w) \cap D| \geq 2$. □

Lemma 4.5 $\gamma_t(G_{n,4}) = \gamma_p(G_{n,4}) = n + 2$ *for* $n \equiv 2 \pmod 4$.

Proof By Lemma 4.2, we only need to show $\gamma_t(G_{n,4}) \geq n + 2$. To this end, let $n = 4k + 2$. We proceed by induction on $k \geq 1$. It is easy to verify that $\gamma_t(G_{6,4}) = 8$ and $\gamma_t(G_{10,4}) = 12$. The conclusion is true for $k = 1, 2$. Assume that the induction hypothesis is true for $k - 1$ with $k \geq 3$.

Let D be a minimum total dominating set of $G_{n,4}$, where $n = 4k + 2$ for $k \geq 3$. Assume to the contrary that $|D| \leq n + 1$. Since any vertex u can dominate at most 4 vertices in $G_{n,4}$ and $|V(G_{n,4})| = 4n$, there are at most four vertices such that each of them is dominated by at least two vertices in D.

We now prove that there exists some $i \in \{1, 2, \ldots, n\}$ such that $|N(v) \cap D| = 1$ for any vertex $v \in H_i^4$. There is nothing to do if there are at most three vertices such that each of them is dominated by at least two vertices since $n \geq 14$. Now, assume there are exactly four vertices such that each of them is dominated by at least two vertices. By Lemma 4.4, there exists two integers s and t with $1 \leq s, t \leq n$ such that two of the four vertices are in H_s^2 and the other two are in H_t^2. Therefore, there exists an integer i with $1 \leq i \leq n$ such that for any vertex $v \in H_i^4$, $|N(v) \cap D| = 1$ since $n \geq 14$.

By Lemma 4.3, $|D \cap H_i^4| \geq 4$ and $D' = D \setminus (D \cap H_i^4)$ is a total dominating set of $G_i^4 \cong G_{n-4,4}$. By the inductive hypothesis, $|D'| \geq \gamma_t(G_{n-4,4}) \geq n-2$. It follows that

$$n+1 \geq |D| = \left|D \cap H_i^4\right| + \left|D'\right| \geq 4 + n - 2 = n+2,$$

a contradiction, which implies that $\gamma_t(G_{n,4}) = |D| \geq n+2$. By the induction principle, the lemma follows. □

By combining the above results in this section and Lemma 2.3, we get the following theorem immediately.

Theorem 4.1 *For any integer $n \geq 3$,*

$$\gamma_t(G_{n,4}) = \gamma_p(G_{n,4}) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 4; \\ n+1, & \text{if } n \equiv 1, 3 \pmod 4; \\ n+2, & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

5 Upper bounds of $\gamma_p(G_{n,m})$ for $n, m \geq 5$

The values of $\gamma_t(G_{n,m})$ and $\gamma_p(G_{n,m})$ for $m \in \{3, 4\}$ have been determined in the above sections, but their values for $m \geq 5$ have been not determined yet. In this section, we present their upper bounds. Since $\gamma_t(G) \leq \gamma_p(G)$ for any graph G without isolated vertices, we establish upper bounds only for $\gamma_p(G_{n,m})$ if we can not obtain a smaller upper bound of $\gamma_t(G_{n,m})$ than that of $\gamma_p(G_{n,m})$.

Lemma 5.1 $\gamma_t(G_{n,m}) \leq \gamma_t(G_{n+1,m})$ *and* $\gamma_p(G_{n,m}) \leq \gamma_p(G_{n+1,m})$.

Proof Let D be a minimum paired (total) dominating set of $G_{n+1,m}$.

If $D \cap Y_{n+1} = \emptyset$, then D is also a paired (total) dominating set of $G_{n,m}$, and hence $\gamma_p(G_{n,m}) \leq |D|$ ($\gamma_t(G_{n,m}) \leq |D|$).

Assume $D \cap Y_{n+1} \neq \emptyset$ below. Let $A = \{j | x_{(n+1)j} \in D\}$ and $B = \{j | x_{nj} \in D\}$. Then $D' = (D \setminus Y_{n+1}) \cup \{x_{(n-1)j} | j \in A \cap B\} \cup \{x_{nj} | j \in A \setminus B\}$ is a total dominating set of $G_{n,m}$ and $|D'| \leq |D|$. Therefore $\gamma_t(G_{n,m}) \leq \gamma_t(G_{n+1,m})$.

The vertex set D' may not be a paired dominating set of $G_{n,m}$, that means, the induced subgraph G by D' in $G_{n,m}$ may contains odd connected components. Let p be the number of odd connected components in G. It is clear that $|D'| \leq |D| - p$ by the construction of D' from D. Therefore, we can obtain D'' by adding at most p vertices to D' such that the induced subgraph by D'' in $G_{n,m}$ does not contain odd connected components. Then D'' is a paired dominating set of $G_{n,m}$, and hence $\gamma_p(G_{n,m}) \leq |D''| \leq |D|$. □

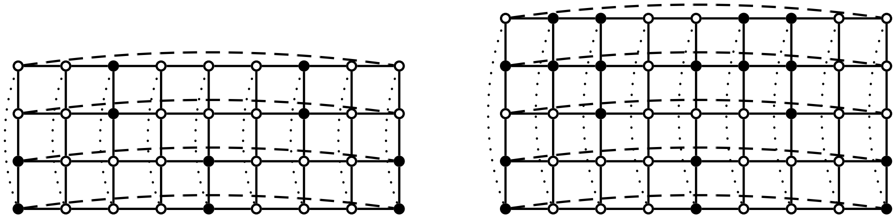

Fig. 3 Two paired dominating sets (*bold vertices*) of $G_{9,4}$ and $G_{9,5}$ in Theorems 5.2 and 5.3, respectively

Theorem 5.1 $\gamma_p(G_{n,m}) \le 4\lceil \frac{n}{4}\rceil\lceil \frac{m}{4}\rceil$.

Proof Let $n = 4a - i$ and $m = 4b - j$ where $0 \le i, j \le 3$. By Lemma 2.3, $\gamma_p(G_{4a,4b}) = 4ab = 4\lceil \frac{n}{4}\rceil\lceil \frac{m}{4}\rceil$. By Lemma 5.1,

$$\gamma_p(G_{m,n}) \le \gamma_p(G_{4a,4b}) = 4\left\lceil \frac{n}{4}\right\rceil\left\lceil \frac{m}{4}\right\rceil. \qquad \square$$

For $n, m \ge 5$, let $m \equiv a \pmod 4$ and $n \equiv b \pmod 4$ where $0 \le a, b \le 3$. We will establish some better bounds of $\gamma_t(G_{n,m})$ and $\gamma_p(G_{n,m})$ than those in Theorem 5.1 for some special a and b. Let

$$D_e = \{x_{ij}, x_{i(j+1)}, x_{(i+2)(j+2)}, x_{(i+2)(j+3)} : i, j \equiv 1 \pmod 4\},$$

where $1 \le i \le n-2$, $1 \le j \le m-3$, and $n, m \ge 5$.

Theorem 5.2 $\gamma_p(G_{n,m}) \le \frac{(n+1)m}{4}$ *for* $m \equiv 0 \pmod 4$ *and* $n \equiv 1 \pmod 4$.

Proof Let $D = D_e \cup \{x_{nj}, x_{n(j+1)} : j \equiv 1 \pmod 4\}$, where $1 \le j \le m-3$. Then, it is easy to see that D is a paired dominating set of $G_{n,m}$ with cardinality $\frac{(n+1)m}{4}$ (see Fig. 3 for $n = 9$ and $m = 4$). Thus, $\gamma_p(G_{n,m}) \le \frac{(n+1)m}{4}$. $\square$

Theorem 5.3 $\gamma_t(G_{n,m}) \le \frac{(n+1)(m+1)}{4}$ *and* $\gamma_p(G_{n,m}) \le \frac{(n+1)(m+3)}{4} - 2$ *for* $m, n \equiv 1 \pmod 4$.

Proof Let $D = D_e \cup \{x_{nj}, x_{n(j+1)}, x_{(i+1)(m-1)}, x_{(i+2)m} : i, j \equiv 1 \pmod 4\} \cup \{x_{nm}\}$, where $1 \le i \le n-2$ and $1 \le j \le m-3$. Then, it is easy to see that D is a total dominating set of $G_{n,m}$ with cardinality $\frac{(n+1)(m+1)}{4}$ (see Fig. 3 for $n = 9$ and $m = 5$), and $D \cup \{x_{i(m-1)}, x_{(i+1)m} : i \equiv 1 \pmod 4\} \setminus \{x_{nm}\}$ is a paired dominating set of $G_{n,m}$ with cardinality $\frac{(n+1)(m+3)}{4} - 2$, where $1 \le i \le n-2$. Thus, $\gamma_t(G_{n,m}) \le \frac{(n+1)(m+1)}{4}$ and $\gamma_p(G_{n,m}) \le \frac{(n+1)(m+3)}{4} - 2$. $\square$

Theorem 5.4 $\gamma_p(G_{n,m}) \le \frac{(n+1)(m+1)}{4} - 2$ *for* $m \equiv 3 \pmod 4$ *and* $n \equiv 1 \pmod 4$.

Proof Let $D = (D_e \cup \{x_{i(m-2)}, x_{i(m-1)}, x_{(i+1)m}, x_{(i+2)m} : i \equiv 1 \pmod 4\} \cup \{x_{nj}, x_{n(j+1)} : j \equiv 1 \pmod 4\} \cup \{x_{(n-1)m}, x_{n(m-2)}\}) \setminus \{x_{(n-3)m}, x_{1(m-1)}\}$, where

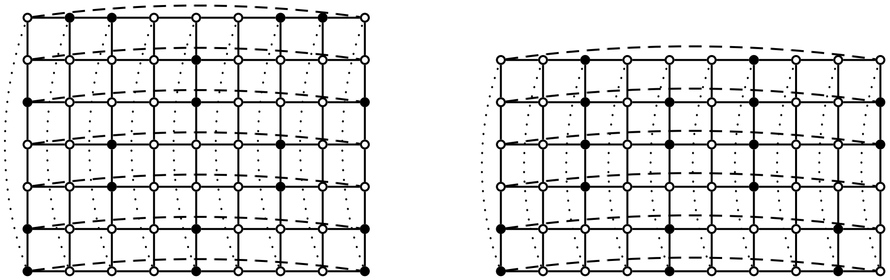

Fig. 4 Two paired dominating sets (*bold vertices*) of $G_{9,7}$ and $G_{10,6}$ in Theorems 5.4 and 5.5, respectively

$1 \le i \le n-2$ and $1 \le j \le m-3$ (see Fig. 4 for $n=9$ and $m=7$). Then, D is a paired dominating set of $G_{n,m}$ with cardinality $\frac{(n+1)(m+1)}{4}-2$. □

Corollary 5.1 $\gamma_p(G_{n,m}) \le \frac{(n+1)(m+2)}{4}-2$ *for* $m \equiv 2 \pmod 4$ *and* $n \equiv 1 \pmod 4$.

Proof By Lemma 5.1, $\gamma_p(G_{n,m}) \le \gamma_p(G_{n,m+1})$. The corollary follows from Theorem 5.4. □

Theorem 5.5 $\gamma_p(G_{n,m}) \le \frac{(n+2)(m+2)}{4}-6$ *for* $m,n \equiv 2 \pmod 4$.

Proof Let $D = (D_e \cup \{x_{i(m-2)}, x_{i(m-1)}, x_{(i+2)(m-1)}, x_{(i+2)m} : i \equiv 1 \pmod 4\} \cup \{x_{(n-1)j}, x_{(n-1)(j+1)}, x_{n(j+2)}, x_{n(j+3)} : j \equiv 1 \pmod 4\} \cup \{x_{n(m-1)}\}) \setminus \{x_{1(m-2)}, x_{1(m-1)}, x_{n(m-3)}\}$, where $1 \le i \le n-2$ and $1 \le j \le m-3$ (see Fig. 4 for $n=10$ and $m=6$). Then D is a paired dominating set of $G_{n,m}$ with cardinality $\frac{(n+2)(m+2)}{4}-6$. Thus, $\gamma_p(G_{n,m}) \le \frac{(n+2)(m+2)}{4}-6$. □

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